Linear Algebra

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Lecture - 42
Rank of a matrix

So, in this course on linear algebra we have been studying matrices now and today's lecture I want to start studying rank of matrices and in particular prove that row rank and column rank are the same.

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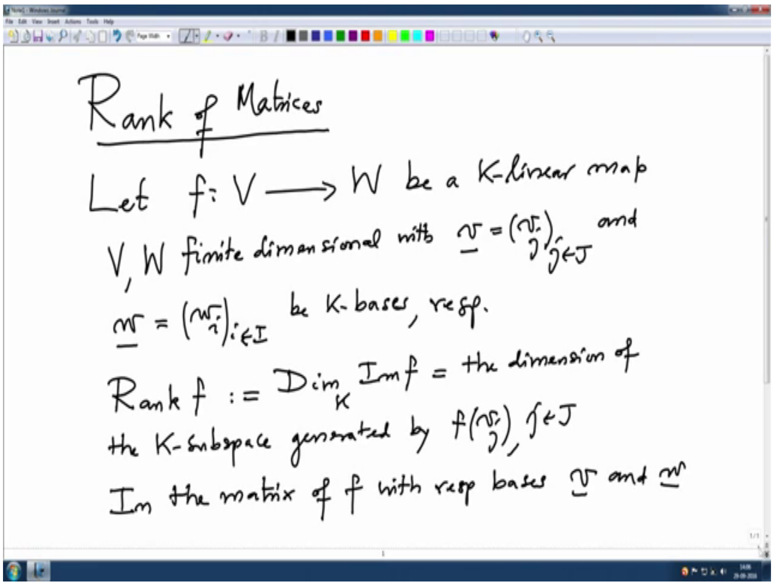

So, let us start building of the things. So, rank of matrices as I said earlier also all the concepts about matrices I want to get motivated from the linear maps and then define accordingly.

So, let us start with a linear map let f from V to W be a K linear map and we assume V and W are finite dimensional vector spaces; V W finite dimensional and we also choose the basis v of v i number it by the finite shade J and w of w; w I number by finite shade I be K basis of V and W respectively. No, we should not calculate this. Then as we know when we studied linear maps we have defined rank of linear map f, this worth by definition dimensional of the image space of f and this is nothing but the dimension of the subspace generated by the images of the basis of v. So, that is generated by f of v j J in J, but you when we defined the matrix that is the; this image of v j under f was precisely we have written in the j th column. So, for that let me also; this in the matrix of f with respect to basis v and w.

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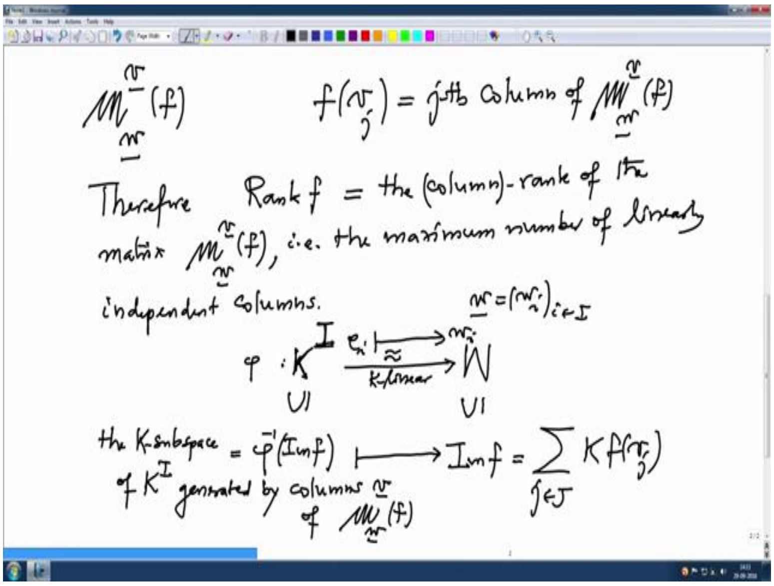

So, that is; and the notation we have given for this matrix was M v w f, this f j f of v j is the j th column of this matrix M v w f therefore, what we have defined therefore, the rank of f equal to the column rank, let us say the column I will put in the bracket, the rank of f; rank of the matrix M v w f. This is the number of linearly maximum number of; maximum number of linearly independent columns. So, I would define these bijectly formulae, but before I define that I want to also set up little bit notation. So, note, here w has a basis W which is numbered by w i and image of f is a subset here; subspace and this is generated by the images of the basis v js v images of the basis v which is v j. So, this is K f v j J in J and we are interested in computing the dimension of this.

So, this basis of W will give you a coordinate system into the using this coordinated vector space K power I and this map is precisely which gives these coordinates that is e i and this map is e I, W i and because a basis map. So, basis this is an isomorphism and this when you write the matrix of f with respect to the basis V and W then what is the image of; if you call this map as f in the inverse not f; f is already used. So, I have to use another letter this is phi this when I take the inverse image of this image here which is a subspace here that c inverse of image f this is precisely the subspace the K subspace of K power I generated by columns of the matrix M v w f this is very easy to see because that is how we have given the coordinates.

