Linear Algebra
Prof. Dilip P Patil
Department of Mathematics
Indian Institute of Science, Bangalore

Lecture - 40
Change of bases

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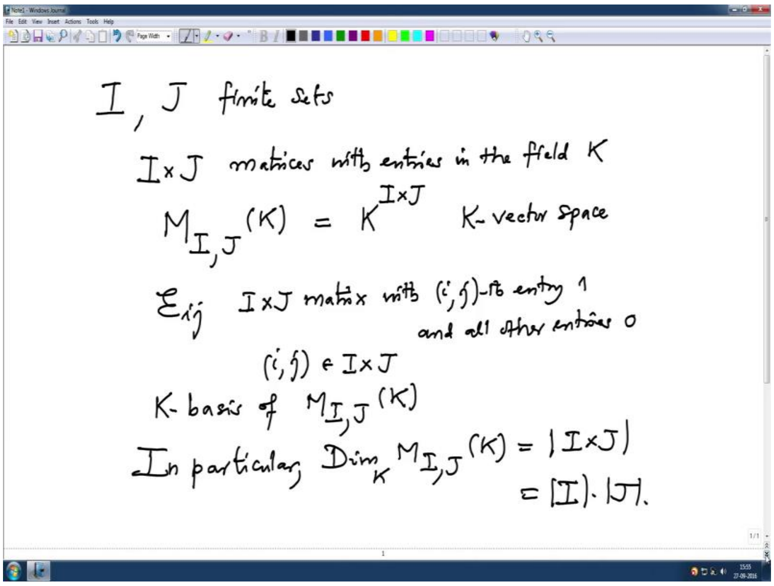

So, let us recall what we did in the last lecture quickly. So, for finite to finite sets I and J, we have studying I cross J matrices with entries in the field K and this set we have denoted by M I, J K we have noted this is this is actually by definition it is I power I cross J this is a K vector space and we found a basis of this the matrix units E i j where this is at I jth plane this is a matrix I cross J matrix with i jth entry one and all other integers are 0, and this is i comma j running in I cross J and this clearly a basis K basis of this vector space M I, J K in particular dimension is dimension of this vector space is cardinality of I cross J which is cardinality I times cardinality J very special case.

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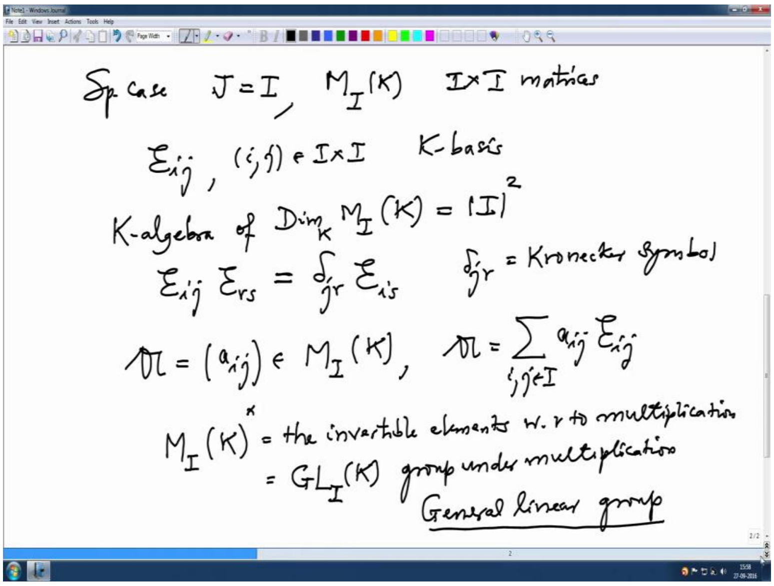

Special case when I equal to J equal to I, then M I K the square matrices I cross I matrices entry is always in the field K, this has a basis E i j you know i j both are varying in I cross I is a K basis and we have noted that this is actually K algebra this is a K algebra. So, we can also multiply this basis elements E i j, E r s this is very important if for calculation this formula this is E i s, I made a space here E i s delta j r where delta j r is kronecker symbol and E i s is a another matrix. So, this is another matrix.

