Linear Algebra
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Lecture – 35
Homomorphism theorem of vector spaces

Now, I want to set up a notation so that things are easier to verify.

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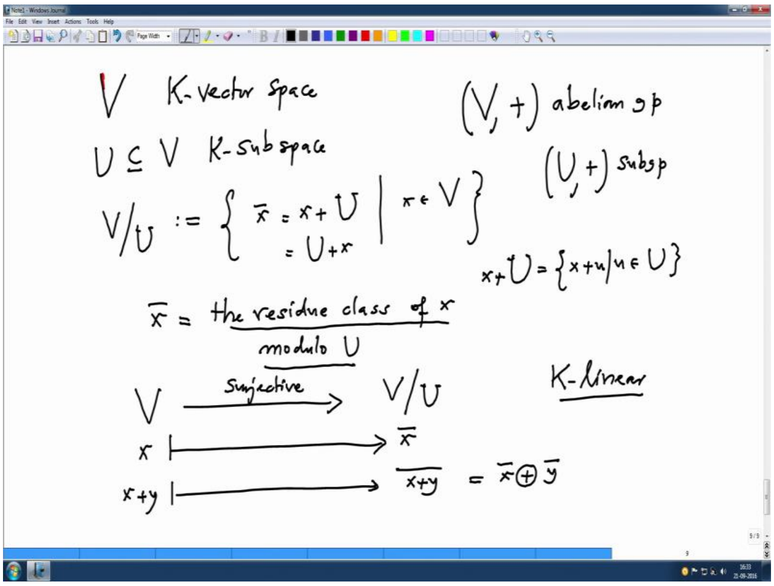

So, V vector space, K-vector space and we have U subspace K-subspace. And then I denote V by U this equation set. The elements of this are precisely the cosides of U I then additive sub group of V. So, that is x plus U. And this x plus U I will write it x bar. And now x is varying U V. Note that some of them may be equal because if the difference belong to U then x bar equal to y bar. And also note that this is also same as U plus x left coset equal to right coset, because we are in a Abelian group V plus; V plus Abelian group and this U plus is a sub group of this which is normal. So, this left coset is same as right coset or even visibility scene, because x plus U is by definition U will equal to also elements x plus U; U varies in U.

So, that is a coset: this is a left coset, this is a right coset. But for any x in V this two cosets are equal and that I am denoting by x bar; x bar is a coset that is what one as to remember. And x bar is also called; I will keep calling x bar is equal to the residue class of the element x modulo U. This terminology is inherited form that from z to z mod n residue class modulo n. So, the same reason I call this x bar as the residue class of x modulo U.

Now what is the map V to? V by U this is the quotient map. Any x here go to x bar. So, the fibers of this map or precisely the cosets, and this map are surjective that is clear. The problem is I want to check now whether this map is K linear. But before I check K linearity I need to ensure that there is addition here and there is a scalar multiplication here. So, this becomes a vector space. And then I want to check this map is K linear. So, let me define therefore, the addition on this V by U and also scalar multiplication. And; obviously, the it will come the addition of V and the scalar multiplication of V.

And what do you want? K linearity means x plus y; first of all x plus y by definition this is going to x plus y whole bar. And what do we are asking is, when you this equal to x bar plus y bar for instance and how we are defining this plus. So obviously, the definition is clear we only need to check that it is well defined. That means, it does not depend on the representative of this coset.

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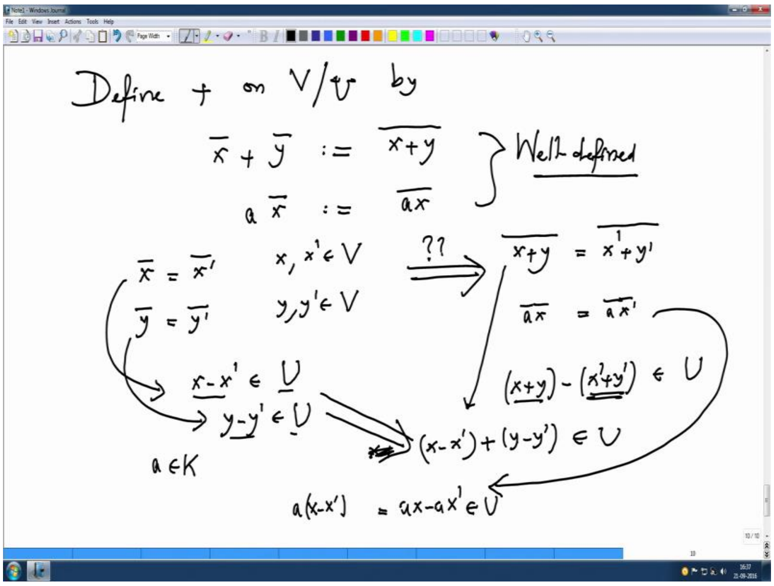

