Linear Algebra
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Lecture - 37
Matrix of a linear map

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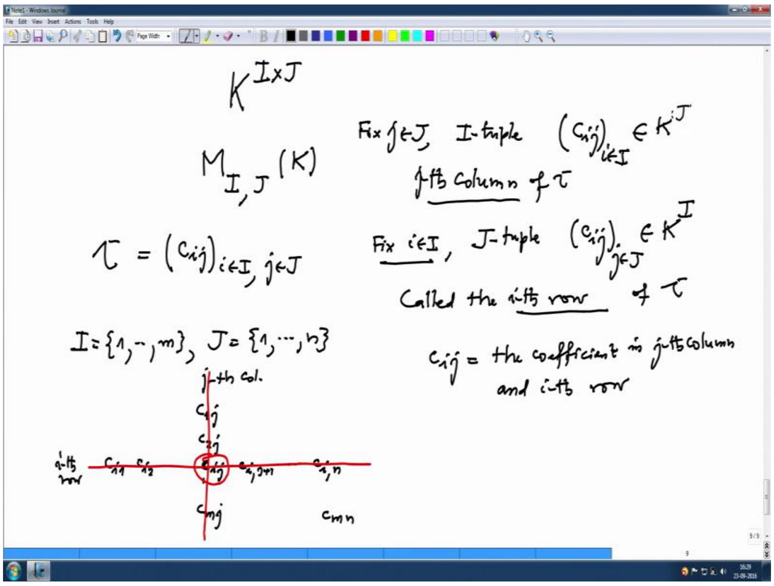

So, the matrices, so instead of denoting K power I cross J, I will use a notation M, M for matrix I comma J of K. So, this is to remember the rows are index by I. I will, I will define rows and column that time I will say and columns are indexed by J and the entries of matrixes are in the field K.

So, for example, if I take a matrix c and elements of this, we are calling it a matrix. So, c; that means, c is the entry c i j, i is varying in I and j is varying in J .If I fix, fix i in I and let j vary so; that means, if you look at the J tuple, c i j remember J is varying. This is an element in K power I, this is called, called the j-th column of c and remember that, the way we have defined matrix of a linear map. If it is coming from a linear map, then only finitely many entries are could be non zero for a column.

So, therefore, c is c i j is the coefficient in j-th column and similarly I should have defined row. So, fix j, fix j in J and then look at the I tuple. Here, that is c i j, I is varying this is in K power i. I think I made a mistake this one is no, no. So, I made a mistake in the notation. So, I want to correct it. So, for a fix i for a fix j this is, this is called j-th column of c and for a fix i the j tuple is called j i th, i-th row. So, this is i-th row of c, this is j-th column fix j, then if it is j-th column fix i then you get i-th row and c i j then the coefficient in the j-th column and i-th row. So, it is also, pictorially also it is usually written the j-th column i is varying. So, it is c. So, let us take simple case where i is standard indexing set i is 1 to m and j is 1 to n.

Then the j-th column. So, j is index here, j-th column starts with c 1 j, c 2 j etcetera, etcetera c m j that is the j-th column and i-th row it starts with c i 1, c i 2 and here it will come c i j, c i j plus 1 etcetera. It, it will go on till c i n the last entry will be c m n and this intersection of this is i-th row, this is j-th column, this is i-th row, this is j-th column and the intersection is this point here. So, this is i-th row, this is j-th column and this intersection is the i j-th entry and, and in principle we have the only assumption we have is in each column almost all entries are 0. That is when we do not assume things are not finite dimensional, but we let us assume it is finite dimensional. So therefore, for a linear map for f V to W and the basis here v 1 to v n and the basis here w 1 to w m.

