**Linear Algebra**
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**Lecture – 39**
**Matrix of a linear map [continued]**

So, come back to second half of this lecture. I will now restrict myself to a linear maps from the same vector space to same vector space. So, those are called endomorphism or those are called linear operators on a vector space v.

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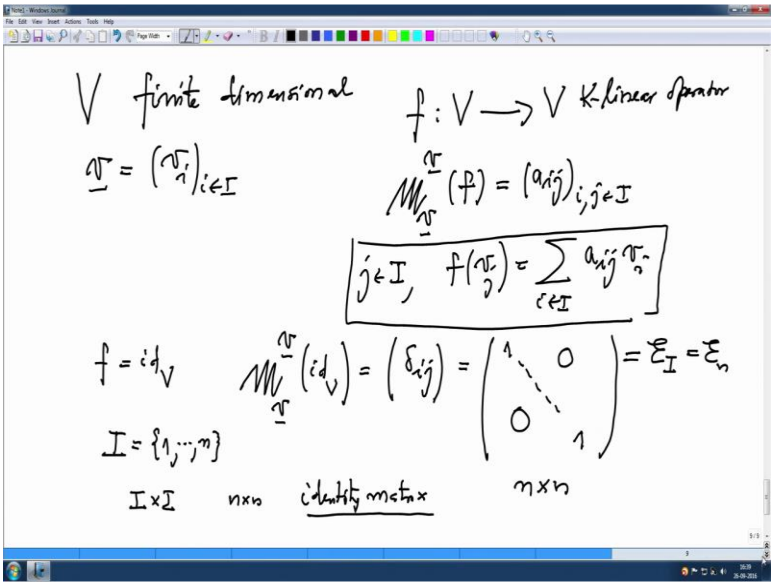


We will assume V is finite dimensional, v is a basis numbered by finite index set i, and we consider a linear operator f from V to V K linear operator, and then we have to this linear operator we have associated a matrix m v v f this was the matrix of f a i j, i j both are elements in I now same basis no.

So, for each j in j j in I f of v j is the jth column what is a i j, v i. So, we have given. So, what is a suppose I take a f equal to identity map i d v, then what is a matrix m v v of the id mat identity map; this is i just have to know where v j goes under identity v j goes under identity to v j only so; that means, this is a matrix where it is a del i j matrix where del i j is a chronicle delta this matrix also in when i is a standard finite set 1 to n, this is the matrix 1 1 1 on the diagonal it is n cross n matrix and everywhere else it is 0 such a matrix is called identity matrix n cross n identity matrix or I cross I identity matrix.

The name again comes from because it is a identity map, and this is the standard notation for this I will use is E I, that is for the identity matrix or e n when i is the standard statement to n and also it is clear now.

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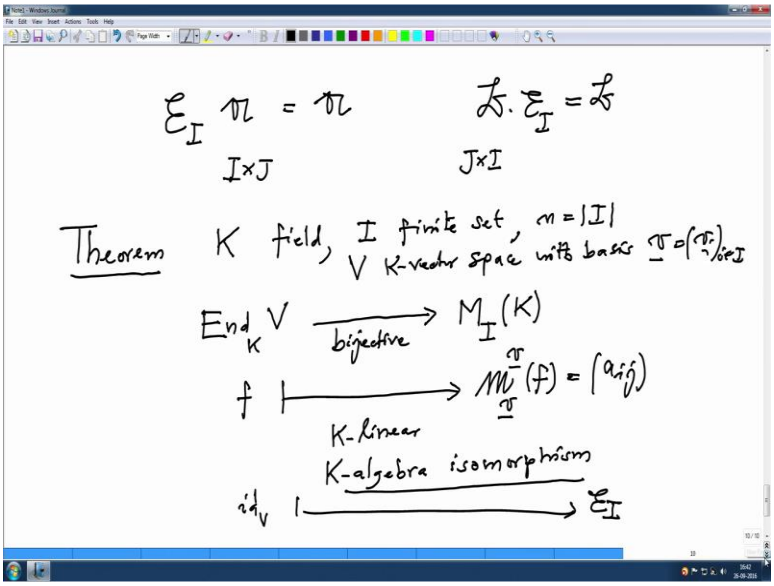


Now, there are the rules what we saw it is clear that if I have E I times arbitrary a as a matrix a then it is a where a is i cross j matrix similarly any matrix b times identity matrix I, E I is same as b, this b now as to be J cross I matrix.

