Linear Algebra
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Lecture – 38
Matrix of a linear map (continued)

So, we will continue this lecture from the last time last time we introduced matrices of a linear map with respect to a basis. So, let me recall quickly.

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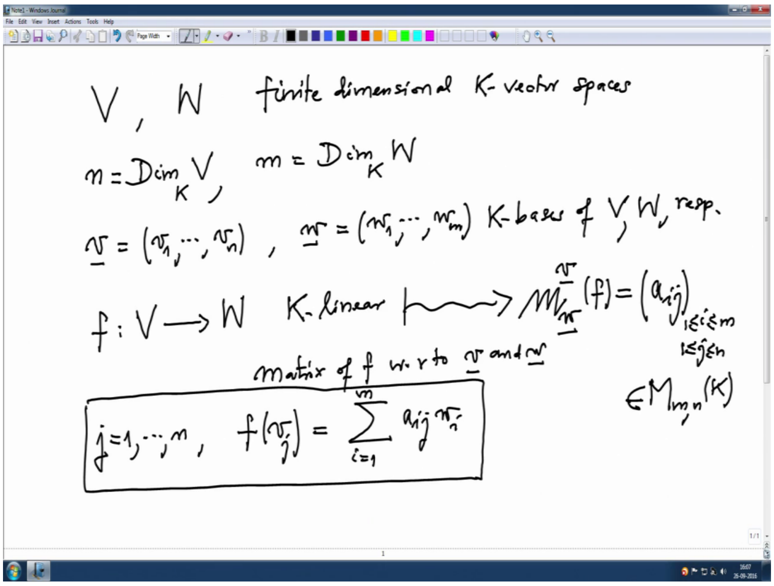

So, situation; we have V; V and W let us assume from the beginning finite dimensional K vector spaces and let us call n to be the dimension of V n to be dimension of W and let us fix let us take v to be basis of v, it will have n l about n vectors v 1 to v n, I want to write in this way because I want to keep track of v is the first elements in a basis and so, on and W this is a basis of w w 1 to w i K basis of V comma W respectively and suppose now I have a linear map f from V to W K linear map then to this linear map we have associated a matrix m v w f is matrix is defined this is m cross n matrix a i j 1 less equal to i less equal to m.

1 less equal to j less equal to n; so, this is a element in m and n K matrix of f with respect to v and w. This matrix depends on v a basis v depends on W and depends on f. So, all this is necessary in the notation and how the edges are defined? The edges are defined by these equations j equal to 1 to n the image of the jth vector in the basis of v under f that is

f of v j this you can write in a linear combination of the basis W of capital W that is written a i j W i i is on 1 to m.

These are the equations they are n equations and they determine this matrix f and now we are going to study many properties of this matrix for example, what happens if you change the basis or what is the matrix of last time we saw matrix of the composition map.

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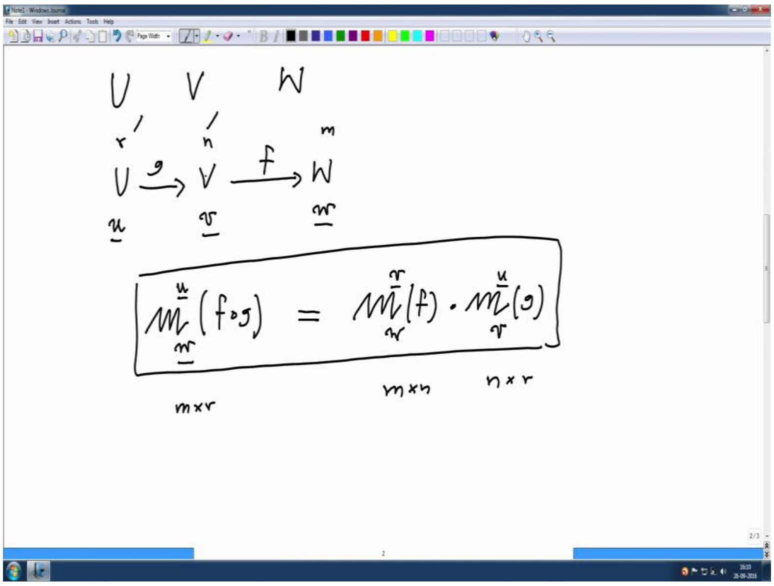