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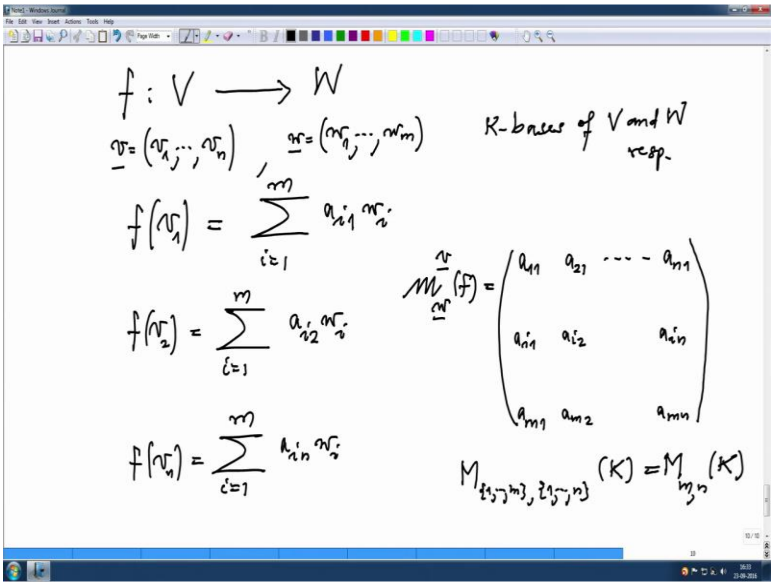

So, I will denote now the like, like this. This is my basis, these are K basis of V and W respectively and it is very important, because I do not want to write the bracket, I want to keep track in (Refer Time: 06:43) later. Because, that will be f of v 1 will be the first column. So, f of v 1 we have written as the combination of w 1 to w m. So, the sum is running from i equal to 1 to m, a i 1 w i. So, therefore, the in the matrix of, matrix of f with respect to v and w f this give the first column a 1 1, a i 1, a m n not m n m 1 these are, this is the column.

The next column will be given by f of v 2. So, f of v 2 is summation i is from 1 to m the next, the second index is for the v 2 that is a i 2 w i. So, a 2 1, a i 2, a m 2. So, that is the second column and so on the last one will be, the vector in this basis f of v n is summation i is from 1 to m, a i n w i. So, a n 1, a i n, a m n. So, this is the matrix and in this case, we will instead of writing M n 1 to n, 1to, M 1 to n. Instead of writing this big, big notation we will abbreviate it by M m comma n K that is what we will abbreviate.

So, the, if you just have this vector space K power I. Then we know the elements here are the vectors that I tuples. So, I tuples are the elements here, but then I want to consider them as a columns.

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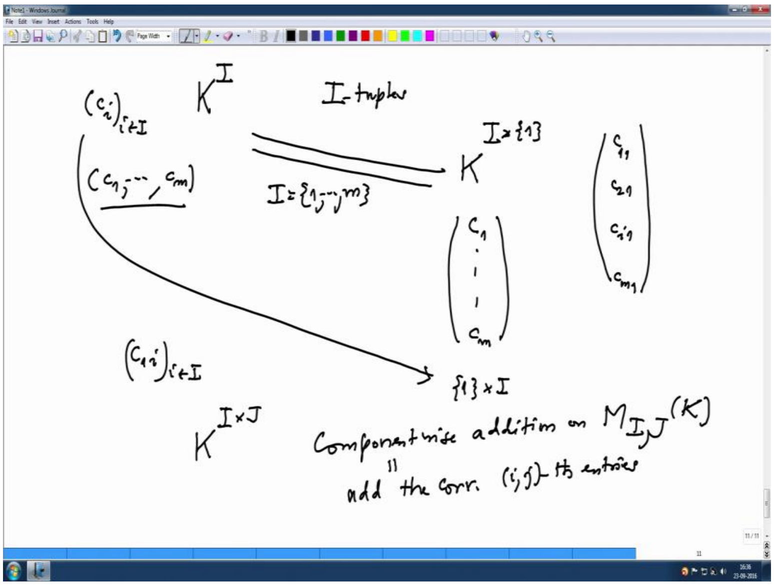

So, instead of writing this I would prefer this to write it as K power I cross singleton 1. So, it will be the column. So, the vector here instead of writing a tuple like this c i i were within I, I will write it as c, c c i 1, c 1 1, c 2 1, c i 1, c m 1 this is when c m this m is in the same. See, instead of writing c 1 to c m when I is 1 to m instead of writing elements of this like this I would write them as like this columns. So, this means this extend the v index is just to say that it is like this columns, these are columns. So, elements of this I will think as columns and the row vectors. If I want to think as a row vector then I would not take I cross 1, but I will take 1 cross 1 and then it will give me a row vector. So, this one I will this one is thought as 1 cross I. So, it is thought as c 1 i, i in I. So, this is a row.