So, let us define addition on V by U by x bar plus y bar equal to x plus y add it in V and take the bar of that. And remember bar means you have taking the cosets. And also you can define scalar multiplication as taken is scalar a times x bar, define it to be first take a scalar multiplication of a on thus element x in V and take the bar of them. And what we need to check that this operations what we defined or well defined. That means, they do not depend on the representative class of x bar and y bar.

So, what does that means let us write down. That means, if x bar equal to x prime bar and y bar equal to y prime bar. So, where x prime are element in V and y y prime are element in V; and what are this equality. Then we should check this is a main problem to check that x plus y whole bar equal to x bar plus x prime plus y prime bar. And also we have to check for any scalar a a x bar is same as a x prime bar. This whole bar here. Then only these definitions will make sense, they do not depend on the representative of the equivalence class. So, what are this mean? This means the difference x minus x prime belong to the subspace U. And this means the difference y minus y prime belong to the subspace U. And what do you want to check? The checking means the difference that is x plus y minus x prime plus y prime they belong to you, this is what we want. And we have given this.

Well, if you stay rated what do we do, you add this equations. If you add this two equations this belong to U, this belong to U, then there is sum will also belong to U; the x minus x prime plus y minus y prime that also belong to U because it is a subspace. But this when you rearrange it you put x and y together you get this and take minus out and this and this together if it this term; so very easy. So, this check that the sum that we have defined on V by U is well defined.

Now for this it is similar for this now I want to check for any a this condition means what, this condition means the difference a x minus a x prime this belongs to U. But this is same as a times x minus x; x minus x prime. And I will given x minus x prime is in U and a is arbitrary scalar and because U is a subspace this times a is also their; that means this is U. So, it is obvious.

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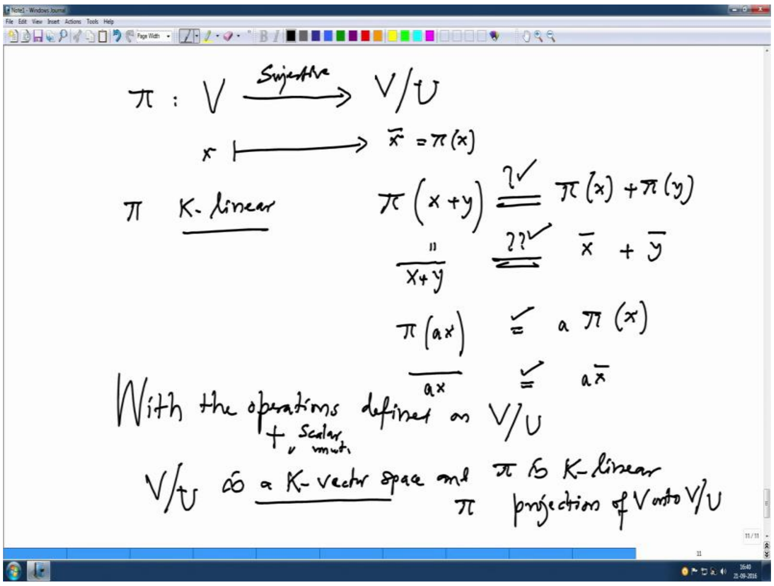

So, we have check that these two operations are well defined. Now, look at the quotient map. So, the quotient map I am denoting by pi; pi is a map from V to V by U this is a natural quotient map that is x going to x bar. Or sometimes x bar also will write it pi x. And I need to check that we have check this is well defined and I need to check pi is K linear. Now I have to vector space is this and this, and I want to check that it is K linear. That means, I have to check pi of x plus y equal to pi x plus pi y. Or in this notation it is this is x bar plus y bar and this is x plus y bar and I want to check this equality. But well that is how we have defined it. We do not have to check (Refer Time: 09:19) because this is how the sum in the V by U defined with this.