So, this are now I want to state a theorem to remember. So, this is a theorem. So, K field I finite set may be you can call n to be the cardinality i and then we have we have defined a map from endomorphisms of the vector space V, V is a v is a vector space K vector space with basis v v i numbered by that finite set i and then we have a map from endomorphisms to the matrix square matrices i cross i matrices the field anything in the field K namely f goes to identity matrix the matrix of f with respect to the basis f this is a i j, and what I want to say that this map is isomorphism this map is bijective, isomorphism of vector spaces that is it is a k linear map and bijective and more over I want to say that it respects addition multiplication.

So, it is also K algebra homomorphism not only it is isomorphism of vector spaces, but it is a isomorphism of rings and remember isomorphism of rings means it respect addition, it respects multiplication in the ring it identity should go to identity and that we have checked just now if your identity map it corresponds to the identity matrix E I, so that

that is very important. So, it is a k algebra isomorphism in particular the dimensions are same.

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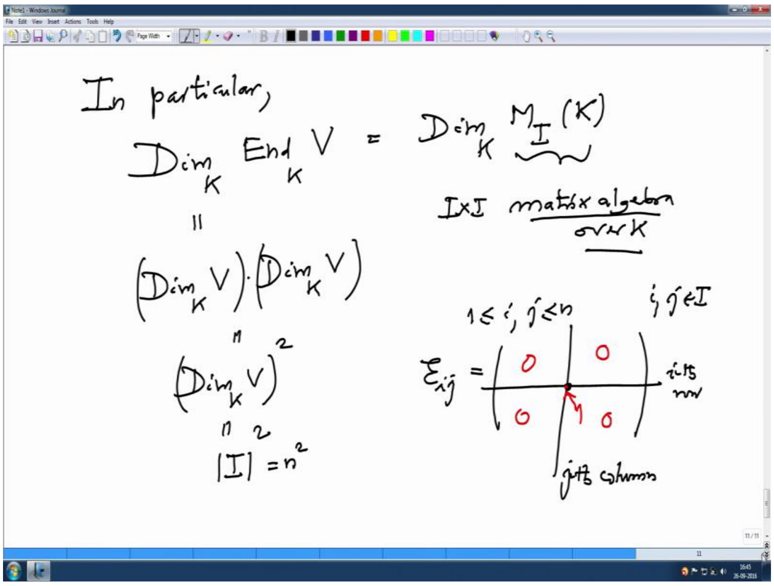


So, in particular dimension of endomorphism vector space equal to dimension of matrix algebra that is a reason this is called a matrix algebra over K, I cross I matrix algebra and this dimension we know this dimension is same thing as dimension V times dimension V. In general we have checked earlier that dimension of ohm v w is dimension v then dimension w. So, this is dimension V is whole square, which is cardinality I whole square by this is also dimension of the matrix algebra, and what is the basis the basis of these should have so many elements.

So, if cardinality I is n then this is this should have n square basis elements. So, let us write down a basis of this matrix as well, now you think in terms of matrices. So, why n square you take any two indices between 1 and n not necessary, or you can simple say i comma j is in I and then to this pair I want to define matrix in it is E i j; this is the matrix where i jth entry is one this is ith row, this is jth column and this entry here is 1, 1 and everywhere else it is 0 this entry here is 1. Then is called a matrix i jth matrix unit and it is obvious that every cross I matrix this is this matrix units are defined for any not necessary square matrices, and this it is obvious that any matrix is a a linear combination of this in a unique way because and it is. So, this is a basis.

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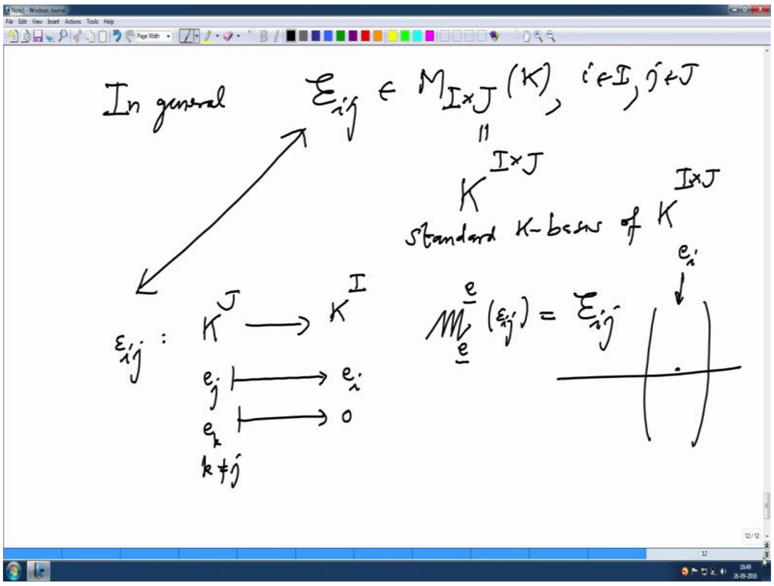