What is I need. So, now, let us prove a theorem, this is only a notations for matrices and it is clear how we are adding matrices. We are, we know already the matrices are elements from this vector space. So, they are maps. So, to add matrices we just have to add the corresponding entries. Component wise addition, addition on M I J K means just add the corresponding entry add, this means add the corresponding i j-th entry and it is clearly an Ablian group we have seen actually this is a vector space we have seen. So, you only have to see it in a new notation nothing else scalar multiplication also component wise. So, it is actually vector space. So, M I J K is a K vector space this is called soon I will, I will we will check where we take K algebra and that will be called a matrix algebra ok.

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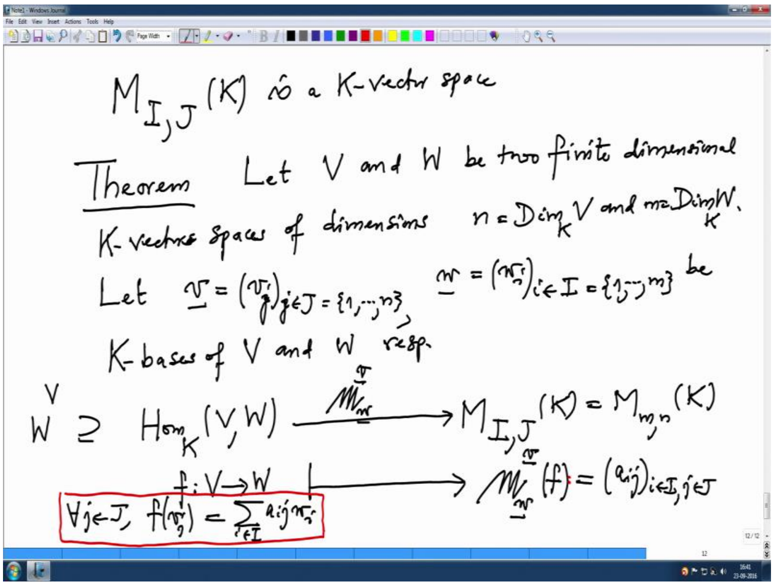

So, right now I want to prove theorem. So, theorem, so let V and W be two finite dimensional K vector spaces of dimensions n, n is the dimension V and m equal to dimension W and when we say basis also. Let v equal to, I still want to write v i, i in I where i is you may think it is 1 to n not I, j; j in J. is 1 to m and w, w i, i in I i is 1 to m be K basis of V and W respectively and we have defined. We have this matrices M I J K this is actually m cross n matrices M m n K this is the vector space we just noted and on the other hand we have this K linear maps on v to w Hom K V W these are set of all K linear maps from v to w and we have noted it is a vector space, this is actually subspace of a very big space W power V. So, we have this subspace, we have this vector space and this vector space and we have a map here this map is m v w what is the map let us say (Refer Time: 16:15); that means, any K linear map f from V to W map to its matrix with respect to this basis, m v w f and you know this is a i j, i is in I j is in J yeah and this a i j's are defined by this equation for each j in J f of v j is summation a i j w i i in I.

So, this equation is they are bunch of equations their j equations as many elements have in j this equations define this matrix. So, we have a map now I am going to prove that this map is a K linear map and then we will check that this is actually a bijective map. So, it will be an isomer (Refer Time: 17:33) vector spaces and therefore, we can compare the dimensions, this dimension we already know this dimension is the product of the dimensions therefore, dimension of a m n K will be dimension of v dimension of w, which will be m n. So, the first we in forget information about this matrices if the dimension is m times n and then we will worry about how to find the basis ok.

So, let us. So, proof the; so the statement first this map, the map remember this map is m for matrix and it depends on a basis v and we said w is K linear and bijective.

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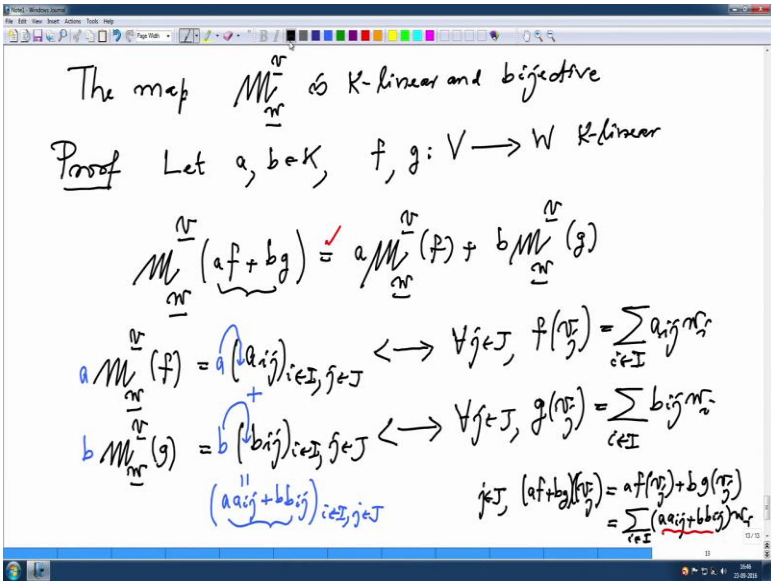