Similarly, we have to check that pi of a x equal to a pi x, but this is in bar notation it is a x bar and this is a x whole bar. But that is how we have defined this scalar multiplication. So this is correct, this is correct, this is correct, this is correct. So, therefore, pi is scaling where. Now, we have the natural map is a K linear map. Now we will start comparing what happens to their dimension, what happened to their basis, and so on and so on? That is what do it. Here I will just make sure that with operations defined on the quotient space V by U namely we have defined plus and scalar multiplication V by U is a k-vector space and pi is K linear.

For this we need to check that the scalar multiplication distribute so or and these and that those four properties of the first lecture of the definition of a vector space; that I am not given to do it here. Pi K linear we have already checked. So, our operation is so very well defined that this map is K linear map and also it is clear that pi is surjective. This pi also I will sometimes call it; pi is called projection of V on to V by U.

Now first of all we need to find kernel of this map. It is surjetcive we know and we want to find the kernel.

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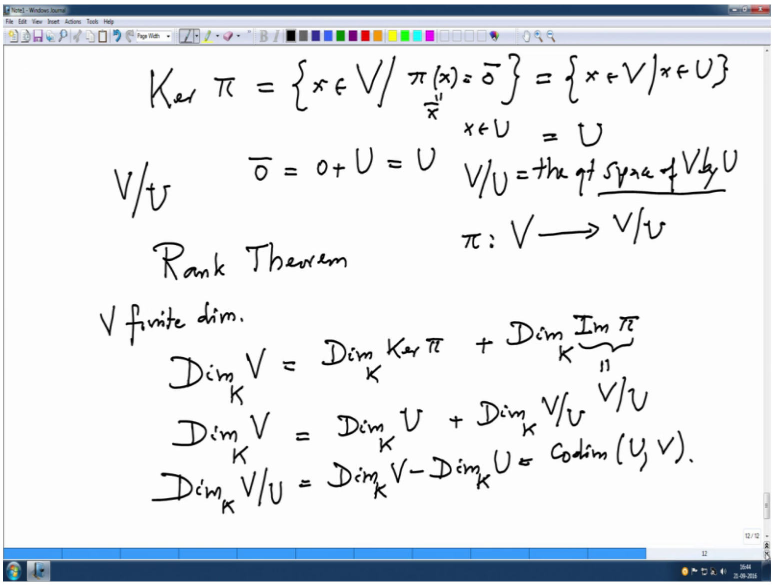

So, kernel of pi; what is 0 in V by U? The element 0 is precisely 0 bar is what? 0 bar is by definition it is U plus U which is 0. So, 0 bar is precisely U, this is the 0 element. So, kernel of pi is by definition all those x in V such that pi of x is 0 bar. But this is same as x bar is 0 bar; that means x belong to U. So, this is same as x in V such that x belongs to U, this is nothing but U. So, U is the kernel of pi.

And therefore, what we will get formula? Then we know now I want to apply the rank theorem. Rank theorem says; the dimension of I want to apply rank theorem to this map pi V to V by U; for this I am assuming V is finite dimensional where assuming V finite dimensional. Then we know dimension of V equal to dimension of the kernel plus dimension of the image. So, dimension of V equal to dimension of the kernel pi plus dimension of the image, but image of pi is everybody because pi is surjetcive; this is V by U dimension is always over k.

And kernel pi we know it is U. So, dimension U. And this is dimension of V by U. So, we have this nice formula: dimension V equal to dimension U plus dimension of this new quotients phase. This V by U is also called a quotient space by U; V by U the quotient space of V by U. And now if I want formula for dimension of V by U I can shift this U to the other side. So, dimension of V by U equal to dimension V minus dimension U. Or remember this is nothing but the co dimension of U in V. Some people also called these V by U as a factor vector space or quotient vector space or residue class space. So, in their also this names are same for the same V by U.

