Linear Algebra
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Lecture – 36
Isomorphism theorem of vector spaces

Welcome to this course on Linear Algebra, and the lectures on Linear Algebra. I want to deduce so called isomorphism theorem from the last theorem which I have proved. Last time I have proved a theorem on homomorphisms. So, let me induce homomorphism. So, let me recall that because I want to deduce consequences from this theorem.

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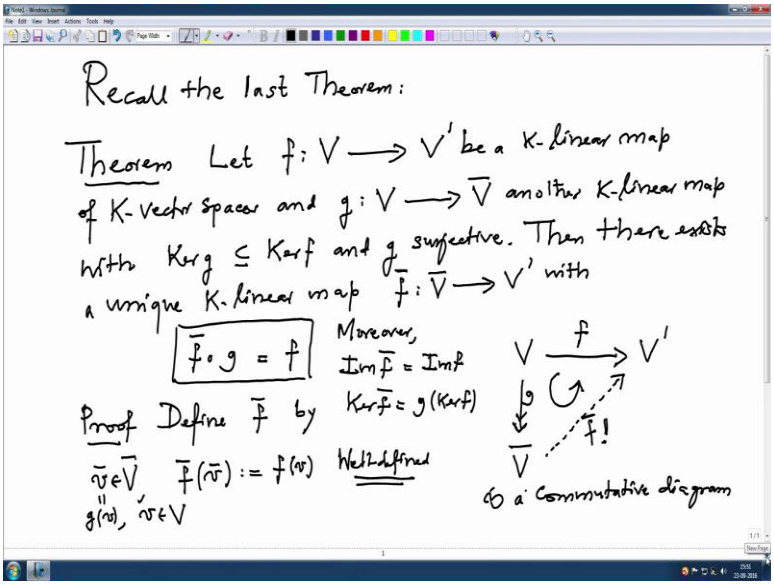

So, recall that; call the last theorem we proved that was the following. Let f from V to V prime be a K-linear map of K-vector spaces, and another linear map. And another one g from V to V bar another K-linear map.

I will try to draw diagram in this corner. So, f is from V to V prime and g is from V to V bar. We assume with kernel of g is containing kernel of f, and assume and g surjective. Sometime surjectivity I will also denote by putting a two arrow heads that mean map is surjective. Then this is given, then there exist a unique K-linear map which I will denote f bar from V bar to V prime.

So, in the diagram this map is dotted map exist that is f bar we are denoting, unique also unique is say usually denoted by this exclamation marks involve with; if I go from V to V prime by f that is one way or I can go also first go by g from V two V bar and then follow it by f bar that should be the same. So, g compose f bar. This equation is also usually saying that this diagram is commutative like commutative of the diagram is indicated by this circular arrow, so that is commutative diagram.

And the proof was very easy. Proof was just defining f bar which is dictated by this demand. So, define f bar by take any element in v bar; so f of v bar v bar. So, take any element v bar in capital V bar and because this map is surjetcive V bar has to be of the form g of v per some v in v an define these to be because of this define it to be f x. Now the only thing we need to check here is this dimension does not depend on the pre image of; this is V not x this is V. This definition does not depend on V which is a pre image of V bar under g. So, that is only thing.

So, V this is well defined; that is one has to check. And then this demand is automatically satisfied by we have defined it like that. Moreover, in this there was another part. So, moreover it says what is the image of this new map f bar; image of f bar is same as image of f. And kernel of f bar is same thing as image of the kernel of f under g; g of kernel f. These are this two equalities are merely verification. So, you take any element here proof it is here and conversely similarly for this. So, this was the last theorem and I want to use this theorem to deduce now some more assertion which we will be very useful for us.

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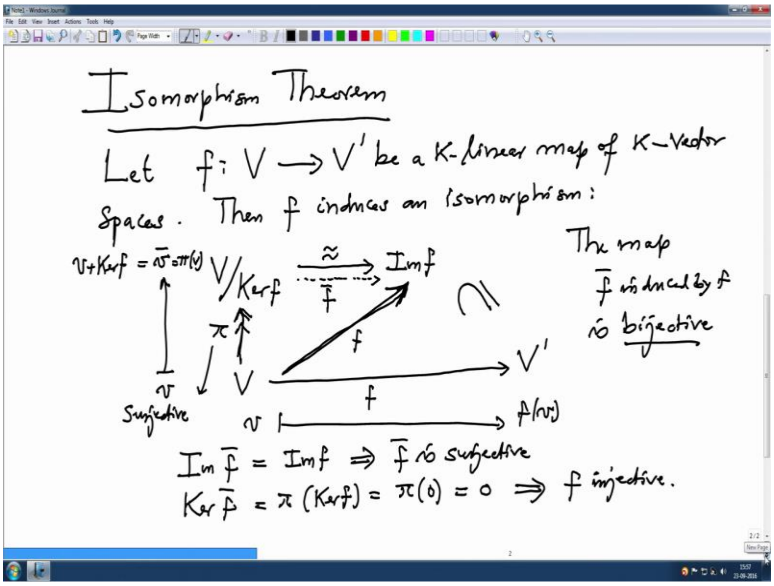