So, proof to show it is K linear; that means what? That means, let a and b be 2 scalars and f and g be two linear maps on V to W, then we have to show that m v w a f plus b g is same as a m v w f plus b m v w g we have to show this. But let us see what is, how did we define m? m v w f if this is a i j, i in I, j in J you are assuming finite and everything. So, this this is defined by the equations for every j in j f of v j equal to summation a i j w that is why we defined this matrix all this coefficients.

How did you define then m v w g this is b i j, i in I, j in J this is coming from the equations for every j in J f g of v j equal to summation i in I, b i j w i that is it. This is how they are defined. How did you, when I multiply by this matrix by a multiply this by a, but our definition says it should go inside. So, this goes inside similarly when I multiply this matrix by b and multiply here and that should go inside and then i have to know what is this and how do I add the entries. So, if I have to these addition this is nothing, but the ij th entry will be a times a i j, plus b times b i j and this is i is in I and j is in J that is ij th entry, this is the ij th entry of the this side, left side and how the matrix in the left, let us compute them i j-th entry on the on the left side this is right side right side i j-th entry is this.

Now, I want to compute on the left. Side left side is what now then the linear map is this. So, I have to evaluate the linear map for each j in J I have to evaluate a f plus b g evaluated at v j, but this the way it is defined this is a f of v j plus b g of v j, but I know f of v j is this, g of v j is this and their addition and then. So, i j-th entry will be per say when I combine this, when I write this i in I a times a i j plus b times b i j times w i when i plug in this equal to this this equal to this and rearrange the brackets. So, i j-th entry is this. So, therefore, on the left side matrix have the i j-th entry is this on the right side i j-th entry is this which are equal therefore, the 2 matrices are equal therefore, this equality. So, that proves that it is K linear.

Now, let us prove it is bijective. So, bijective means it is injective and surjective.

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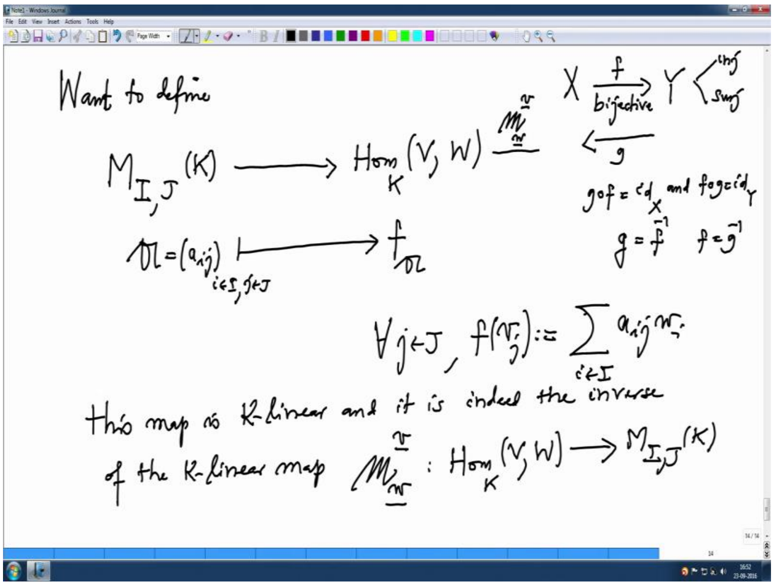

So, either remember whenever we want to prove some map x to y x is a set y is a set and f is a map and whenever you want to prove this map is bijevtive. You check it is injective and surjective, but better would be the produce a map in the other direction better would be produce a map in the this direction, let us call it g and we have to check that g compose f is identity map on x and f compose g is identity map on y and advantage of doing this is not only you prove that the map is bijective, but also you produce a inverse.