So, I want to write couple of statements which we will very useful for studying more linear maps later for example. For example, what happened to the arbitrary homomorphism? So, what happen? So, I want to analyze the following situation V.

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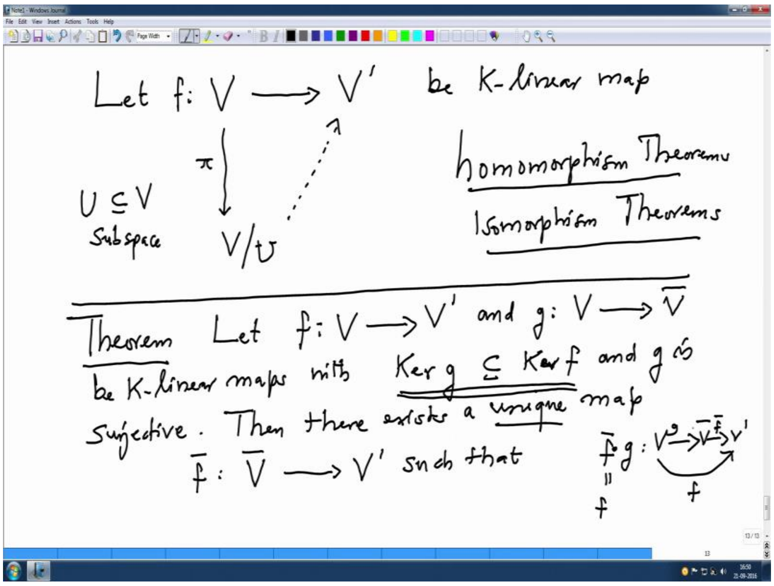

So, let V to V prime f be K linear map. And suppose I have U subspace given U is a subspace of V, then definitely we have a linear map here V to V by U this quotient space of V by U and this map is pi. Now how we are you going to compare this? That is what I want to analyze. When can I define map from here to there. So, I am looking for a map from here to here so that this diagram is commutative. That means, when I go from V to V prime by f I should be able to go from here to here like this composition. So, that is precisely the content; it always it may not exist always. So, under what condition if exist and if f is injective is this map injective if f is surjetcive is then map injective and so on and so on. All these topics comes under homomorphism theorems or also isomorphism theorems.

So, I want to state the very precisely the proofs are; once you state them very precisely the proofs will come obviously; so little more general situation than this. So, I want to state it has a theorem. So, theorem let f is from V to V prime and g is from V to V bar be K linear maps. V V prime V bar there all k-vector spaces, and I have two linear maps with the condition that kernel of g is containing kernel of f. It will be clear why this how this condition will be use once we write the statement. And it is also assume g is surjetcive. Then there exist a unique map from where to where from that map I will call it f bar from V bar to V prime such that if I go V to V to V prime and know if I go from first apply f bar that is from V bar to V prime, no first apply g sorry; g V to V bar and f bar which is V bar to V prime. So, this is this composition this goes from V to V bar first that is g and then V bar to V prime this is f. So, this is f bar composition g.

On the other hand I have also given map f here to here, these two maps are same. So, the statement is there exist a unique map V bar to V prime such that f bar composite g is same as f. So, before I go on first of all this settle this problem. See because what will I do I want pi is given to us. So, this is my g in the statement. So, what condition I need to have a map from here to here namely the kernel of g which I know it is pi which is kernel of pi is U and this kernel should be content in kernel of f then only they will a map here. So, I this is condition will ensure that this map is well defined. So, this I have stated in a general maps here such that this; I want to write down this means the diagram is commutative.