So, in general let us write down in general E i j where i cross j matrices i in I, j in J only i jth entry is one and every or else it is 0 it is like a standard basis. If you think this as K power I cross J this is a standard basis of K power I cross J standard basis K basis of K power I cross J, and which map do they correspond you should always think both ways. So, they should corresponds to an linear map f from no I want to give some other name E i j let me give E i j small e i j, this should be a map from or not small e i j is small epsilon E i j this should be map from K power J to K power I i j finite sets if not you put a round bracket here.

Here standard jth basis e j that should be where that should go to the 1, and all other guys should go to 0, and e k goes to 0 for k naught equal to 0. So, if you write down a matrix of this with respect to this, this is a standard basis that is precisely this matrix units because this will be the jth column jth column here is precisely you will have 1 in the ith place. No no no this is not, this is not quite correct what I said not one, but e e i where i this e i is a standard basis here, when this e j is a standard basis here. So, that is sometimes a problem when you have the similar vector spaces standard basis (Refer Time: 12:10).

So, this one and remaining guys go to 0 vector. So, therefore, ith jth column in this matrix of again I have written this K wrongly this should be small i j, this is the vector the jth column is the standard vector e i; that means, at the ith position here one; that means, ith row as one and this is a jth this is this these are the matrix units. So, and we know the dimension we know this, now, if you want to make a calculation for example, in the ok.

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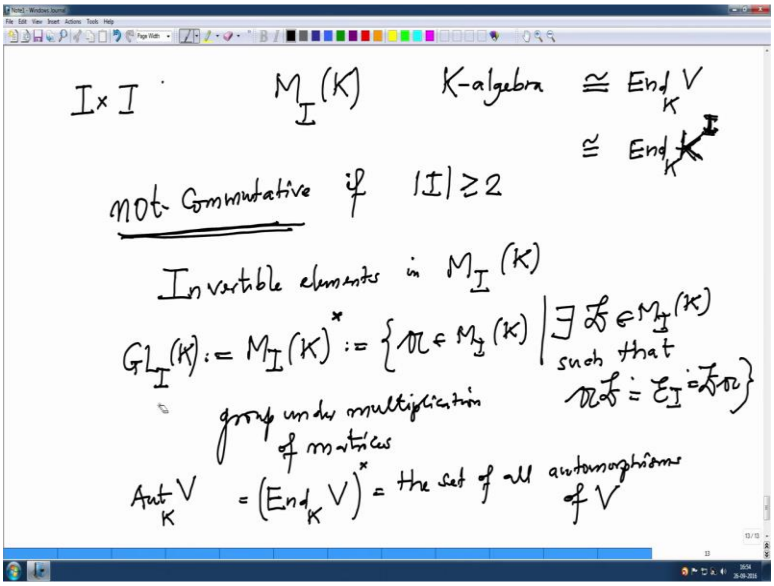


So, that is that is all then I want to go on to now the invertibility. So, now, we have seen we have seen that if I take the I cross I matrices square matrices M I K, this is the K algebra we know how to add, we know how to multiply, we know how to scalar multiple this is a scalar algebra and this k algebra is isomorphic as a algebra 2 and K V where V is finite dimensional vector space V basis is numbered by i or also we can think more concurrently this is a isomorphic to end K power I these vector space K power I. This is the standard model for a vector space which as basis numbered by this set I.

This is a K algebra, so therefore, and it is it is not commutative if cardinality I is at least 2, this one can see it very easily from you can we can write down two by two matrices for example, whose they do not commute the matrix multiplication. So, it is not commutative that is not so good, but on the other hand we have to deal with it to study the linear operators on a finite dimensional vector space. So, therefore, it makes sense to talk about the invertible elements.

So, invertible elements, elements in M I K what are they? They are matrices it should have inverse; that means, you remember a notation in general M I K and then you put a cross here this is the set of all matrices a such that there is there exist a matrix b in M I K again such that both the products a b make sense that should be identity matrix and also b a both should be identity matrix and because it is non commutative algebra both these equations are necessary, this and this because we do not know this is commutative or not.