So, the first one: now this is usually known as isomorphism theorem. From a given linear map we want to get an isomorphism. So, let f from V to V prime be a K-linear map of K-vector spaces. Then f induces an isomorphism which will be denoted by f bar; that is from V modulo kernel of f to image of f. Remember we have defined quotient space of V modulo of subspace; kernel is a subspace of V and this image is a subspace of V prime. So, we have gone V mod the kernel. So, this is an isomorphism.

And how are apply the homomorphism theorem. We have given f that is from V to V prime, this is we have given. But actually the image of this f really goes inside the image. By definition this is any V goes to f of v, but f of V is already in the image. Therefore, it is indeed f; you can think of a map from here to here. So, I will again denoted by f only. And there is a natural map here pi. This is a natural surjetcive map, this is any v here goes to v bar. v bar is now by definition this we are denoting by b v plus kernel f, or this is also because this we are denoting this map has pi, this is also we are denoting by pi of v.

So, we have this map pi and V have seen that pi map has kernel of f and it is surjective; pi surjetcive. So, our just theorem what we have recalled that say that I definitely have a map here and that map I call it f bar; this is f bar. And I want to check that that map is bijetcion which K-linear is fine. I want to check that the map f bar induced by f is bijective.

But you have noted in earlier theorem that image of this f bar is same as image of f. So, we have noted above that under this circumstances image of f bar is image of f, but image of f is whole thing here. So therefore, f bar is surjective. We have also noted what happens to the kernel of f bar; kernel of f bar is pi of the kernel of f, but pi of kernel of f is kernel of f is 0, 0 element here; this is pi of 0 which is 0. So, we have checked that there this means f is injective.

So, that proofs f is bijetcive. From a given K-linear map we can get an isomorphism by going mod kernel and the image is the image of f. So, this is called an isomorphism theorem. And if you say from this now the immediate corollary is.

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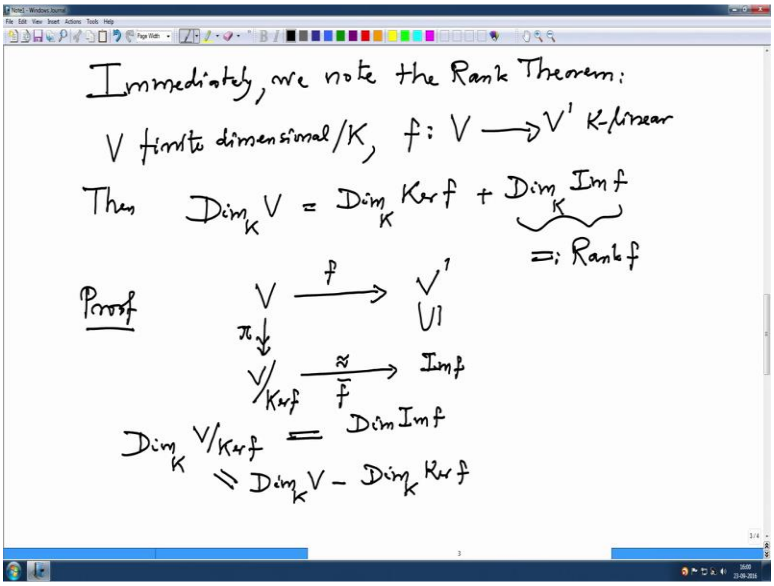

Immediate we can deduce immediately; we note what we have proved earlier the rank theorem. So, what was the rank theorem? V finite dimensional over k and f is a linear map from V to somewhere V to V prime K-linear map then dimension of V equal to dimension of the kernel of f plus dimension of the image of f and this we call it rank of f.

So, to proof this: by isomorphism theorem we know that this f will induce map isomorphism f bar; f bar will be from V mod kernel f to the image f. This is a pi map here natural canonical surjection and f is define this diagram is commutative. So, in particular dimension of the quotient space equal to dimension of the image because it is an isomorphism. So, dimension of quotient space V by kernel f if therefore equal to dimension of the image, but dimension of the quotient space we know it is dimension of the big space minus dimension of the kernel. And therefore, we get this formula. So, the rank theorem is trivial, if you proof the isomorphism theorem.

