Linear Algebra
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Lecture – 41
Computational Rules for Matrices

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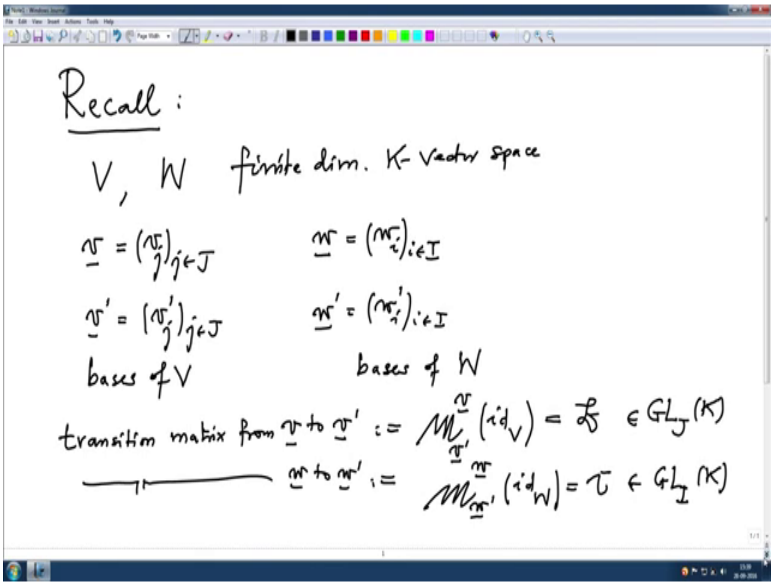

So, let us recall from the last lecture what we did in last time I want to recall quickly. So, recall what we did was we try to analyze what happens to the matrix of a linear map when you change a basis. So, let me write down the exact situation V, W finite dimensional vector spaces and we have V is a basis of V numbered by a finite set J, there is another basis V prime numbered by the same index set J and also there is a basis W of W i basis of W, and W prime which is a basis of W there are two. So, these are basis of V now these are basis of W, and then each one each pair will give us a transition matrix. So, transition matrix from V to V prime this is the matrix V denoted by M V from V is a top index to V prime of the identity map of V and these matrix are denoted by b.

Similarly, the transition matrix from a basis W to W prime, this is by definition to the matrix from W to W prime of the identity matrix of identity map of W, this let us call it C and both these matrices are invertible matrices. So, this is in GLGK and this is GLIK this we saw last time.

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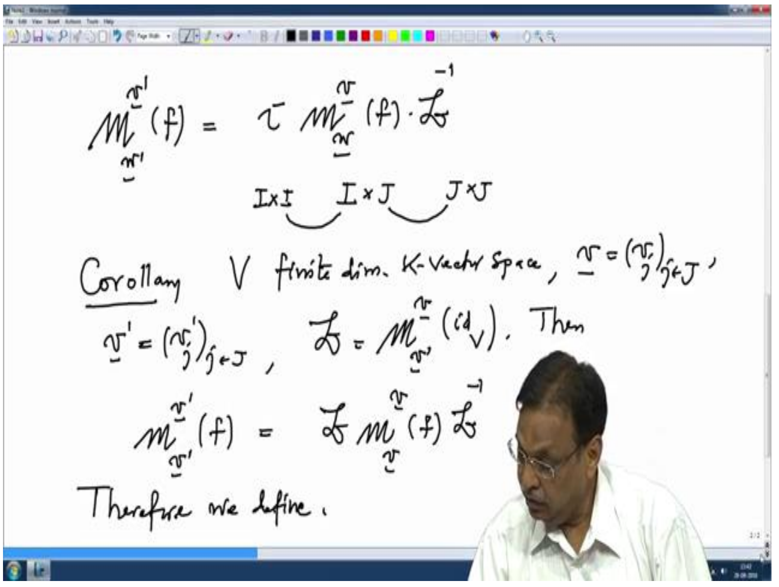

And now I want to write down the relation between m m v w f and m v prime, w prime f. So, this is equal to this is I cross J matrix. So, this is b inverse here and c here, remember this is I is I, c is I cross matrix, and b is J cross J matrix. So, inverse is also j cross j matrix and this makes sense because this columns is rows and this columns is rows and this equality this is what we have check the equality; and equality was very easy to check because there I just put C equal to identity and then this was very easy to check.

