Linear Algebra
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Lecture – 04
Definition of subspaces

We have seen some examples of vector spaces and I would it till like to see more examples more examples.

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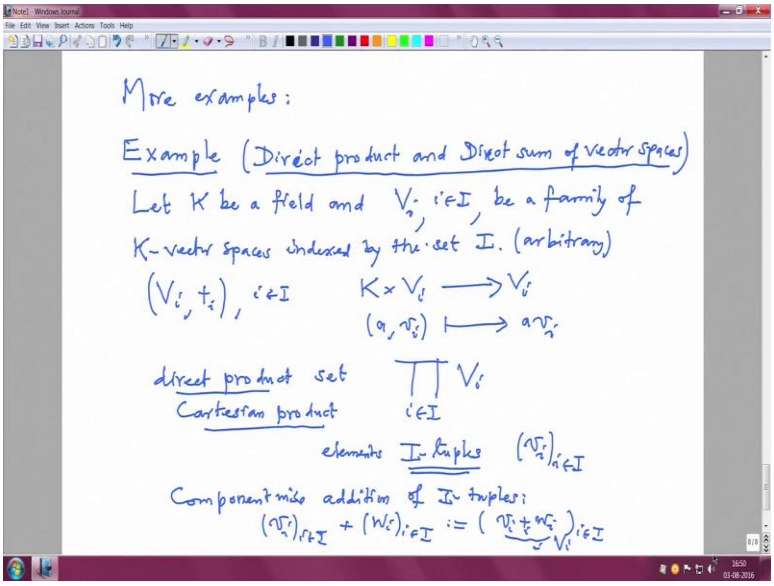

So, this example in this example from a given family of vector spaces over the same field K, I will construct a new vector space which we will be called as a direct product, and then we will construct another one which will be called at direct saw. So, the title of the example is direct product, and direct sums of vector spaces. So, let K be field and V i, i in I be a family of K vector spaces indexed by the set I.

In this case I could be finite or I could be infinite also. So, i's arbitrary. So, on each V i we have a each V i is an Abelian group strictly speaking for each i this Abelian group structure on V i would be different. So, strictly speaking I should write sum suffix i these are Abelian groups, and each of them as a scalar multiplication each of them has a map from K cross V i to V i these are called scalar multiplication, and to let me remind you that we are denoting the same a comma V i going to a V i. So, it is clear when you write scalar multiplication which sector space we are doing the scalar multiplication.

So, now consider the product direct product that is usually denoted by this product V i, i in I. So, the limit on this is also called a Cartesian product. So, you can think of elements of these Cartesian product as a tuples. So, they are I-tuples; that means, like this V i, i in I the elements of this set are I-tuples you see if the set i is countable then you can still be write this as 1 to n and so on continued to the tuples, but when set i is typically let say real numbers or intervals, then we do know which element will come faster and which element will come later.

So, only then you think of them as a tuples with i-th position, and we do not know which is before earlier position and later position. So, the positions are numbered by the elements of i. So, on these; obviously, there is a group structure is a obvious that is component wise; component wise addition of I-tuples that is if I have 1 I-tuple V i another I-tuple w i then the result should be another I-tuple and what is the i-th entry here you add the corresponding i-th entries. So, V i plus w i and this sum we are taking in a vector space V i shows this. So, this sum I will say in the vector space V i.

So, this defines an addition and it is also again very easy to check that with this addition these Cartesian products become Abelian group. If you want the inverse of the tuple V i you just take the inverse is of V i at the it H position and the additive identity will be just 0 0 0 0 0 every where all the tuple all the entries are 0 0 0 at the i-th place should mean 0 vector in the vector space V I, scalar multiplication also component wise also.