So, this is very helpful for the calculation because any matrix say square matrix for example, a i j in M I K we can write it as a equal to summation entry a i j tens matrix units i j running in i. So, this are the coefficients. So, there are uniquely determined because of this basis and so on and when you multiply another matrix we just have to perform these operations. So, it will be easier for a calculation. So, also now we have seen this is a K algebra and the dimension of this K algebra is mod I square cardinality I square and we have also seen last time invertible elements in this. So, M I K cross these are the invertible elements, elements with respect to multiplication and this set precisely denoted by GL I K this is a group under multiplication of matrices and it is called general linear group, actually this name is better if you look at it as endomorphism because that is a linear map. So, they this what linear comes from there.

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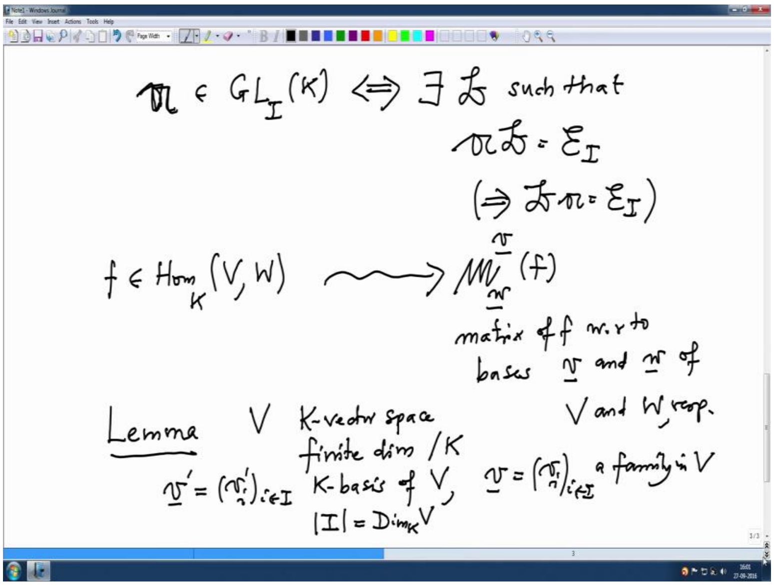

So, we have seen last time that if you want to check the matrix is invertible a matrix a belongs to GL I K, if and only if you only have to check one equation they exist a matrix b such that a times b is equal to identity matrix, then it will automatically imply b times a is also identity matrix this we saw last time because for a finite dimension vector space the map is linear map is injective if and only if surjective, if and only if bijective. So, called pigeonhole principle for linear algebra.

Now we are going to continue this and we I want to analyze actually now you remember for a any f in the Hom k V, W we have attached the matrix m v w f and this matrix of f with respect to the basis v and w of v and w respectively. I want to now analyze how what happens if I change a basis what happen to this matrix, this matrix depends on a basis. So, I want to analyze what happen to when we change a basis. So, I am going to do it first for a square matrices.

So, let us see first I want to write lemma. So, V is a vector space finite dimension K vector space, finite dimensional over K and I have given the basis V prime v i prime i in I this is a K basis of V, and another family index by the same indexing sets that is v, v i, i in I is not a basis of family in V indexed by the same. So, they have the same number of elements. So, cardinality of I is the basis, cardinality of I is the dimension then what is the statement.