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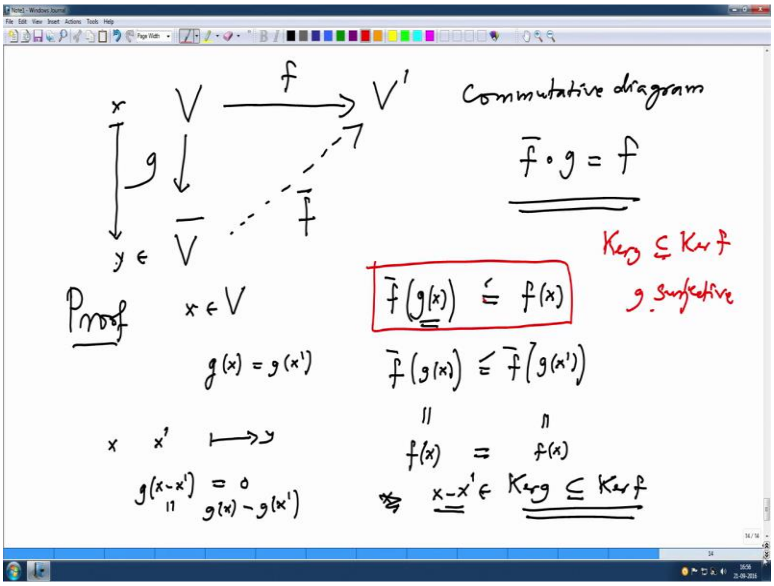

That means the diagram like this V to V prime, this is f, this is g, and this is the map we are defining this is I call it f bar. This diagram is commutative diagram. That means, from here when this is only one possibility from here to here namely f; from here to here there is another possibility here, so this is g compose f bar. And that can only happen here you see there is no other there is three points here, but this is only possibility two ways: here there is no possibility here also no possibility. So, we want to check this so that existence of f bar we have to write. After we prove the existence of f bar we will see. Actually this condition will prompt to work do define f bar.

So, I want to define f bar and this should satisfy. That means, for every it is a proof; for every x in V I have given this f x this is an element in V prime and also I have given g x g x also I have given that is an element here. And I want to define this; f bar of this. Whatever it is that should be this because I that is the demand. Of course, I can defined it like this, but for that I need every element of V bar should come from V that precisely means g is surjective; that was our assumption here, see g is surjetcive, and here also pi is surjetcive. Not only that, but you have to check little bit thing that suppose somebody may have two elements; surjectivity means that given any x bar here given any y here, y is coming from x, but it may happen that the same y is coming from two of them.

So, it may happen that g of x also equal to g of x prime, but then we will have to check that f bar of g of x is same as f bar of g of x prime. This means what? This means this is equivalent to checking f bar of g x minus g x prime is 0, that is because f is linear, f bar is linear. That means, this should be in the kernel. But this, if you want to say that it is well defined no sorry you can do it much speaker. See you want to check these are well defined means this equal to f x and this equal to f x that means, we need to check that the kernel of f should contain kernel of g.

Yes, so g: see x and x prime go to the same y that means g of x minus x prime is 0 it goes to 0 because this is g x minus g x prime and both are y. That means this means x minus x prime belong to kernel of g that is given. And kernel of g is contained in kernel of f that is a given condition. So, when this happens f should be equal. And once f's are equal this is equal this is well defined. So, this condition I will just colour it here, this condition here make sure that the map we defined by this formula is well defined. It is defined because g is surjective, and well defined because kernel g is containing kernel f and g surjective, surjectivity defined this and this condition checks it is well defined it does not depend on this x we choose it.

So, that proofs this. So obviously, you can apply this to the map pi, pi is surjective and kernel of pi is U and then you need a condition that whenever you have a linear map whose kernel contains given U then it will go onto V by U. So, let me write this explicitly as a corollary.

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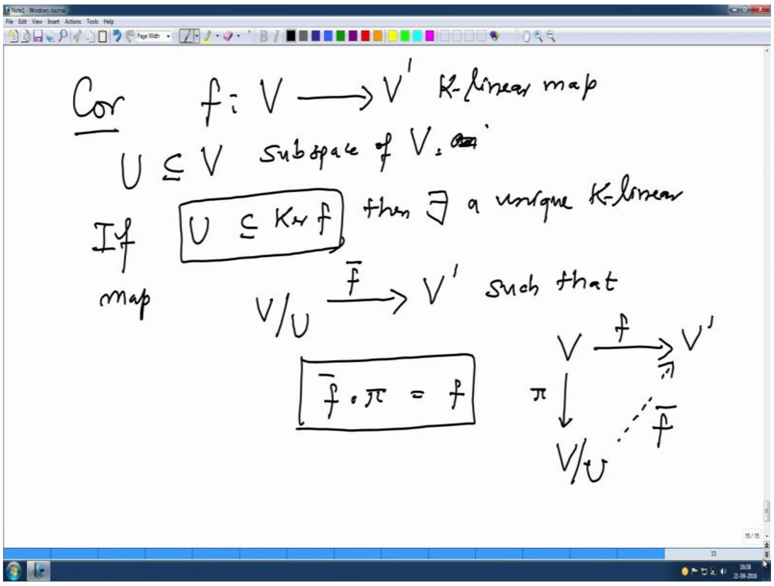