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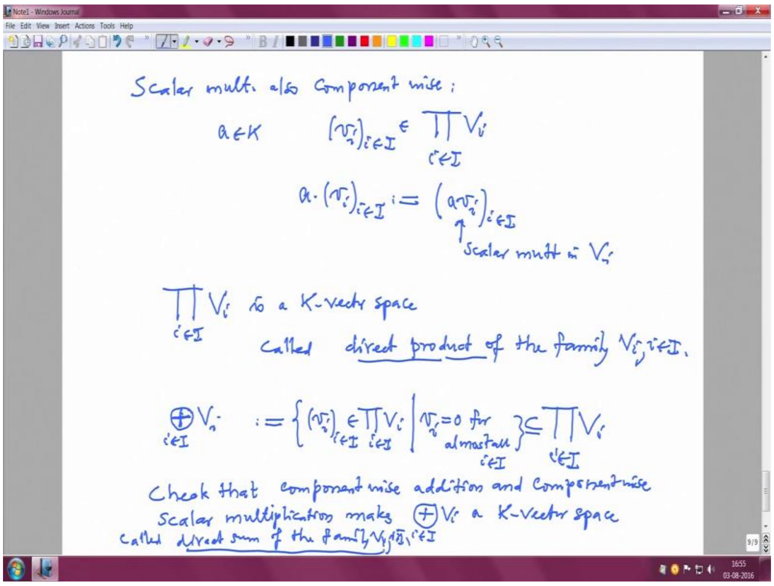

Component wise that is if I have a scalar a and if I have a tuple V i this is in the product set Cartesian product, then a times V i a time the tuple we say means by definition it is to push that scalar inside, so a times V i.

And this is a scalar multiplication is V i. So, with this Cartesian product becomes again a vector space this V i is a K vector space; here i should mention that I am not checking the properties that we need for vector spaces 1 to 4, but they are so natural that more or less their obvious, but the first time participant should check this on their own this this product vector space is called direct product of the family V i, i in I. Now I want to construct a new vector space from there from this direct product. So, we have this Cartesian product. So, I do not take all I-tuples I will consider only those I-tuples V i in the product which have the property that almost all V i are 0 for almost all i in I.

That means only finitely many i-th entries are non-zero vectors in V i's, all other i-th entries in this tupple are 0 vector in the corresponding vector space V i. So, this I want to give a name for this subset this is a subset of this this is for example, this name I want to give is sum plus with a circle around it and i in I this is V i. You will soon you will see where I do I denote it by the sum. Now what do you need to check that actually it is the same addition and same scalar multiplication in to the component wise also make this as a vector space. So, I will just means check that component wise addition and component wise scalar multiplication makes this direct from a K vector space.

These vector space is called direct sum called direct sum of the family V i i in i let us little bit illustrate.

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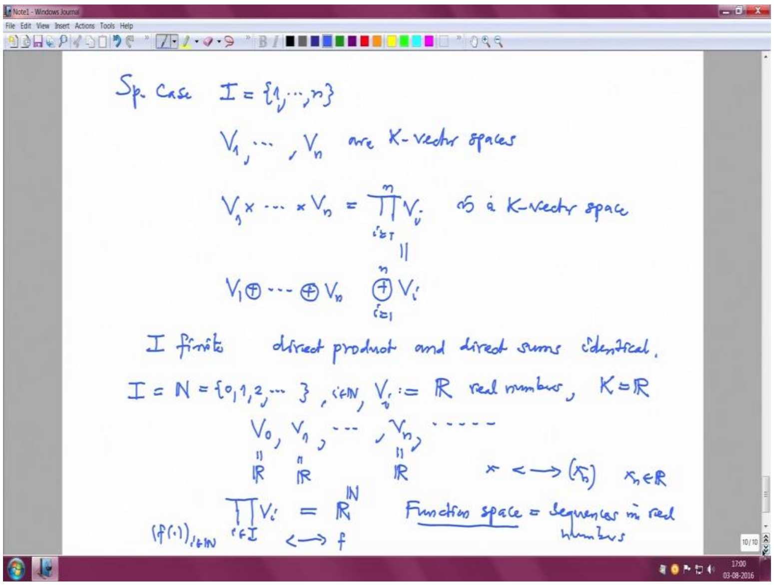

For example when I is a finite set special case i finite which i is 1 to n and; that means, what we have given vector spaces a numbered by V 1 to V n are K vector spaces then we have on the product V 1 cross cross cross V n this is a product set, this is a K vector space; component wise addition and component wise scalar multiplication. In this case what is the difference between these two there is no difference because they have only finitely many vector spaces are given. So, every tuple is almost all vectors in the tuple of almost are 0 only finitely mean you can be non-zero, so where now everybody is allowed.