So, last time, let me recall what we proved last time. Last time what we did was we have these 3 vector spaces V W X or U W U, 3 vector spaces U V W that is what is I want to keep this n notation and we have linear maps g from U to V and linear map from V to W and their basis here u v w and this dimension I; we do not need to call what dimension is they are numbered by a suitable index finite index in set then last time, we proved that matrix of the composition that is f o g with respect to the basis u and w, this is a map from this composition u to w.

So, this basis to this basis that is this on 1 side and the other side what can you do we can take a matrix of f with respect to v w and matrix of g u v and we have noted that matrix multiplication then you define I will recall that that is same as this. So, is a matrix multiplication is defined. So, that this formula is correct. So, in general let us recall what is a matrix multiplication this is very important formula and that also motivates, why did the matrix multiplication is defined in that way. So, that this composition formula is correct.

So, let us recall the matrix multiplication. So, first of all we see here this is built in here this is if this where n dimensional this is n dimensional and this is r dimensional then this is this matrix is m cross r, this matrix is m cross n and this matrix is n cross r. So, to define a product of 2 matrices we need the columns of the first matrix and the rows of the second matrix they are numbered by the same set in particular they have a same cardinality you see this.

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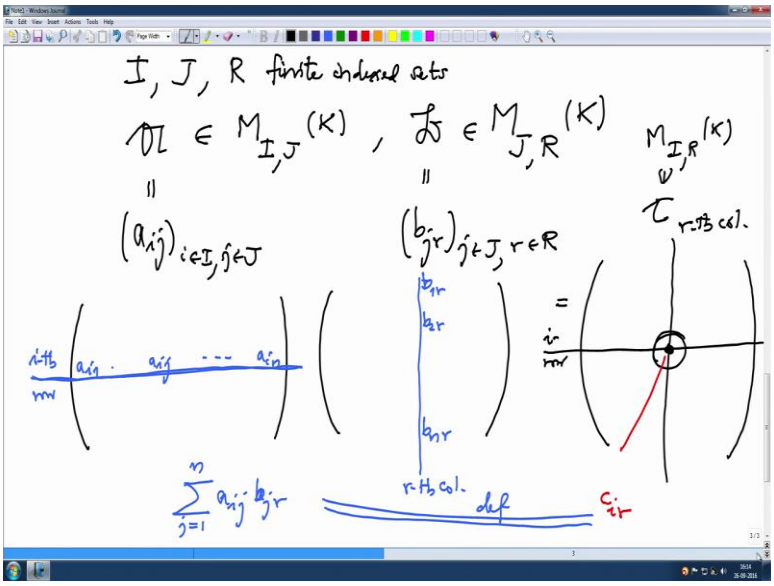

So, we are going to define a product of 2 matrices a and b. So, 1 matrix is a, I am going to denote matrices by the gothic letters these matrix is M I cross J matrix I J K, I J and now what do will; I do the R; they are finite index sets . So, the rows of this matrix is denoted by I numbered by I and the columns are numbered by J and number matrix b, b this should be now the role of this should be numbered by J and the columns are numbered by R. So, this looks like a i j i in I j in J and this is b j r j in J r in R and we want to multiply by multiply these 2 matrices.

So, what we do is we want to write the product; product is the matrix c let us say. So, the c will be M, we forget the J and I cross R matrix. So, I have to explain you what is the entry here this is ith row and this is jth column no rth column; rth column, this is ith row and this entry, we want to explain. So, that let us call that entry to be c i r and that is then as follows you take the ith row here this is ith row and take rth column this is the ith row ith row of a. So, this is a i 1 a i j a i n my speed gets reduced because I have to think which indices and this one is b the rth column.