So before I go on to the definition formal definition of the similar matrices let me write one corollary. Now V is finite dimensional vector space K vector space, and we have basis one basis is V numbered by V j j and J, another is V prime which is V j prime j and J and as usual b is a transition matrix; b is matrix of V to V prime of identity matrix then this is a transition matrix then the matrix of f with respect to a basis V prime to V prime f this is b times m v v f b inverse, just put in this c equal to b and b equal to w, and w equal to v, basis w equal to v and basis w prime equal to basis v prime and this is just a special case.

So, this allows us to give a following definition therefore, we define these matrices are similar; that means. So, we define last time in the last lecture in the last few minutes I had goofed up little bit. So, I wanted to correct it.

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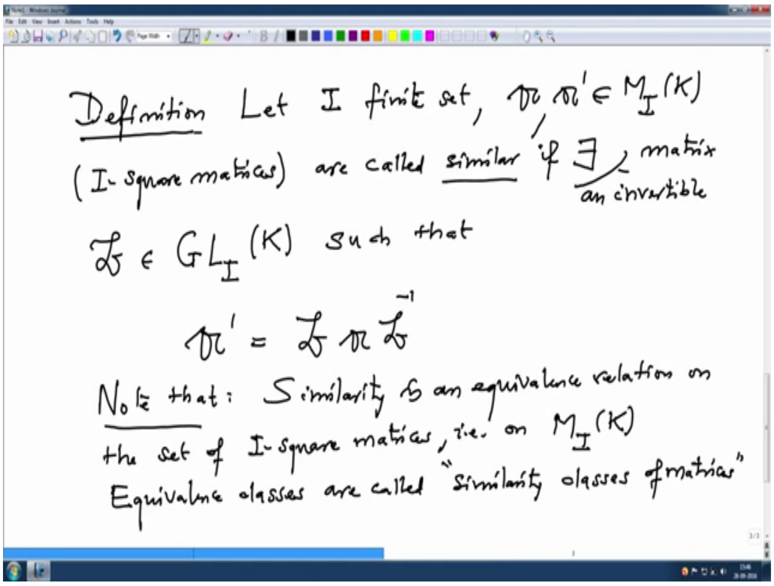

So, a definition now I will write the definition purely in terms of matrices, always I will follow the philosophy whenever I want to define anything about the matrices first I will look into the linear map, and their matrices with respect to the basis and from there I get a clue what to define or what is worth to define. So, let I be a finite set and then we are looking at two square matrices a and a prime belong to m I K these are square matrices, these are also called I square matrices. Two of them then are called similar if there exist matrix b which is invertible there exist allow me to change here, there exist an invertible matrix b in therefore, it should be of the same order and it is in GLIK, mean by this is a group under matrix multiplication such that a prime equal to b a, b prime then we call a and a prime to be similar.

Now, first important thing to note is the following, first important thing to note is similarity. So, note that similarity is an equivalence relation on the set of I square matrices, that is on MIK and therefore, the set of matrices we will broken into the equivalence classes, and this equivalence classes I will keep calling similarity classes. So, equivalence classes are called similarity classes of matrices, this is it is really easy to prove it is similarity then equivalence relation because clearly it is reflexive and symmetric etcetera that is because the GLI is the group.

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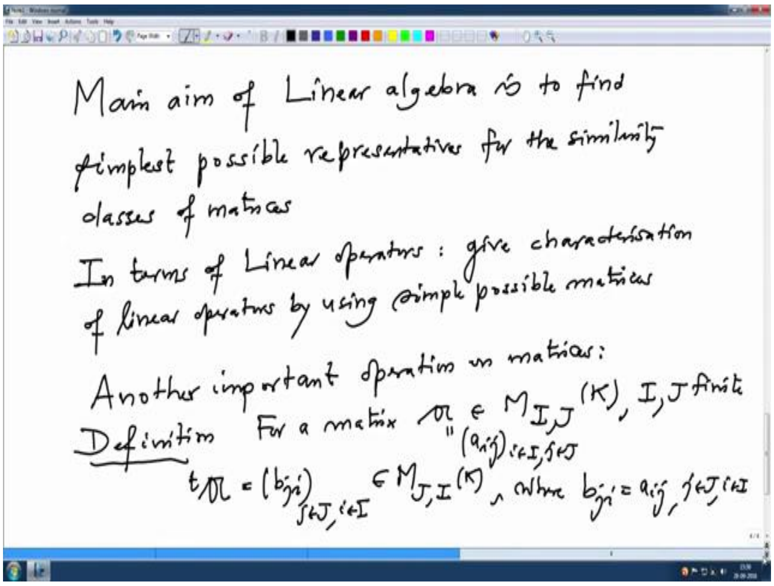