So corollary, if f is a linear map from V to V prime K linear map U is a subspace of V. If U is contained in kernel of f, then there exist a unique K linear map from where, now from V by U to V prime. That I have call it f bar such that pi compose f bar is same as f. So, in the diagram if I have to draw V is here V prime is here, this is f, V by U is our quotient space, this is our pi map and then I am defining a map here that is f bar so that this diagram is commutative mean this equality. This is very very useful; that means you are passing out on to the U and looking at the new linear map.

This is very useful because, in some practical problem when you are proving something by induction on the dimension, and then you will choose subspace U and U pass on to this V by u, but we have to choose with this condition and then you can pass it on, then you get a new linear map. So, that this linear map is converted to the original linear map f by this diagram; this is very very useful. And we can add something more to this statement. So, moreover I will add it now this is on the theorem. So, going back to theorem I will draw only the diagram in from the theorem.

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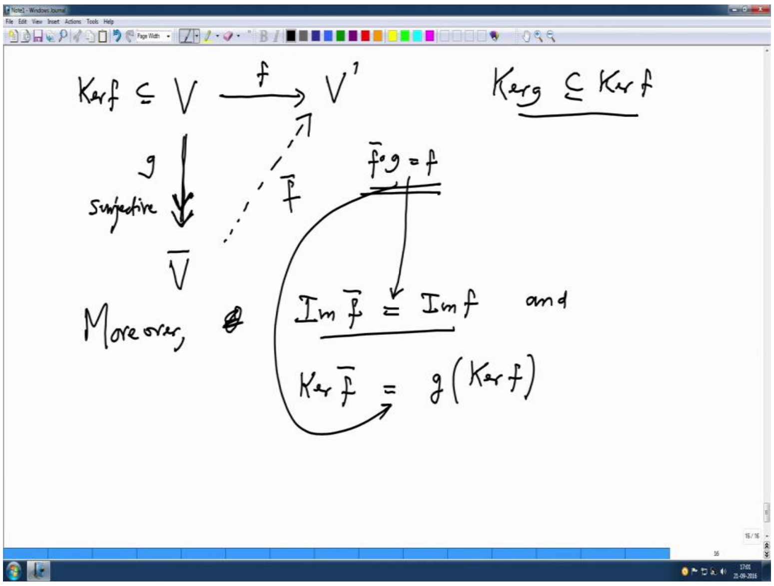

So, V to V prime, this is a linear map f and g is given this is surjective here g; sometimes surjetcivities also denoted like this double arrow here this means g is surjective. And then we know that when do such a map exist f bar f bar exist with this diagram commutative; that means g compose f bar is same as f that is only when the condition is kernel of g should be containing kernel of f; this is important. But now I am adding something more to this.

Moreover, the image of this new map is same as image of f; image of f bar is image of f. And what is the kernel of the new map? Kernel of f bar is the image of the kernel here g of the kernel of f; kernel of f is a subset here subspace here. So, you see the new map we know its image and kernel very well in terms of the image of f and kernel of f. This equality will be very useful again. So, let us check them. Well, this is a follow from this equality, because you somebody then the image then it is f of x, but then the f of x is f bar of g of x. So, it is in the image of f bar. This equality just says this immediately.

Now, this one also is coming from here. See suppose somebody is in the kernel of f bar; suppose let us take this side g of somebody in the kernel. And I want to check it is in the kernel of f bar, so apply of f bar that. So, f bar of that will be g of that, but this is f of that since (Refer Time: 32:15). So, this is also immediate from this equality. So, this method of inducing and passing map from a original vector space to the quotient space is very nice process it keeps stack of the image kernels etcetera and also it will be the dimension drops.

So, I will stop here.