So, b 1 r b 2 r etcetera b whatever n. So, I do not know what will I write here the number of n n n r. So, you multiply this the next this and take their sum so; that means, we are adding a I j j is an a b j r and this sum is ending over J from 1 to n and this is by definition it is this here c I r this is a definition that is how you multiply matrices and this funny way is because we want a nice formula.

So, that is what in general one defines and these one this is row this is column normally I want to write vectors in a vector spaces as a columns and not the rows because a calculation is easier with the columns at least one thing one can also do with a rows equally, but one can do better with a columns I think now. So, we are going to get information about a map about a map from the; it is matrix that is the whole idea in this lecture.

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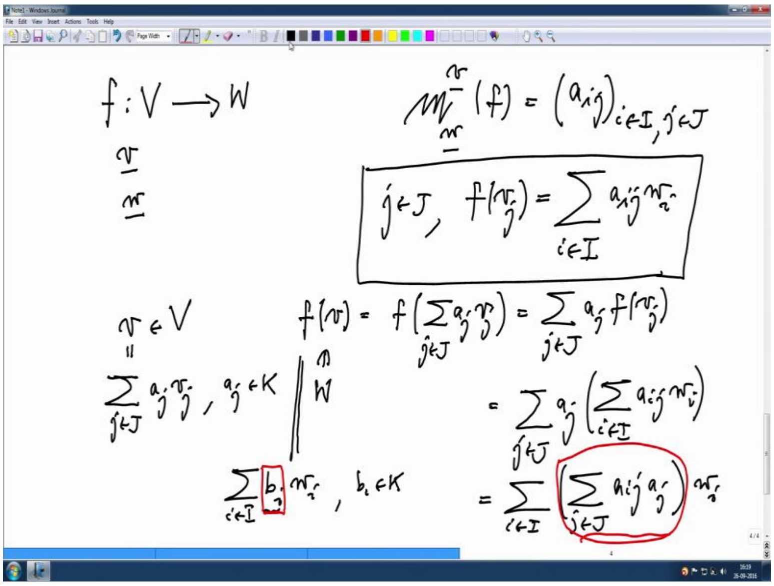

So, f 11 and V to W on the other hand we have this matrix f M V w f and if you one should write it every day at least one case a i j i is i is in I J is in J, these are defined for any j in J f of v j this is jth vector in the basis of V is written in terms of the basis W of W capital W this these equations.

This gives you back the linear map. So, V is a basis here and W is a basis of W, now the first thing I want to note that how do suppose I have given a vector v in V then I know what is f of v f of V. So, first take that V and write in terms of a basis. So, this V you write it as summation a i a j v j j in J a j that is scalars we are assuming finite dimensional. So, j is a finite set what it is I do not have to say a j are finitely many are non 0 etcetera. So, this then I; what I want to compute is f of v f of v of course, we know because f is linear, this is a f of summation a j v j which is summation this is our j this is a j out f of v j and f of v j you can substitute from here.

So, this will be summation j in J a j then summation i in I, this we know there a as a w, but they are finite sum. So, I can rearrange them. So, I can interchange them for instance. So, this is same as summation i in I summation j in J and I club this together. So, that I will write it as summation a i j a j and then club it like this and times W i and this on one hand, this on the other hand. On the other hand this f of v whatever it is, this is in W so; that means, I can write this into a basis linear combination of this basis small w.

So, this is on the other hand, this is equal to summation some constant that is numbered by now i b ; W i b i are some constant and now these 2 equalities W i is a basis. So, corresponding coefficients are equal so; that means, for each i in i b i equal to this insert thing. So, in this b i equal to this quantity with this quantity for each i so; that means, let me write that.