And what is the main aim? Main aim in the linear algebra, main aim of linear algebra is to give or to find simply as possible representatives for the similarity classes of matrices that is a in terms of linear operators now in terms of linear operators give a characterization of linear operators by using simply as possible matrix matrices. So, each similarity the similarity class if you know one matrix then we know all the matrices, and you want to choose in each class the simple as possible that is the main of studying linear algebra and all these all the machinery is centered around studying this problem, and as we go on I will take, I will make more and more stronger remarks about this statement and this we will it in the next chapter, and how will you do it we are going to attach to a matrix. So, in a linear operator or polynomial and therefore, we are attaching the zero of that polynomial and therefore, we are attaching the multiplicities of that polynomial and whether these data is enough to characterize them that is a question ok.

Another important operation on the matrices another important operation on matrices is the following. So, let us write it as a definition for a matrix a in I cross j matrix MI, JK I J finite sets I j finite I define the matrix in the other order. So, transpose matrix transpose I write on the other side I do not want to on this side because you will confuse with the power. So, this is if this is a i j, i in I j in j I want the matrix which is which has the rows are numbered by j and columns are number by I. So, this is b j i where J in J, i in I this b j i is nothing, but a i, j in J (Refer Time: 14:24) this is called this.

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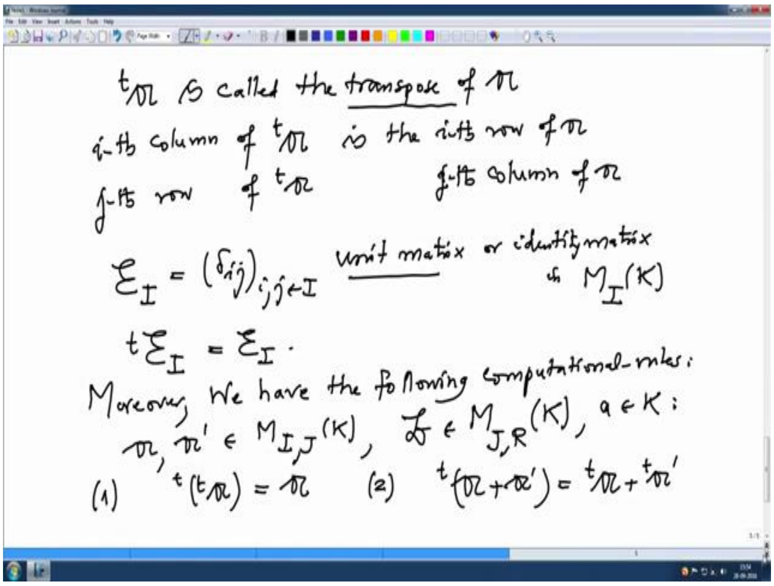

This matrix W matrix is called the transpose of n; now where is question I have cropped up what happens many invariants like rank (Refer Time: 15:00) and so on about the transpose. So, jth column of the transpose of the transpose ith column; ith column of the transpose is the ith row of original matrix and jth row of transpose (Refer Time: 15:43) the jth column of n rows and columns are interchanged.

So, it is basically like taking a reflection around the main diagram roughly speaking of. So, the unit matrix or identity matrix we have denoted by e suffix I, this is the matrix delta i j, i j are in I this is they are called unit matrix or the identity matrix, matrix in M I. We have seen that this M I K is a K algebra and this identity unit matrix is the identity in this K algebra with respect to the multiplication of the matrices, and the transpose of this unit matrix is itself. Moreover we have the following computational rules moreover we have the following computational rules. So, all these are now in this a and a prime are two matrices of the same order I comma J, and b is a matrix in the number the row numbers are given by the J and column numbers are indexed by the R and a is a scalar; then one first of all the transpose of a and again the transpose of this, we get back the original matrix.