So, therefore, in this case direct product and direct sum are same. So, in this case when i is finite the concepts direct sum direct product and direct sum these two concepts are identically, but obviously when I is not finite let us take 1 case i not finite. So, let us take a very simple case where i is infinite. So, the simple is infinite. So, it is the set of natural numbers which is the simplest one this is 0, 1, 2 and so on. So, in this case we have given and let us take a case where all our vector space is V i for any i in n the vector space is V i let us take them all to be real numbers real numbers, and let us take the field also to be real numbers . So, each V i is R-vector space and we have given countable many they are numbered by V 0, V 1 so on V n, and goes on and each 1 of them is R.

And the product means what now Cartesian product means product of V i i in n now you can because all these are same. So, this you can think it is mapping from into R because each mapping each mapping f will correspond to which tupple as I said f of i, i in n given n tuple you have a mapping from in to R and given a mapping you have a tuple numbered by n. So, these answers they are same. So, and what is the vector space to shear component wise addition and component wise multiplication. So, this is nothing, but in earlier example we have taken arbitrary set i and the field, and then the functions from I have to that field the same vector space structure here.

So, this is nothing, but a function space. In fact, if you see what are the elements, elements are actually re sequences in the real numbers. So, this is consisting of this is sequence element are sequences in real numbers. So, elements are precisely the functions and functions from n to R are precisely the sequences from their sequences in R each sequence in R x n where x n's are real numbers.

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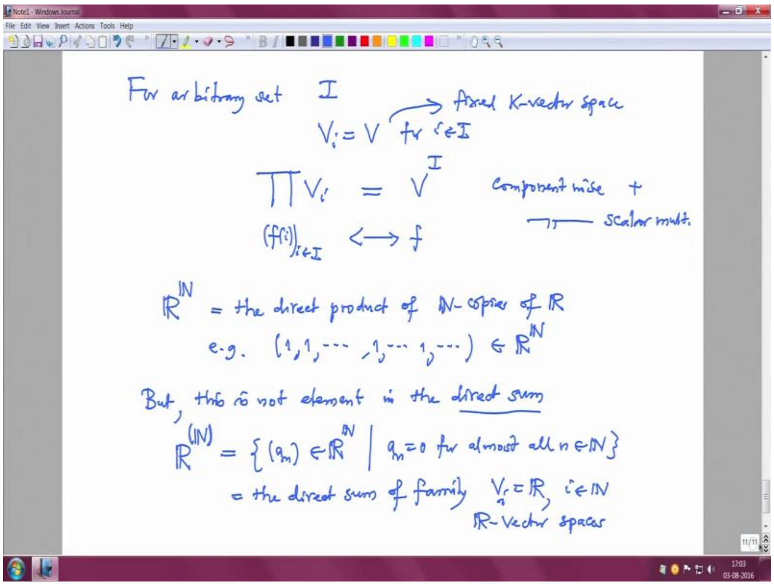

So, you can think them as functions x their value of x at n is x n. So, and there in nothing special about this one it is for a. So, note that for arbitrary set i if I take a family V i to be a fixed vector space V, this is a fix vector space.

Fixed K vector space, then product set Cartesian product you can think it is a functions from i to V any function here f you can think a tuple f of i, and any tuple you can think in the function and what is the structure here of vector space it is a component wise addition and component wise multiplication. Addition and component wise scalar multiplication, these new vector spaces are very useful going back to earlier example where in this R power N. So, what is the direct sum this is the direct product, this is the direct product of N copies of R. So, for instance the sequence the constant sequence 1, 1, 1, 1, 1 all the way 1 this is an element here.

But this is not element in the direct sum because for an element in a direct sum the characteristic property is only finitely mean entries in these tuple should be non-zero, but in this all the entries are non-zero. So, it cannot be element in the direct sum. So, in these case the direct sum I am giving to denote by R power N, but i should distinguish. So, I put a bracket here round bracket. So, this is the set of all sequences n in R power N such that a n equal to 0 for almost all N, this is precisely the direct sum; the direct sum of the family V I, V i each V i is R and i is varying in n R-vector spaces these two concepts are very important. So, let me again repeat the notation for the special case the particular case.