So, g will be the inverse of f or f is f is inverse of g. So, the advantage of proving this sometimes it is easy to write down the formula for g, but sometimes it may not be easy and it is always better to give a formula that will give you algorithmic proofs. So, I am going to produce a map in the other direction. So, we want to want to define a map from, from where from the matrices M I J K to Hom K V W in such a way that each compose it is identity. So, in such a way, so if you call this, what will I call this map this map let us call it as. So, if a matrix is a I am going to denote matrices by gothic letters. So, a is a i j then I am going to define a linear map which depends on this a. So, let me write it as f of a f suffix a, this is what I want to define. I want to tell you what if suffix is.

And we have to check that if I take this map and take the matrix of that map with this (Refer Time: 25:40) whatever basis. So, remember here we have fix a basis. So, we want to write the inverse is, inverse with respect to that basis know. So, this followed by m v w in the matrix a, it should become identity map and also this followed by that should become identity map. So, this is very easy I given a matrix i is in I j is in J and I want to define a linear map. Well, if I have to define a linear map it is enough that, if I give values on the basis that is the theorem we proved in the beginning. So, I just have to give for every j in J, what if f of v j? I have to define what it is and what should it be it should be an element of w and how do I define element in w by using the basis w i it is a combination of w has, but that I will use this v i js. So, define a i j w i. So, this gives a unique linear map, with these definition and you see the same equation will give you this matrix back. So, this map is indeed the inverse of this this map is K linear and it is indeed the inverse of the map of the K linear map m v w this map which is a map from Hom K V W to M I J K.

Now, see whenever I want to study a linear maps between v and w, I will choose a basis of v and w and pass on to this matrices and whenever I want to compute efficiently I would like to choose basis also efficiently. So, that in the computation this coefficients a i j's they will be, many of them will be 0 or many of them will be easier to compute, many of them are once and this will help us to make a efficient computation with linear maps. Now, in addition to this now I also want to study what happens to this maps under composition so; that means, I want to analyze the following situation.

So, now I want to study, we know composition of maps composition of linear maps K linear maps is again K linear map is again K linear map.

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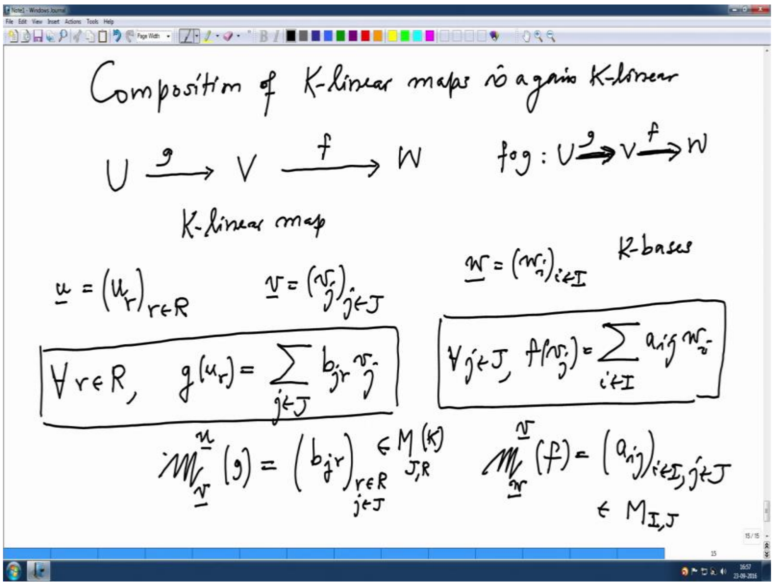

That means if I have a vector space U, I have V and I have W these are vector spaces and if I have a K linear map here and a K linear map here this I call it f, this I call it g these are K linear maps then our composition we are denoting by g o f first no, no in the other way f o g, f o g first g that is from u to v and then v to w this is this is the first is f and then first is g and then f.

Now, to study g we need a basis of U to study f we need a basis of V and to write the images we need a basis of W. So, we will choose a basis u r r in R this is a basis of u the basis of v is v j v is v suffix j j in J this is u underline and here w underline is w i i in I these are K basis of of u of v of w and how do you study g? We study g by taking the values of this basis and writing as a unique combination of this basis; that means, for each r in R, for each r in R g of u r you are writing in terms of vj. So, these we are writing I will write its bs b, remember this r is the second index and the j is the first index. So, this sum is on J b j r v j clear. So, this is how you write the g is uniquely determined by this equations these are the determine these are the equation which g is uniquely determined.