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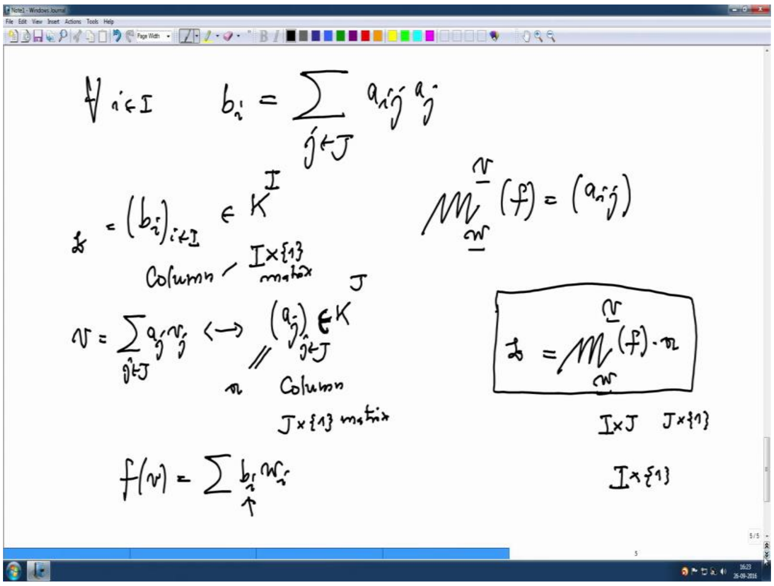

That means for each i in i b i equal to summation is over j j in J a i j a j, this we know from the linearity of the map, but I want to explain this in terms of matrices, now think of this b b i i in ,I this is elements in K power i these are the coordinates of f of v with respect to the basis W and i want to think it as a column.

On the other hand what we have we have a matrix M V w f which is a i j matrix and we have the v v, we have written coordinates of v v, we have written as summation a i v j. So, this corresponds to the vector a j j in j in K j K power j, but this also I want to think as a column. So, think also this as a column means it has rows as many rows as cardinality J.

So, therefore, this is a actually j cross 1 matrix this column vector and this column vector it has I rows. So, this is I cross one matrix. So, now, I am saying look at M V w f times let us call this, this column vector as let me give some name to this small gothic k. So, this times small gothic next is this is I cross J matrix this is J cross 1 matrix. So, the result should be multiplication is defined because the number of columns have the first number of row or the columns are index numbered by a the set j and here the rows are numbered by j.

So, this makes sense and the result will be I cross one matrix. So, the candidate for that is this vector b small b. So, the result is this 2 are equal. So, this nice formula so it computes the image vector, so, it computes b computing b is writing f of v in terms of b right this is what we have. So, computing these coefficients means we are computing values of a arbitrary vector v. So, that is that is how we will use this for calculation.

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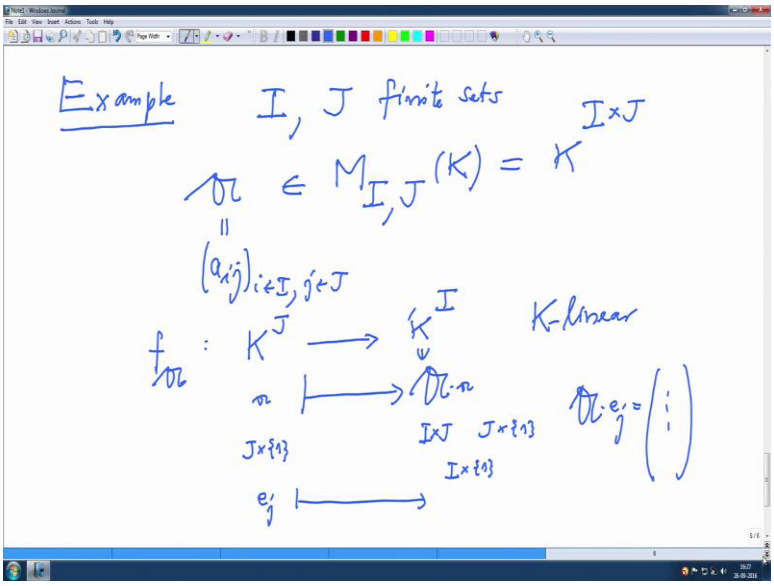