So, both the equations are important we cannot say only one equation; in commutative case one equation is enough because other equation is a consequence from the commutativity. So, this is and remember this I want to denote by GLIK this is a definition of GLIK and this is a group under multiplication of matrices that is true in general we have seen if we have a ring r then r cross r cross the set of all invertible elements in r with respect to a multiplication from a group under multiplication of the ring, and that group is called a unit group of the ring a.

We could have done this and remember a notation for the corresponding end V K the linear operators on V which are invertible they are called automorphisms. So, they are aut K V this is the set of all automorphism of V and they will corresponds to each other that is one thing, another thing I want to make a comment for the future see a definition of GLIK is all those matrices which are invertible; that means, there is a matrix b.

So, that a b equal to identity matrix which is also b a; now it is not very easy to check for a matrix to be invertible from this definition because we have to find a matrix b and you have to multiple both sides and then check both the sides are identity, this is lot of lot of checking is involved if one want to check quickly then after one or two lectures I am going to introduce a concept of determinant, and that determinant will easily check how do you check that the matrix belong to the GLIK or not and that that will be at the cost of the determinant computation of the determinant.

So, these I will do it when it is the necessary so, but before that I want to little bit put hands on the calculation with the universes.

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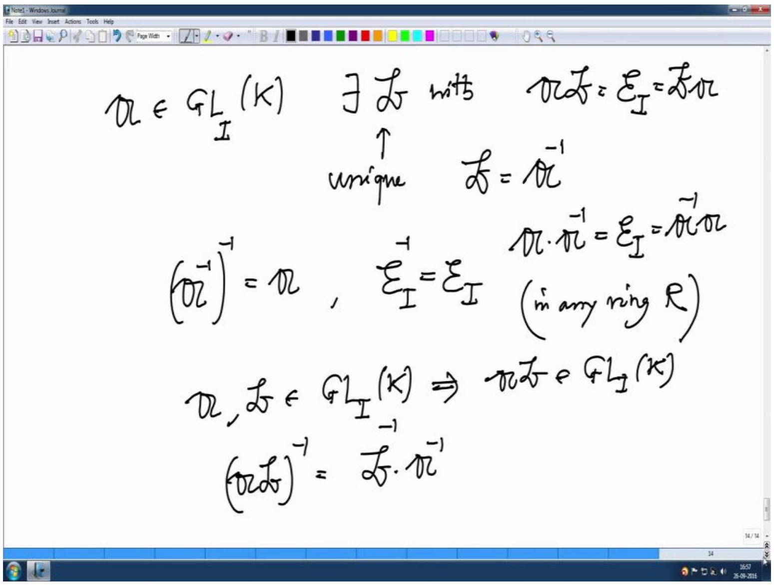

So, if a is in GL I invertible matrix; that means, there is a matrix b with a b equal to identity, and also b is identity and you say this b is uniquely determine unique, the same proof as how we proof in a ring if a inverse exist it should be unique that therefore, this b is denoted by a inverse a power minus 1, and then these two equations becomes a times a inverse equal to identity matrix also equal t o a inverse times a; and again standard formulas like inverse and then take the inverse of that we get the original matrix, this is nothing to determinant I say this is in general rule in any ring whenever inverse exist the inverse of the inverse also exist and you may get back the original element.

Similarly, inverse of identity matrix here identity matrix itself, this rules are nothing to do with matrices this is in general rule in any ring R, similarly how would I compute a if you know both a and b are invertible pre matrices invertible then their product is also invertible, this is again no big deal it is a group under a multiplication and what is the inverse then and a b inverse is b inverse times a inverse in the other way product, this is because it is not commutative in a commutative ring this will not be the case ok.