Second if I add a and a prime addition is possible because they are of the same order, and then take the transpose that is same as transpose as a, plus transpose of a prime this means the addition of matrices respect the transpose.

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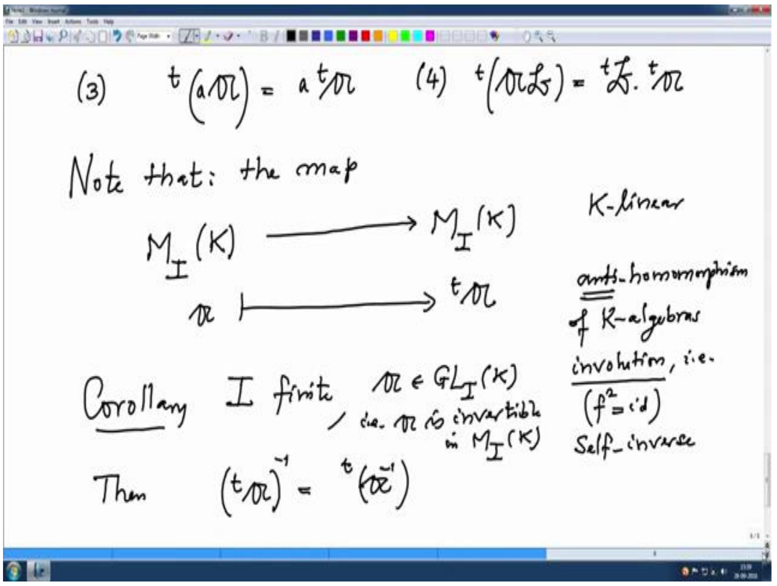

Third one if I multiply a matrix a by scalar a and then take the transpose, this means it is same as a times small a times the transpose, and the fourth one transpose of a times b note a and b are not the same order matrices. But the product makes change because of our assumption on b that b as the rows of b are numbered by the same as numbered by the columns of a, and this is same as first you take the transpose of b then multiply by transpose of ,a this product also makes sense because rows and columns get interchanged. So, these rules are very easy to verify I will not verify them, but I will note the following important thing note that the map from square matrices I cross I, to square matrices again this is a K algebra this is a K algebra and I am giving a map from this K algebra to that K algebra. So, that a going to transpose of.

So, it showed that this map is K linear first that is the property second and third and. So, if you call these as a i want to I do not want to give a name. So, this map is K linear and fourth says that it does not respect the multiplication, but it reverses. So, it is sometimes it is also called antihomomorphism antihomomorphism this anti refer refers to the term of multiplication operation antihomomorphism of K algebras, and more than that it is involution; that means, square of that is identity so; that means, it is self inverse. So, this means a map is called involution when its square is identity, so they itself inverse. So, that is also called self inverse. So, this has that property. So, sometimes it is useful lot of such antihomomorphisms come in actually study of cryptography h.

So that was about this, now the immediate corollary another observation from this rules are I want to write it is a corollary. So, corollary again I is a finite set and then we want to say something about this group. So, a is an element in the group GLIK; that means, a invertible matrix. So, that is a is invertible in M I K. It is an invertible element in the K algebra M I K, then on one hand you take the transpose first and then the inverse note that if a is invertible the transpose is also invertible that will follow from one of this rule you see if a is invertible there is a matrix b. So, that this a b is identity, and the transpose of that will be the other way so; that means, this inverse is nothing, but first you take the inverse and then take the transposition, these two operations are commutative. So, inverse and transpose they are commutative operations, they are the reverse this is again very simple to check from this rule. So, I will not try to formal proof.