So, next example I want to write one example. So, example it is as we have seen it is not really necessary to numbered columns rows and columns by the standard indices 1 to m or 1 to n. In fact, it is better if you index by arbitrary finite sets because in that case we do not have to think where it starts and where it ends. So, I will think I J finite sets and a is a matrix; a is a matrix numbered rows are m numbered by I and columns are numbered by J with entry in the field K this remember this also is nothing, but K power I cross J.

So, it is really a function from the product set I cross J to the field K this a matrix and let us write this as a i j entries are a i j i want to think this as a linear map. So, I want to define linear map f this f will depend on this matrix. So, I will write that f suffix here and this is a map from where to where this is a map from K power J to K power I this are finite dimensional vector spaces because I J are finite sets and these are the standard vector spaces E j e e; there is a standard basis here there is a standard basis here and if I want to define a linear map I just have to give values on this basis or directly I give take a column vector I will think the vectors here as columns and vector here are also columns and I directly want to give a map.

So, what is a map take the column vector small a and map it to it should go to our column vector again and what capital a times small a this makes sense because you see this is this a is I cross J matrix and this a was here. So, it is J cross 1 matrix. So, this time this is J cross 1, the result should be I cross 1 matrix that is what we wanted it. So, this is indeed an element here. So, this is the linear map corresponding to this matrix and another thing one should note here that it is not really necessary these are finite sets in if they were not finite I will put 1 bracket here until if we make sense because the matrix multiplication will still be defined because the columns will have only then finitely many non 0 entries.

So, and where as we can also we can also check where the standard basis E j goes where do this go that also you can compute this that will be computing this is easier now because E j has only 1 non 0 entry and then which what will be that column you can easily figure it out. So, I leave it for you to check what is the column a times E j this answer should be in terms of the entries of a. So, we check what are the entries? Here conversely you already know that we have defined matrix of a linear map.

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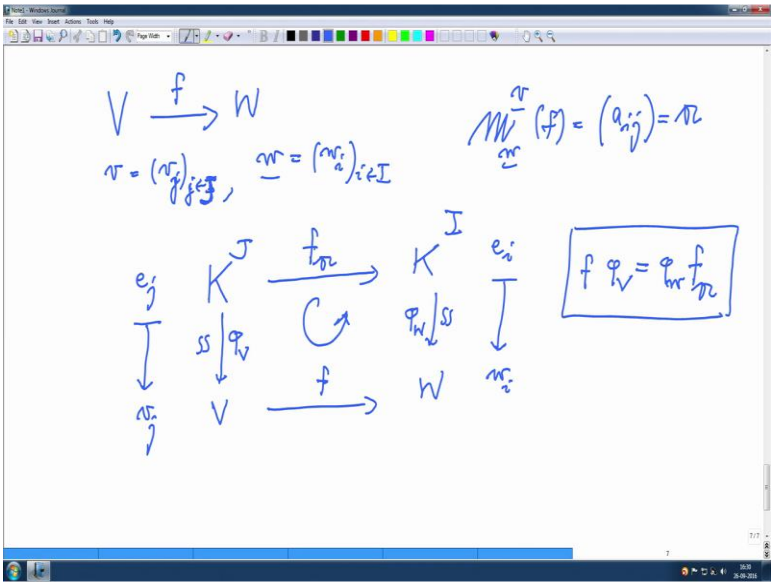

So, if you have f from V to W; f and here V is a basis V i i in i here, W is a basis W j [FL] this is v j j in J, this is W i i in I and then you have this map f.