And for f for each j in J f of v j is the summation i in I a i j w i this is this these equations we give a matrix f f, this will give you matrix from basis v to basis w f this is a i j matrix i in I j i J and this will give you a matrix b matrix m from the basis u to v g this is by definition b b j r r is in r j is in j. So, this is, this is how many cross how many matrix? This is m j r matrix and this is m in matrix in m i j right and you want to define; we already know how to define composition. I want to write down the matrix of this (Refer Time: 33:47). So, what will be the matrix of this? So, the matrix of this will be I just have to evaluate f o g on u r and write in terms of w's.

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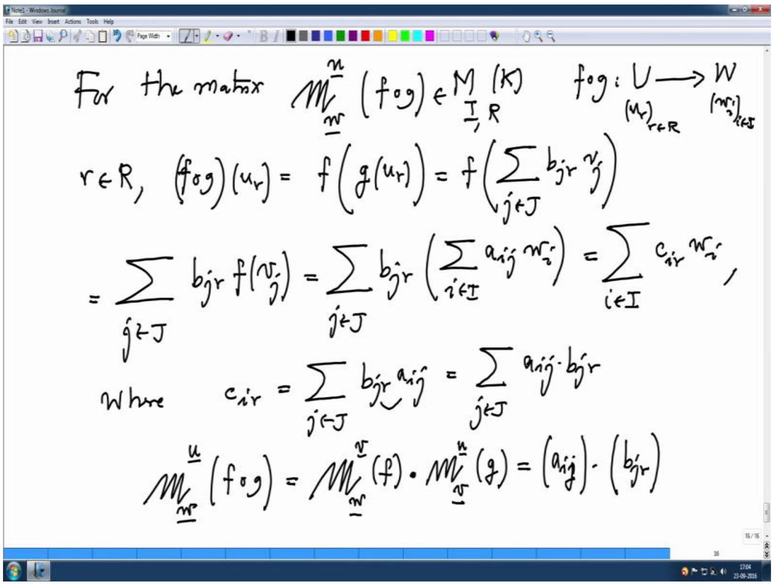

So, for the matrix that is R cross I matrix of no. So, first let me write the matrix and then we will decide what is the order for the matrix m from u to w; that means, and of the map f o g; that means, i have to take for any r in R, I have to evaluate this linear map f o g on u r and this I want to write it in, in the form w i in the coefficients because f o g is a map from u to w. So, here basis is u r and here basis is w i. So, this will be i cross j matrix. So, this will be M no I cross R matrix i comma r K this will be that matrix the first indexing set is the where a vector space goes in.

So, I have to compute this and in term of the matrix of f and matrix of g. So, this is by definition first you have apply g. So, f of g of u r, but g I know how to write f of u r that is f of summation j in J b j r v j that is how we have written g of u r and now f is linear. So, this will be equal to, I will push of inside and linear these are scalars. So, they will come out. So, this is summation j in j b j r will come out and then I apply f to v j, but f of v j I know. So, this is same as summation j in j as it is let us write b j r and then I have to write this in terms of w I’s. So, that is written as summation over I a i j w i both these are finite sums and I can rearrange and let say when I rearrange I will write the coefficients. So, c this will depend on i and it depends on r. So, c c i r w i and this sum is running over i in i where ci’s have collected coefficients. So, where c i r i have to collect i and r is fixed and j is varying. So, this is summation j in J b j r times a i j, but because we are in a field. So, I would like to inter change this. So, that. So, this is same as a i j times b j r and the sum is running over j in J.

So, you see this when the j is running this is the index, this is the column of the first matrix not the first. So, which is the first matrix now a is. So, which is the first and which is the second.

So, therefore, I want to say that what we have proved is the following m u v u to w f o g is nothing, but m, m u which is f f is the second one u, f is the second one know. So, that is v w f times m u w not u w, u v. u v g where I say I want to define 2 matrices by this formula. So, let us spell out that. So, this was this matrix this was a i j a matrix of f of a i j right and the matrix of g was b j r. So, the definition of 2 product of 2 matrices I want to define by this formula. So, for that first of all require that the first matrix the column index should be same as the row index of the second matrix and then we some or we multiply each row and then the column multiply the entries and sum it up and that will be called i r th entry of the new matrix

So, with this I will stop, but next time we will see some concrete examples and define formally the matrix multiplication and remark that that matrix multiplication is indeed comes from the composition of maps. Composition is more easier to study then the matrix multiplication.

Thank you.