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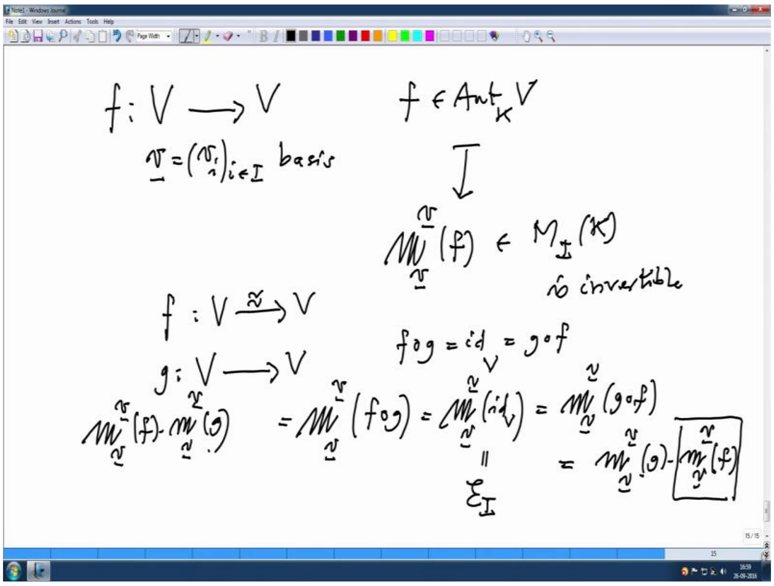

Now, therefore, it becomes question is given a linear operator f V to V on a finite dimensional vector space, and if we are given a basis v numbered by finite set v i i in I basis. Suppose we know this f is auto automorphism then we definitely know the matrix of f, f with respect to this basis which is a matrix in MIK this is invertible, and how do how does and check this. So, because f is an automorphism, f compose there is a map in the other direction g V to V this f.

So, that f compose g which is identity map on v, similarly g compose f is a matrix identity matrix on v that is a definition because this is an automorphism. It have a inverse map and that inverse map is nothing, but j now when you apply the matrix apply this map m v v, that is m v v of f o g equal to m v v of identity, and that is also m v v of g o f, but m v v of identity is identity matrix and this we know it is product of this matrices and the other hand it is also this m v v g times m v v f and t his therefore, this matrix is invertible and this is the inverse of that.

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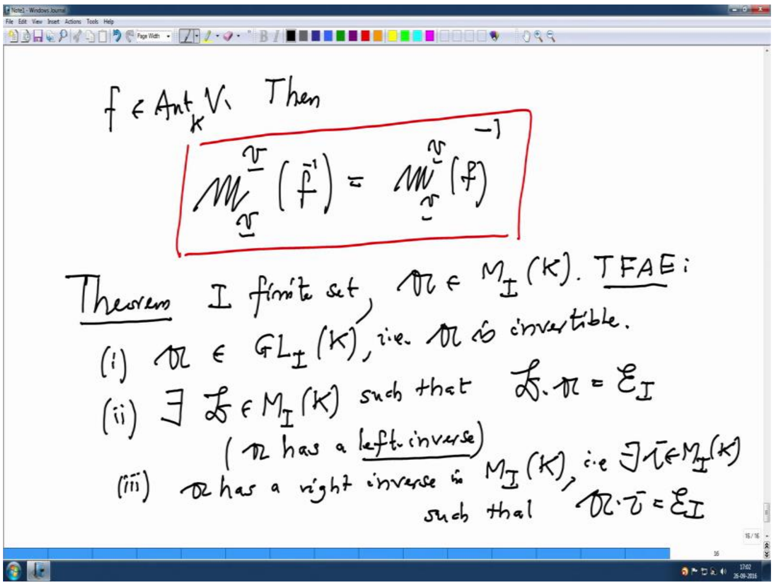


So, therefore, what we have proved is let me write on the next page it is important for a long, what we have proved is if f is an automorphism then m v v f inverse is equal to m v v f inverse of that. So, inverse I can take out. So, this is very important.

Now, I just want to do little bit little bit about matrices. So, just now I said that to check a matrix is invertible we have to find a matrix b. So, that a b is identity matrix and b a is also identity matrix; still I want to do it little bit economical. So, that is where I state this theorem. So, theorem that is here the linear algebra is useful. So, I infinite set, and a is a matrix in MIK then the following are equivalent.

One a is invertible a belongs to GLIK, that is a is invertible; 2: their exist a matrix b in GLI not GLI their exist a matrix MIK such that b times a is identity matrix; that means, also this is also shared in general that a has a left inverse, in general left inverse of an element in a ring a is an elements when you multiple that element by left it becomes identity, the other side we do not know. So, such an element is called a left inverse.

Three: a as a right inverse, in the algebra MIK that is there is a matrix C such that a times c is the identity matrix. So, this three conditions are equivalent so; that means, we have a checking is reduce to only one equations, but this is special only for this algebra and that is because.