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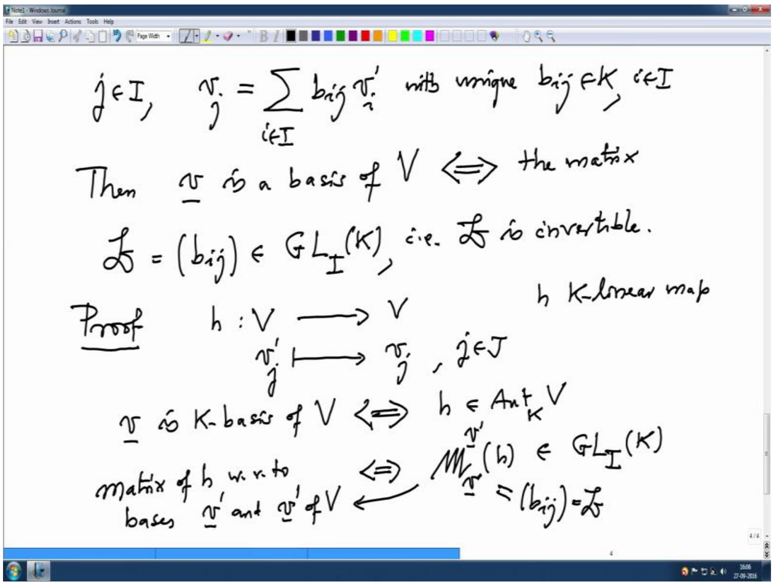

And I am looking at for every j in I this jth member in the family v j this I can certainly write it as a linear combination that I am calling it b i j v f i, i in I with unique b i j in k i in I this is because this v i prime is a basis; then this v is a basis of V if and only if the matrix b which is v i j this matrix is invertible; that means, it is GL I K that is b is invertible.

So, again this is not so difficult, let me indicate the proof look at the map from V to V if I want to define a linear map from V to v i only have to give it a value on a basis elements. So, v j prime goes to v j j in J, this defines a linear map let us call it h h is linear. And we have seen earlier v is a basis the image is a K basis of V, if and only if h is an automorphism this is we have seen earlier a linear map is an automorphism if and only if it map basis to a basis, but this means the matrix, matrix of v prime with respect to v prime to v of h this is invertible matrix GLIV GL I K, but what is this matrix? This matrix is precisely the b i j matrix which is b, that is because we have this equations the jth element is written in terms of this.

So, that is oh may be I have to little bit careful with the basis. So, this is this with respect to the basis V prime to V prime, because this goes to the jth column and that column we are writing again in terms of v i prime. So, this is a matrix. So, this is matrix of f a matrix of h with respect to basis v prime and v prime of V. So, that was the proof of the lemma.

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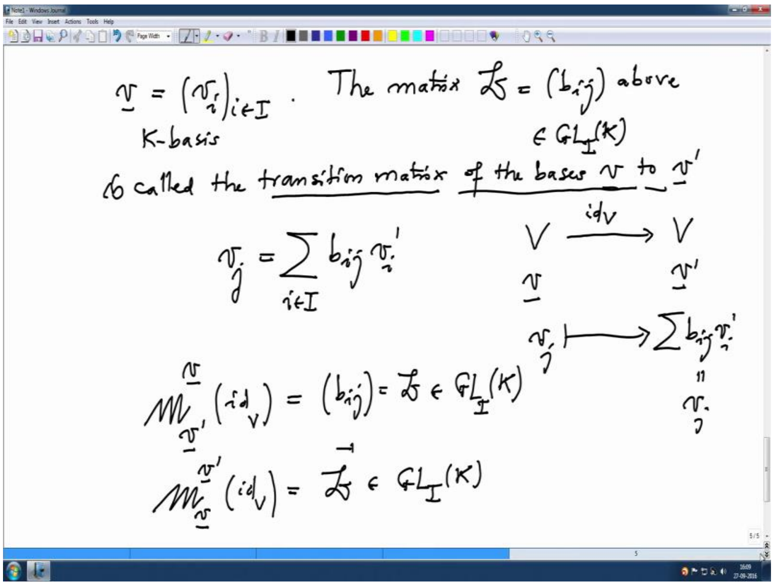

Now, let us continue with the same. So, now, I in the above situation we have two basis v is a basis, that is v i, i in I and the matrix b in the matrix b above b is b i j above which is in GLIK is called we are assuming now this a basis is called the transition matrix of the basis v to v prime.