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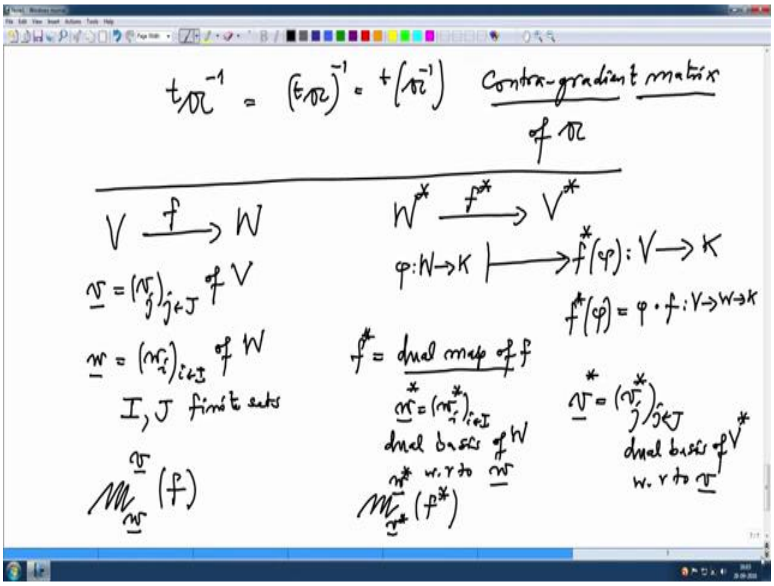

So sometimes this matrix this transpose a transpose inverse, so I would have without loss I will denote this without putting a bracket, because whichever I would do it first it is the same. So, this is same as either you take it transpose a inverse or you take it transpose of a inverse, this we have checked it its same. So, this is this matrix is called contra gradient matrix contra gradient matrix of a. This is this can safely defend only for invertible matrices and this is very useful in physics, when I do a geometric part that time I will show you the use of this contra gradient matrix.

Now the next I want to analyze if you have a linear map. So, next problem I want to analyze is if you have a linear map from V to W f, then we know the dual if W star to V star is a dual map f star recall how did we define dual map; that means, we should move where the form on phi (Refer Time: 24:54 W; that means, it is a map from K linear map from W to K; where does it get the associated under f star. So, f star of phi should be a linear form on v. So, that should be it should be mapped from V to K and which map f star V f star of phi is f composed with phi it makes sense first you follow V to W that is f and then W to K by phi. So, this is a definition of the star map this is called a dual map of f star f of f denoted by f star, f star is called a dual map.

So, now, the problem is we know we have we know that if I have given a basis v, v j of basis of V and basis w, w i, i in I of W we are assuming both finite dimensional. So, I j finite sets and then this basis will give a dual basis. So, we will give a dual basis V star n which is V j star j in J this is a dual basis of V V star with respect to the basis [vocalized-noise], and similarly I will get a W star this is W star these are the coordinate functions of the basis. So, this is a basis of dual basis of v with respect to the basis w; and now on one hand we have a matrix of f respect to this basis. So, m v w f, on the other hand we have this matrix m now we have to be careful upper we have to write w star, v star of f star and; obviously, we are asking the relation between the two right see the strategy when you write these indices in this notation matrix of f, the top one coming from where the map is defined and down one is where the map is going into. So, the relation is one is a transpose of the other.

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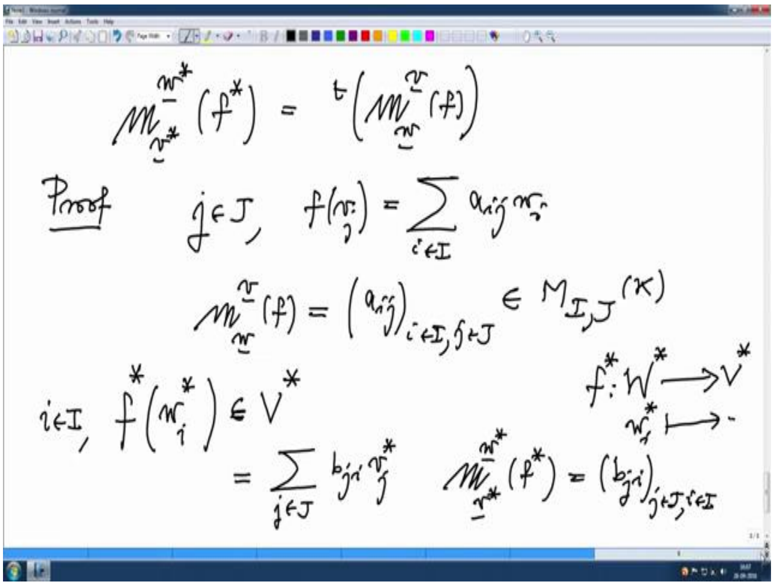

So, this m w star v star f star, this is nothing, but the transpose of the matrix m v w f.