So, on the other hand, so we will get as matrix; matrix of f with respect to this basis this is a i j if you call this as a then we have defined the matrix a linear map corresponding to this that linear map is nothing, but f. So, let me draw a commutative diagram. So, this is K power j to K power I, this is the map we have defined using this a that is f a on the other hand, we have this map here V to W that was a given linear map f and this are as isomorphic because here the basis standard basis v j goes to v j since v j is a basis here this map is e j is as basis. So, this map is injective because v j there is a basis and it is surjective therefore, this map is an isomorphism similarly this map is an isomorphism where standard basis vector e i goes to W. So, this is an isomorphism this is a linear map f given linear f and this is what we have defined.

So, this diagram is commutative this means we go from here to here and then go from here to here the result is same. So, I have to give the names for this. So, let me give name for this is where this is phi of W and this is phi of v this map which is an isomorphism e j goes to e j this map is e i goes to W i. So, this diagram is commutative means f a follow it by phi W there is one side on the other side start with phi v and then go by f. So, these 2 things are same this is this is very very easy to check it is just have to take e j and see where it goes under both the maps and then you will see they are they are equal alright.

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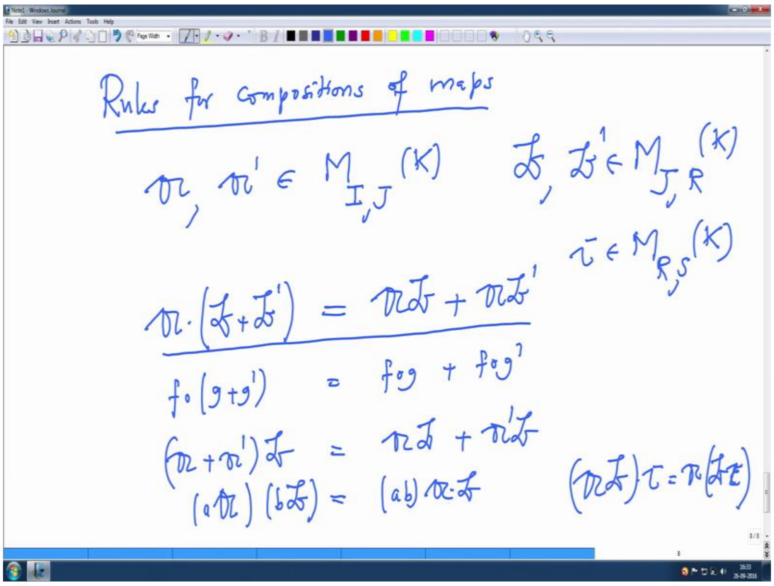

Now, the rules remember we have a rules for rules for composition of maps we can transfer them to the rules or the matrices and I will write the rules. So, the; they will be applied to check the check these rules you just have to apply to the instead of matrices the linear maps defined by those matrices for example, I want to write the following rules it will be very easy to check them. So, I am not going to check these rules.

So, suppose a and a prime are 2 matrices of the same size that is i cross j and we have b and b prime of M J R where the where the columns here and the rows here they are indexed by the same finite set j then I want to write and c is another matrix which is from

R S R is the rows index indices and x is the column indices finite set all sets are finites then I want to write such equation a times b plus b prime, this make sense because b plus b prime may change because they are in the same rows and same columns a times that is also make sense because the number of columns here or the number of rows here or columns are indexed by j rows are also indexed by h. So, this makes sense.

On the other hand, I could multiple the first a b; a b prime and add them this result should be same. So, this will corresponds to what f compose g plus g prime equal to f compose g plus f composes g prime this is very easy to check, but this is this is little more difficult to check because that matrix multiplication is so natural.

Similarly, a plus a prime times b equal to a times b plus a prime times b similarly if we have a scalar a multiply by matrix a; that means, you push that scalar in every entry similarly scalar b times matrix b this should be same as scalar multiplied in the field times matrices multiply one more I will write a times b and then times c this is same thing as a times b c this is associativity of product all these rules were very clear for the composition of maps so, but. So, I am not going to check this with the numerical examples we will take a break and then after the break we will continue.