So, remember b i j are defined by this equation, b i j are defined by the equation v j equal to summation b i j v i prime this is i in I. So, if is the basis, think of it is V here V here if the basis v if the basis, v prime and then this v j goes to this b i j v i prime, but this is v j. So, therefore, this is really the identity matrix. So, therefore, what we get is the matrix of with respect to the basis v and v prime of the identity matrix identity linear map of e is this b i j matrix; which is we have called it b and this is invertible this matrix is called a transition matrix of the basis v to v prime. So, and what if the other way then; now the matrix of the basis v prime to v i d v that will be the nothing, but the inverse b inverse which is given in G L I, because we see when you compose this you get the identity, but its v to v. So, it is clear it is the identity the inverse of this matrix.

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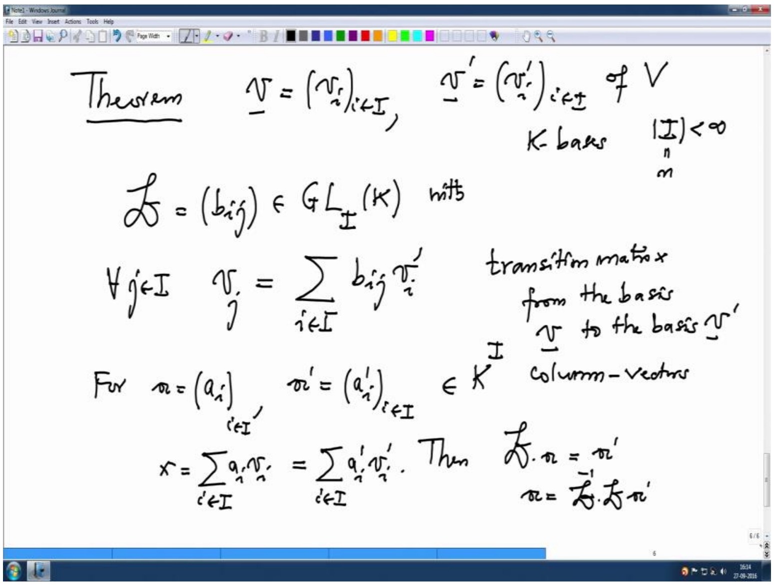

So next now we do it more. So, now, again let us do it. So, let me write it as a theorem, now let me restate everything again. So, v is a basis v i, v prime is a basis v i prime of V, K basis and of course, I we are assuming finite. So, this is m then it a n dimensional and we have a matrix b, b is given this invertible matrix is given with the jth vector here v j is a combination b i j v i prime i in I, this is given for every j in j every j in i, this is the transition matrix may be it is better to say from the basis v to the basis v prime, see it goes v get transfer of v s into v primes ok.

So, if you now put the column vectors I have to be careful with the index small a, this is a i and small a prime, which is a i prime this is a column vector think of this is in K power I columns column vectors for the column vectors; let us look at vector x see these are the coordinate these are the coordinate tuples. So, I wanted to write the vector corresponding to this that is summation a i v i and is of course, i in I and column vector if it is also written in terms of this is now in terms of the basis v i, v i prime this is summation a i prime v i prime. So, the same vector we have represented in terms of this basis v and in terms of basis v i prime then what statement is if I take the matrix b and multiply this column vector a you get a prime and therefore, and also how do I compute this a back from b and a prime this a equal to you multiply this equation by b inverse on both side. So, you will get b inverse b a prime; this is just easy computation when you multiply you just have to compute this, and then you will substitute let me write the proof one line proof.