Also, we can deduce the dimension formula easily now. Let me make it clear. So, either use rank theorem to proof this formula or proof this formula to use a rank theorem; it depends what one assumes and what one proofs.

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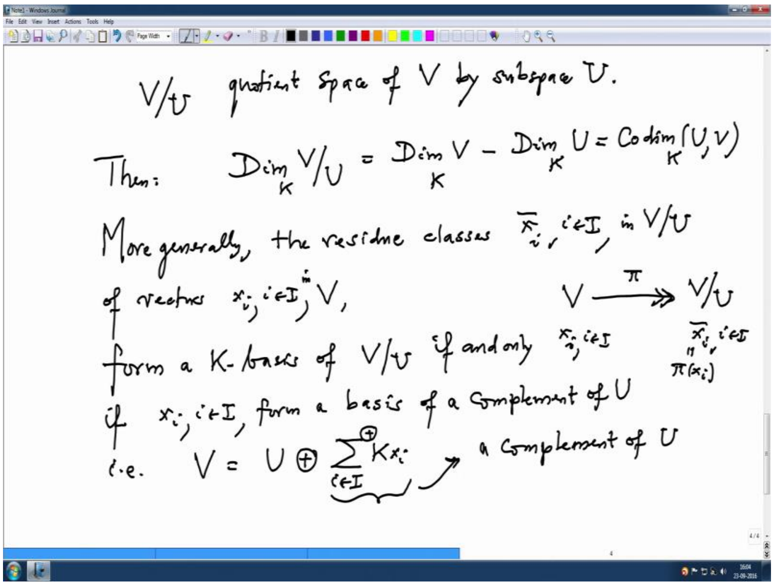

So, think to notice: if you have a quotient space V by U the quotient space of V by subspace U then dimension of the quotient space is the dimension V minus dimension U. And remember this we have called it a co-dimension; co-dimension of U in V.

Actually we can also say something about a basis. So, more generally the residue classes; remember the classes here are we are referring do to as a residue classes they are the co sets of U in V. Residue classes x i bar in V by U form; so these are the residue classes of vectors x i i in I V in V. So, we have V are where here we have exercise and we have a natural map pi map that goes to V by U these map is surjective we know and this exercise go to x i bar here or we are also denoting by pi of x i. And any element of V by U is coming from somebody from pi.

This residue classes form a k basis of V by U if and only if the x i i in I form a basis of a compliment of U. So, that simply means that along with U this guys form a basis of V. That means, V is U direct sum k x i, and this is also direct sum, this is a compliment, this is a compliment of U. In particular you get this formula again, because we know this is a basis, so if we count them if in the finite dimensional case these are finitely meaning and then that we will give you this co-dimension.

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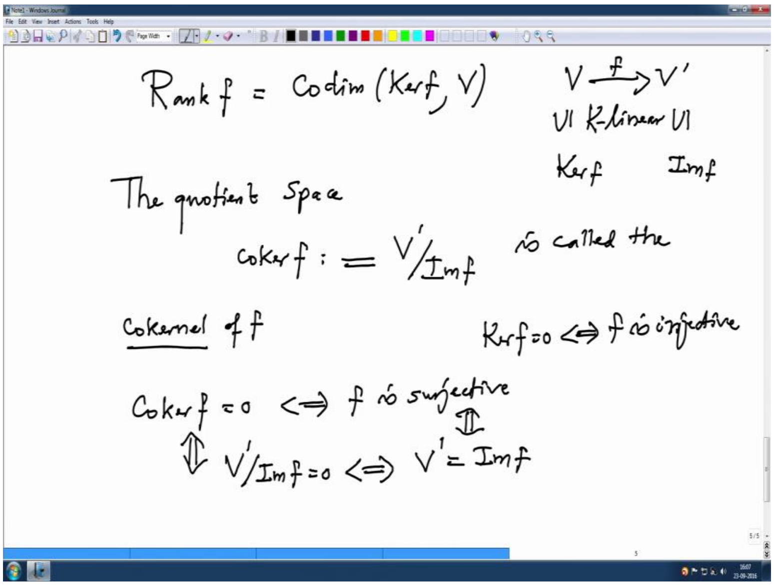

So, in the new in this co-dimension notation rank f rank of a map is co-dimension of the kernel. So, when f is a map from V to V prime. Now I want to introduce a new vector space here out of f we get it. Suppose f is a K-linear map from V to V prime, then what we know that we have to subspaces attach to f: one is a kernel f and then the other is image f. This is a subspace here, this is a subspace here. And we have gone mod this and found co-dimension of this and this. So, we could do the same thing here. So, the quotient space V prime by image f this is called the cokernel of f denoted by coker; is called the cokernel of f.