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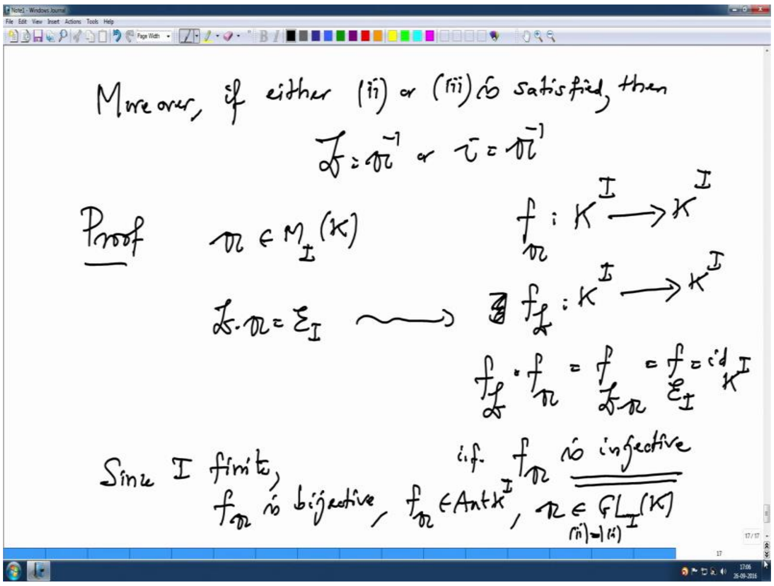


So, proof. So, let me complete still more statements more over if either. So, if 2 or 3 anyway we were going to prove their equivalent, and 2 and 3 are numbered by not by this, but roman two or three is satisfied then b as to be a inverse or c as to be a inverse.

So, what does this means is if two is satisfied that b is a inverse of a, if 3 is satisfied c is a inverse of a and proof, proof is very very simple instead of a matrix say I am going to think a linear map f a this is from K power I to K power I and this matrix a is the. So, for example, invertible that will mean this is a automorphism. So, and left invertibilty means b times a is identity matrix should mean there is a map the map is of course, the map is f b, f b if i compose this maps f b compose f a this is nothing, but f of b compose a b times a, which is f of identity matrix which is the identity map of K power I.

So; that means, this map that is the map f a is injective, because when somebody goes to 0 then I compose with this and it still go to zero, but this is the identity. So, that the element itself is zero. So, this condition the compose this map is identity is equivalent to say that this map is injective, but we were in a finite dimensional vector spaces in injective linear map is bijective. So, since I is finite f a is bijective; that means, f a belongs to (Refer Time: 30:05), but that will mean the matrix which is a will belong to GLI; that means, a is invertible.

So, this proofs two implies one, similarly two implies, three implies one because instead of injectivity the other compose is identity will mean that it is surjective map, and finite dimensional vector spaces surjective, injective is equivalent to bijective. So, similarly so this proofs two implies one.

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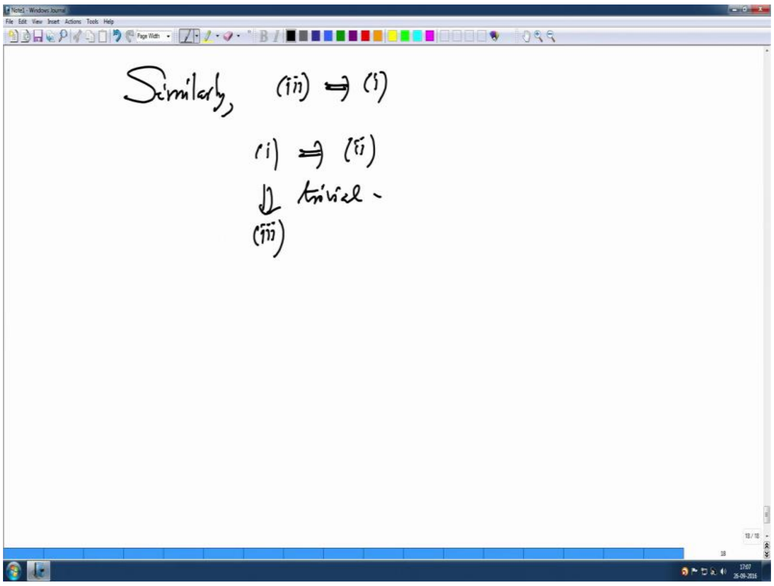


Similarly, three implies one, and of course, one implies two, and one implies three, they are trivial I think will stop continue next time.

Thank you.