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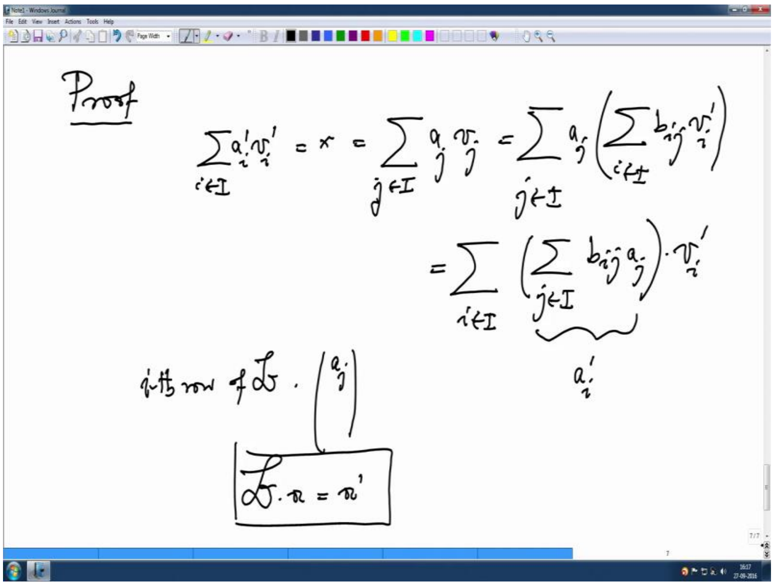

So, proof. So, start with x on one side it is a i prime v i prime, on the other side it is summation a i v i. But this is same has now I am going to substitute this v i v i equal to remember I just want to remind you what I am going to substitute I am going to substitute v j equal to this, this for each v i. So, that is summation i in I, a i and v i is substituted now I have to use index j. So, j in I, j in I and this v i is b i j v i prime I going to change the indexes I and j or maybe we do not do that.

So, let me erase this thing and we do it. So, we do the exactly what we have done. So, for that I have to change this index. So, this is J in I, a j, v j; v j I am substituting that will be summation j in I a j as it is and then v j's are summation i in I, b i j v i prime, but this is same the finite sum. So, I can inter change. So, this is I can do it summation i in I, i inter change the sum and put this. So, inside it will be summation j in I, b i j a j and whole thing multiply by v i prime. So, therefore, this vector is a i this coefficient of v i prime is a prime this. So, that is precisely the row ith row of the matrix b, ith row of b multiply by the column this column a a j. So, you will get this sum that is precisely ith entry so; that means, this matrix capital matrix b times the column vector a equal to a prime this is what we have to prove; and once you prove this b is invertible. So, you can multiply this equation by b inverse from this side and get what we want.

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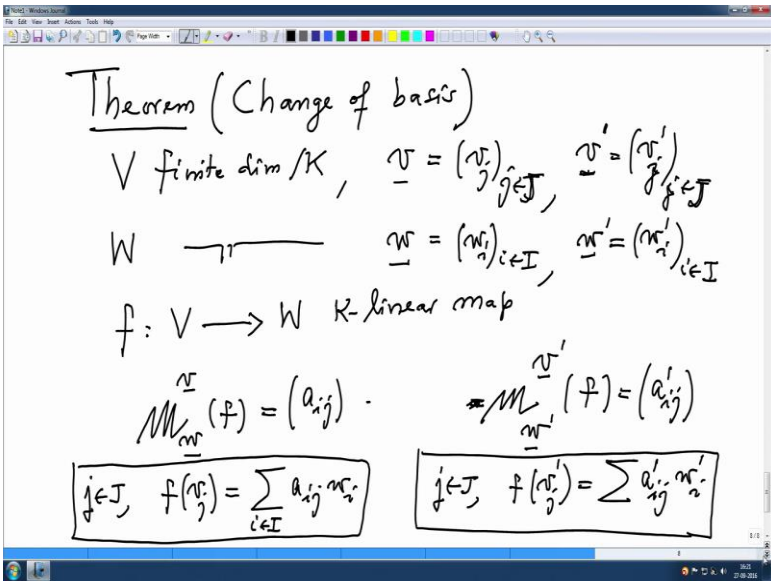