So, this measures how for f is some surjetctive. If you remember this kernel measures the infectivity of f kernel is 0 if and only if f is injective. So, kernel f is 0 if and only if f is injective this we have seen earlier. So, then this subspace measures how for f is from infectivity, if it is 0 it is injective if it is one dimensional then it is close to injective, not injective but close to injective, now more the dimension that more the away f is from injectiveness. Similarly this cokernel measures surjetctivity of f. So, cokernel f is 0 if and only if f is surjective. This is very easy, because cokernel 0 means this is equivalent to saying V prime by image f prime is 0; 0 space. This is a 0 vector space means V prime has to be equal to image of f, but this is precisely surjectivity.

So, with this I want to stop this topic quotient spaces, and now I will go on to matrices. So, now it is a big topic it will go on for several lectures. I will call this section has Matrices.

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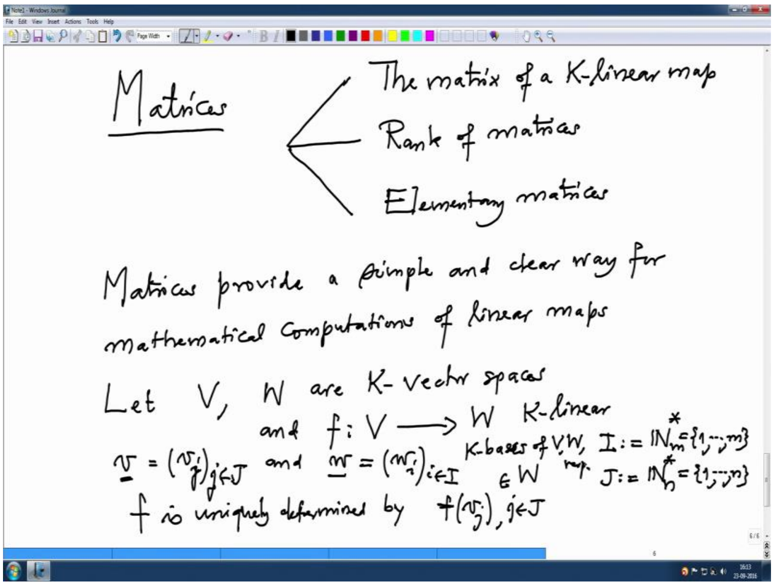

And I will divide this big section into three sections: first of all, all the motivation to do matrices will come from linear maps. So, first I will study the matrix of linear map K-linear map. Then I will study rank of matrices. Then I will study elementary matrices. This is the three topics which will mainly.

So, matrices they provide simple and clear way for mathematical computation; computations of linear maps. First, before actually defining formally what a matrix is I would like to start with a linear map. So, let V W are k-vector spaces and f is a linear map from V to W k-linear. To start with I do not want to assume their finite dimensional, but eventually I will assume their finite dimensional. And we have seen that if I have a basis V, now I have two right basis V. I will write underline V and this I will denote V i; I will denote by round bracket.

So, this also there by eventually when I assume i is the finite set I want to give the order to that finite set. So, which element come first, which element come later and so on. For that reason I am instead of denoting by curly bracket where usually the set is denoted by I want to denote by tupple. And similarly W under line W is.

Now, let me change this i to j. So, this is V j j in J, and W i i in I. So, I and J are index sets eventually we will settle down to where i is equal to 1 to m that is this set. Remember this is general notation from 1 to m and j is 1 to n. This is what we will do it eventually, but instead of writing this I will keep writing I and J. So, we have a basis and then we have noted that f is uniquely determined; f is uniquely determined by its values on a basis of; that means f is uniquely determined by this f of V j j in J.

So, for this reason I am going to write these vectors there vectors in W now; these are in w. So, I want to write them by using this basis W of W, these are k basis of V and W respectively.