Proof is very easy to check let us write down the proof, remember the matrices this matrix of f with respect to the basis is defined by this equations j in J, f of v j equal to summation a i j w i, i is in I and these coefficients define the matrix m v w f is by definition then the matrix a i j, i in I, j in J, this is M I K j. So, if you want to get the description for this matrix or ith entry of this matrix then we have to evaluate f star on the dual basis. So, we need to find a formula for f star of now the orders are interchanging rows have becomes columns. So, f star of w i star you have to find out this. So, for given i in I we want to find this formula in terms of v j star and those coefficients will give the g I jth entry of g ith entry of this matrix.

So, this is now I these are what vary this element this element is a linear form this is V star because f star is a map linear map from W star to V star and W i star are the coordinate functions here. So, when I go under f star it will be somebody in V star. So, therefore, I will have to evaluate to understand this, I will have to evaluate or when you directly write this I have to write this as a coefficients this I want to write in terms of the basis j in J, b j I V j star and the matrix with this equations the matrix of f star with respect to this, this required matrix is nothing, but b j i, j in J, i in I and we need to verify now what we need to verify that b j i is a i j. So, we need to check b j i equal to a i j for all i in I and j in J.

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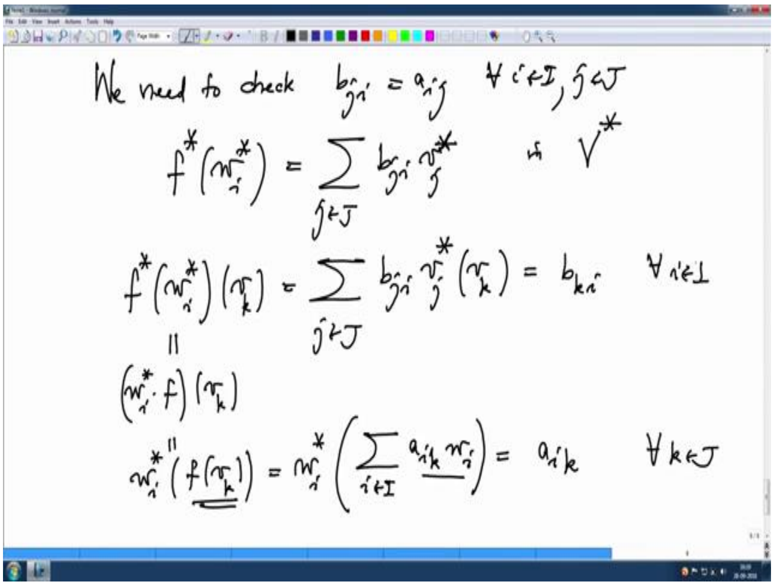

So, let us let me write that equation again we have written this equation and we want to calculate b j i both are linear forms on V this equation is in V star. So, I will evaluate to check the equality means they are equal on every element of the basis of v. So, I evaluate these on V j f star of w i star evaluated at V j, this should be also this other side j and j b j i, V j star I have evaluated an these whole thing I should have taken the some other index, let me write the index write the index k somewhere; this is k and this is evaluated at v k. So, the only term which will survive in this sum is j equal to K term they are the coordinate functions.

So, j equal to K term will survive; that means, this is b k i, on this side on the other side what is the definition of this that is the definition of this is a composition right this is a composition w i star f evaluated at v k, which is w i star of f of v k, but this is now we have written f of v k we have written this is w i star of v k we have written as summation i in I a i j no v k. So, a i k w i this is this I have written yes this is now when I put this again. So, only term which will survive is I term. So, this will survive is a i k. So, therefore, we have proved b k i is equal to a i k this is true for every k in j and for every i in I. So, that check this condition. So, therefore, we have checked that the two matrices are equal; we will take a short break and then come back.