So, that was the lemma now I want to write the full force theorem that is change of basis change of basis; V finite dimensional over K, and given two basis one is v, v j and another basis v prime which is v prime i, i is in I. Now see if the same index in set, but I am using the different indexes just to distinguish and W again finite dimension K vector space, w is one basis which is denoted by no this also j w is denoted by w i, i in I and another basis w prime which is w prime i this is this is i in I; now this one is this one should be j, this one also should be j, this is j and this is j. So, then what do you what do you want to prove and f is a linear map given to us f is a linear map, V to W; K linear map then what we wanted to find out the relation between the following two matrices one is; obviously, m v w f the other is I use. So, I use this basis these two basis this is matrix of x this equal to these two basis.

Now, matrix of f with respect to these two basis, now I wanted to write down the relation between the two and just to remind you this matrix is define by the equations for j in J, f of v j equal to summation a i j w i, i in I this is this equations are the defining equation for this matrix which is matrix a i j; and here jth vector here. So, j in J, f of v j prime is summation a prime i j, w prime i, this equations will give this matrix this is a prime i j matrix; both are the matrices of the same this is I cross I matrix this is also I cross no this is I cross J matrix, this is also I cross J matrix and relation between them is the following.

So, this is equal to no not this, this is equal to this matrix. So, I have to write the next page I will write ok.

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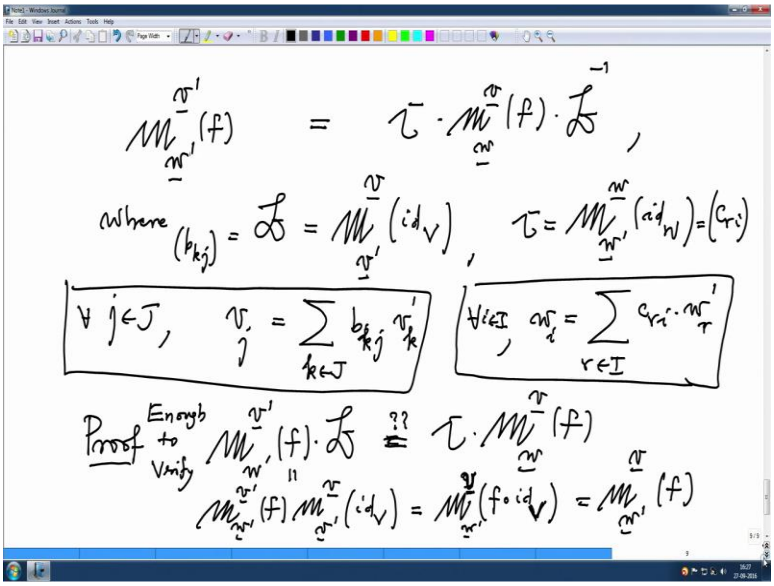

The relation is this mu v prime w prime f this matrix v w f this is a matrix c times this times b inverse this is the relation, where b matrix, b is the matrix of the identity m id v from the basis v to v prime, and c if the matrix of transition matrix of the basis w to w prime and. So, in particular this such matrix is a called similar so we will define it formally after this, but again when you write a matrix and you want to check this equalities it is very convenient to note what are the equations here like I wrote here these equations; like that what are the equation for b and c for b its identity.

So, this guy v j for each j in J, v j we have written in terms of v prime because this is identity v j goes to v j. So, this is equal to b now what do I call it b, b i j, b k, b k j v prime k, k is varying in j. So, we have written vectors in the jth vector as a combination of this and this will be the jth column of b jth column of b. Similarly c is are defined by the equations w i vector we have written this is not f this is id this is id w, w i for each i in I, w I is c r i; r is in I hence w r prime this is the equation which will tell you this matrix is c r i and these are the equations which will tell you this matrix is b k j and we want to check this.