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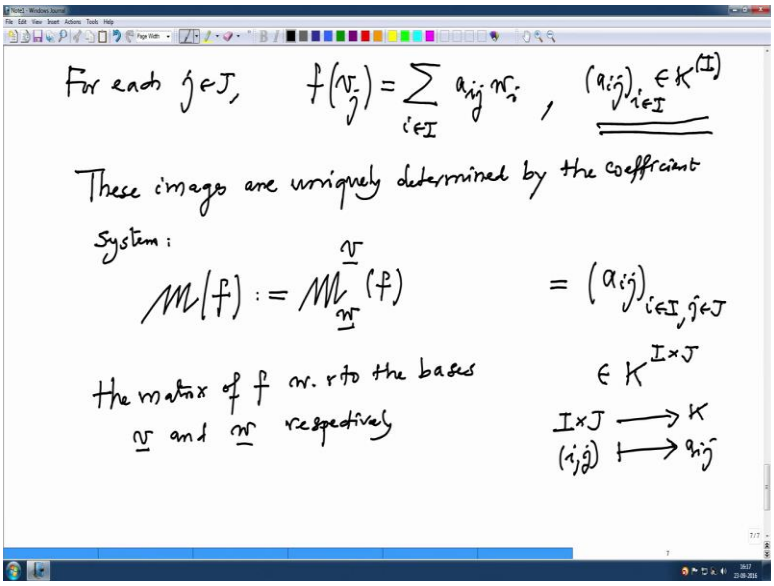

So, what we do? For each j in J we write f of V j; f of V j we know it is in W therefore it is linear combination in W i's; so their some coefficient, those coefficients will depend on the j also. So, a i j W i i in I. Where these a i j is are what is varying is I is varying j is fixed. So, this is i in I and this is in k power round bracket i, I hope we remember this notation k round bracket i. Therefore, this images this V f of V j's are actually determined by this coefficients because we know the coefficients are unique, because W i the basis.

So, these images are uniquely determined by the coefficients systems. So, that I am going to denote by m f, but to remember that I got this is coefficient system is a i j; in this i is varying in I and J is varying in j. And more actually, because these are only finitely meaning in each if we fix j in J there finitely meaning of them are all 0. So, what is the m f? So, this is by definition it depends on the basis V and w. So m, I have to write V and W in the notation V W of f is this. And where is this? This is actually in k power I cross J. Think of it is a function from I cross J to k or tupple i coma j goes to a i j. And this is called the matrix of f with respect to the basis V and W respectively.

The rows are numbered by i and columns are numbered by j. So, if in case if i is not finite; i j not finite you remember this condition that say that if we fixed j; that means, if we fix any column it has only finitely many non 0 entries. But this I am not going to do it for infinite I will assume and their finite soon. So, this is the matrix which we get out of f, and if you know this matrix f is uniquely determined.

So, now let us formally defined what a matrix is and that will match with this definition.

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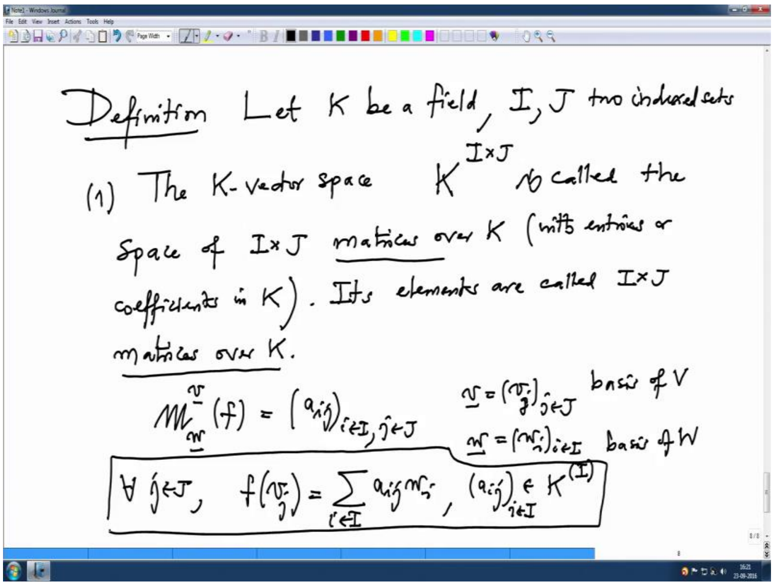

So, definition: to understand the linear map we got these coefficients and we put them in that form and that actually gives a clue what to define. So, let us assume now onwards it is not really necessary. So, let k be a field and I and J are two indexed sets. So, one the k-vector space k power I cross J, we have seen k power any set is a vector space. In particular k power this I cross J is a vector space this vector space is called the space of I cross J matrices over k, or also with entries or coefficients in the field k. And its elements are called matrices its elements called I cross J matrices over k.

Therefore, in what we have defined above is that is really the matrix. So, m f m V is on the top, because we go from V to W. V to W f this is the matrix a i j i in I, j in J and V is a basis V i V j, and W is a basis W i, this is a basis of V, this is a basis of W. And with respect to these two basis we have defined a matrix of f, with respect to is to basis. And think to how their remembered is f of V j for each j in J f of V j is summation a i j W i i in I; a i j are in k power this is varying I is varying k power on bracket i. So, is a one should remember f.