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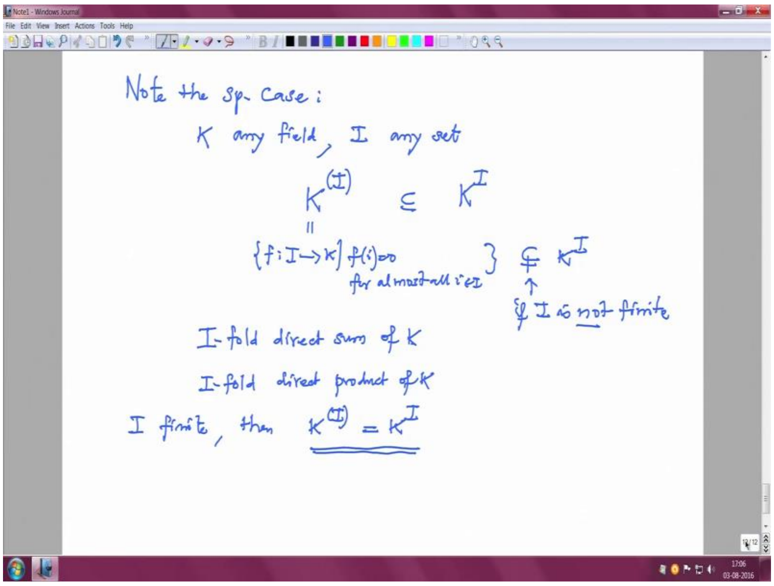

So, again note the special case take K to be n field, and take I to be any set.

And remember on the function space is K power I, they are all functions from I to K and among those functions I take the functions who has only finitely many non-zero elements in the corresponding tuple that I am denoting K power round bracket I, this is a subset here in the notation this is all though the f from I to K such that f of i it could 0 for almost all i in i and this is this is proper, proper subset if I is not finding. So, this direct this I-fold direct sum of K and this is I-fold direct product and of course, when K is finite both these equal I finite then K power I round bracket I equal to K power I this you will see that this notation is very very convenient.

So, I want to make on the remark 1 remark I want to make this is little bit more general.

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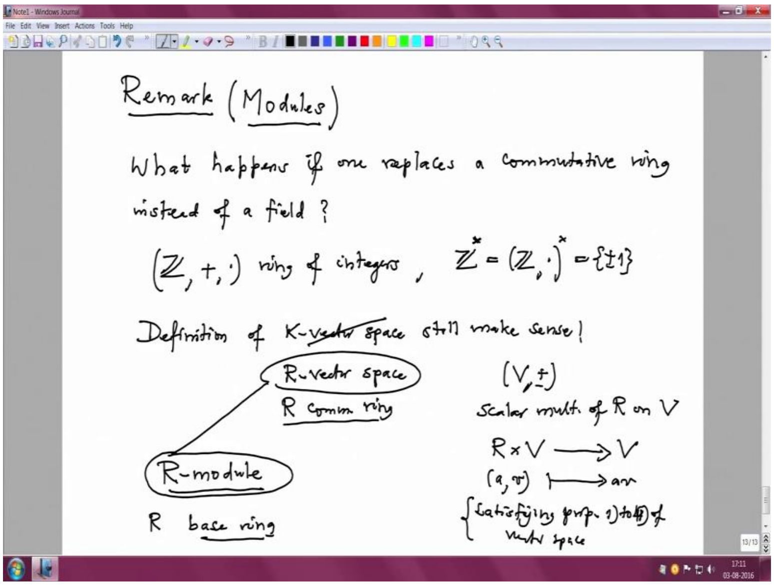

So, it is a concept of modules we will not deal much about modules in this course, but it will be very convenient to know what it is; because and this is a right place because all that remember that to different vector spaces we need a scalar field. So, one might ask what happens if the scalars are not field they do not form a field, but for example, they are only integers more generally what happens what happens if one replaces a commutative ring instead of a field.

Well, obviously, the study will become more complicated because we know in a commutative ring all non-zero elements did not have their inverses. So, in particular when you want to study linear equations we cannot simplify by using the by cancelling the elements dividing by them and so on. So, the study will become more a strictly even for even if you take so called ring of integers, because this ring the only invertible elements when i write cross your in veritable elements with respect to the multiplication there are only two of them plus minus 1. So, if your coefficients are not one and bigger numbers, then you cannot cancel them, and then the study will become more complicated but the definition still make sense.