So, to check this both c and b are invertible matrixes. So, b inverse make sense. So, I can multiply this on this side and check. So, it is enough to check that. So, proof enough to check that this multiplied by b on the right. So, m v prime not v prime this is w prime; v prime w prime f times b, equal to c times m v w f we have to check this, but what is this? So, you have to check this,, enough to verify this equality this equality, but I am going to substitute what is v v is b is this. So, this is m v prime,, w prime f and b is I am just substituting this m v prime i d v that is this; I we have to check this equality right. So, I am going to start from this side this is I am just substituted b by definition, and you remember this is the same basis. So, this is same as m, this is identity of composition, this is f compose i, d v and now this will be from v prime v prime get cancelled and this will be from v to this get cancelled this one v prime to w prime, that is how we proved for the composition, but this is m as it is v w prime this composition f only; what is the other side is one side, but what is the other side that is this side.

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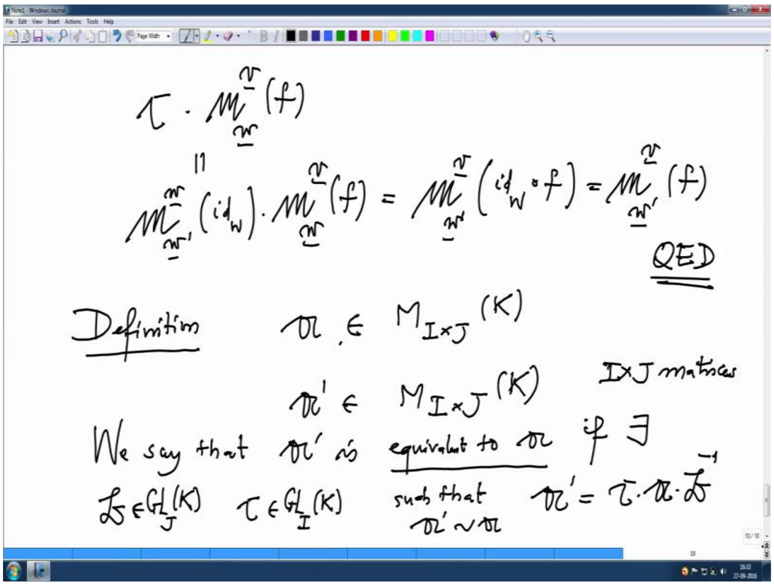

Let us write down right side also that is C, times m v w f, but this is c is the matrix of c is the identity matrix of c is the matrix of the identity map with respect to w and w prime. So, that is w, w prime id w m v w f this is same as m; obviously, this will be i d v on the left composed w and then same [FL] you cancel this w and write from v to w prime, which is m v w prime f, but this is what the other side is also just now I noted see the other side is also this. So, this side is also this. So, we proved both sides are equal. So, this proves.

So, QED its little computational one has to do that. So, more important is we have noted what happens to the matrices when you change the basis, and that allows us to define in general this definition. We have two matrixes a and b, oh! a is in M I cross J, b is also in not b I do not want to use the b, the matrix a prime a prime that is also in the two matrices of the same order, I cross J matrices; we say that a and a prime are similar or we can be little bit more precise we can we say that a prime is similar to a, if there exist I will write their order little later b and c two matrixes such that a prime equal to c a b inverse; actually it is not good to use the word similar, but equivalent let us say because similarity will be reserved for the square matrices.

So, let me just say equivalent. So, that and now I can write down what are the orders. So, this product to be defined you need columns of a and the rows of b have the same index in set. So, b should be from m this square matrix and it should be invertible. So, b j k and similarly c should be in M I K, actually invertible. So, it is even better. So, GLJ and GLI with this now immediately after that I will say this also sometimes related by a equivalent to this b e a equivalent to a prime equivalent to a, immediately after the when we come back after the break we will check that this is an equivalents relation.