So, definition of K-vector space still makes sense. So, when I say this is the K means no K is not trivial, but R-vector space let us. So, R is any competitive ring. So, what is an R-vector space is an Abelian group V plus we thus scalar multiplication of R on V. So, that is a map from R cross V to V, I will use the same letter a comma V with a V and satisfying properties 1 to 4 of vector spaces. Saying these simply means that the ring structure Abelian structure on V and the scalar multiplication on V all these three are compatible with each other, it because in the definition if you note whenever use the fact that K is a field we have only use a fact that it is a ring.

So, but to distinguish the two concept one calls instead of R-vector space to distinguish or to remember that our base or the scalar scalars they need not form a field, but they form only a commutative ring. So, that will be noted by saying R-module. So, I am instead of using the term R-vector space 1 usually use the term R-module where R is any commutative ring and in this case one also calls R is a base ring or even in the case of vector spaces 1 call that K as a either a scalar field or also called a base field. So, this study of modules is; obviously, much more complicated then the study vector spaces, and we will not be ready to give even for the simplest commutative ring Z ring of a d s the study is not as a simple as that of a vector space.

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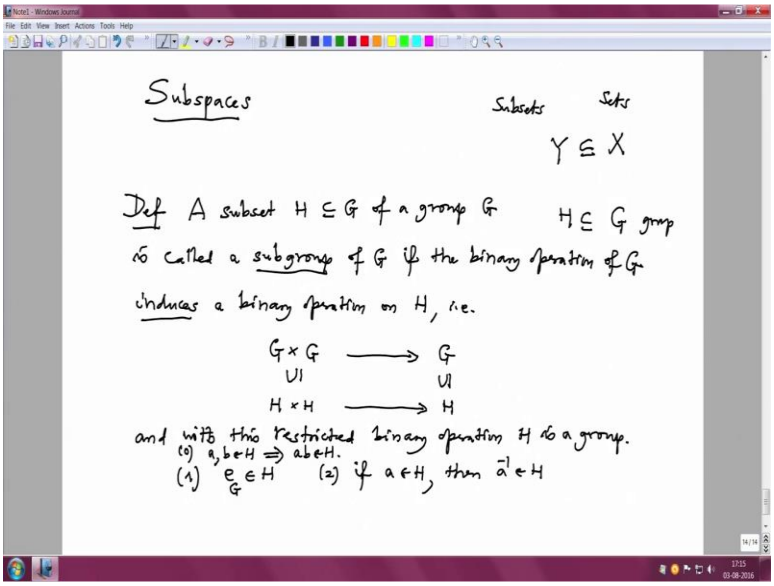

So, I just wanted to make this remark and go on. So, the next one I want to introduce they can set up the sub spaces which is so far we have sets and subsets, if X is a set Y subset just mean that every element of Y in every element of X and there is no other structure on the set that we are considering. So, for example, when we say group G is a group and then subset H when do you call this H has a sub group of G when the same operation of G when we restrict to this H with that operation H should become a group in which own right. So, let us record this formally. So, definition subset H of a group G is called sub group of G, if the binary operation of G should induces binary operation on the subset H.

That simply means see here you have a binary operation it is a map from G cross G to G, these map we can always restrict to the subset H cross H, because H cross H is a subset here, but it is image should go inside H then only it binary operation induces binary operation on H and with this binary operation this restriction by a restricted binary operation H is a sub is a group; that means, it should have identity, it should every element the first of all the binary operation is associative is obvious because it is restriction, it should have identity, but it is the obvious that identity of G will also survey as the identity in H, and inverses should also belong to H.

So, how do we check some bodies is sub group that we have to check that if you have an element in H the inverse of that element, which is a priori in element in G should also lie in H then only an identity should belong. So, the two thing we need to check I will just note it identity of G, e G should belong to H and two if a belong to H, then a inverse should belong to H and; obviously, I have the before that I already we have assumed that I will write it as a 0 which says if a and b are two elements in H, then a times b should be an element in H again this 0 condition means the binary operation induces on H. So, may be this is a good time stop we continue next time.

Thanks.