So, this therefore, if I want to compute the rank of a linear map, I will have to choose a suitable basis of V, suitable basis of W and write down the matrix and that matrix should from that matrix it will it should be easier to check the rank of the matrix.

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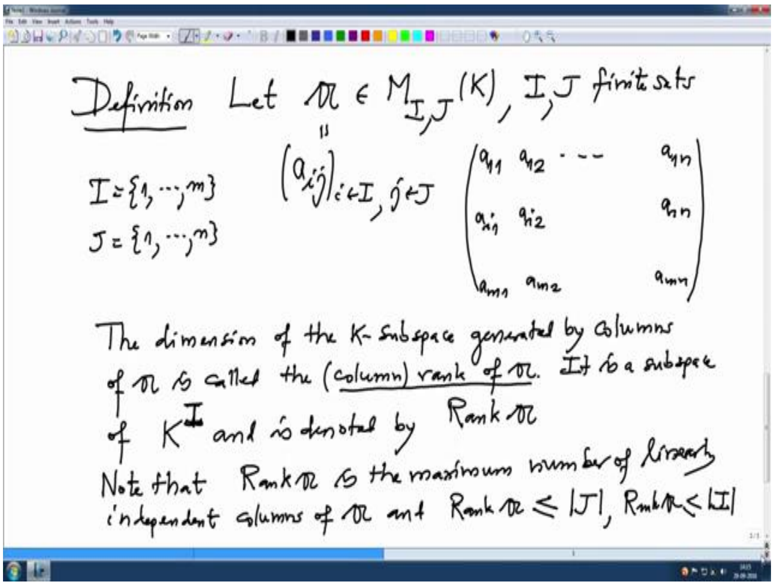

So, definition formal definition, now I will define in terms of a matrix directly. So, definition let A be a matrix we chose a numbered by I and columns are number by K and assume, I J are finite sets. So, a is a i j i in I j in J and we have written this a as like this. So, in case of standard numbering system this is the first row it is a 1, a 1, 2, a 1 n i th row is a i 1, a i 2, a i n and the last row is m a m 1, a m 2, a m n and the columns are the images of the basis. Now in even i is 1 to n j is 1 to n, no, I is 1 to m, j is 1 to n then the dimension of the K subspace generated by columns of A is called; I will just call it rank, but it is defined by using columns. So, in the bracket I write column rank of A and it is a subspace of whom it is a subspace of K power I therefore, rank and I denoted by I is denoted by rank I.

So, first thing to note is note that rank of A is the maximum number of linearly independent columns of A and these total number of columns is n which is cardinality J. So, and rank of A is bounded by cardinality j and also rank the columns is a subspace of

K power I therefore, the subspace cannot have dimension more than cardinality. So, rank of A is bounded by cardinality I.

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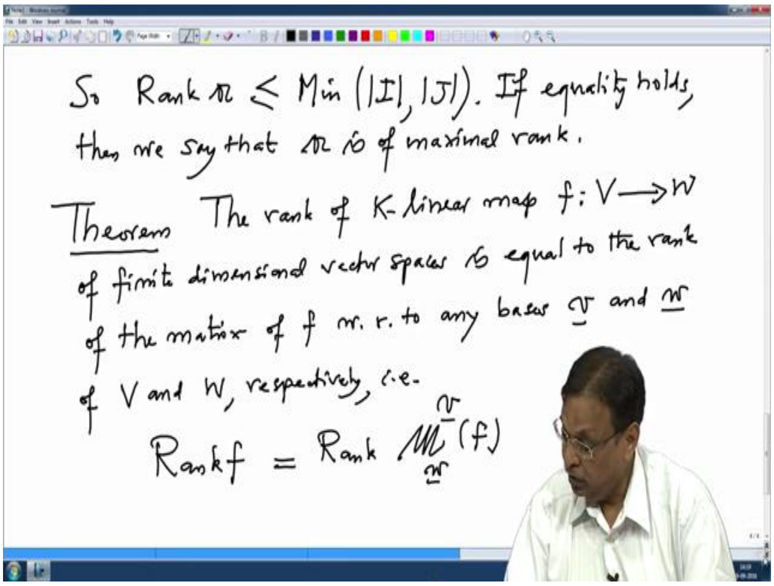

So, rank of A is less equal to min cardinality I coma cardinality J minimum and if equality holds, here if equality holds then we say that a is of maximal rank. So, just remark before this definition we note that the rank of linear map is nothing, but the rank of the matrix associated to that for some basis for any basis so. In fact, I want to note this, this is very important.

So, I let us note it as a theorem. So, the rank of K linear map f from V to W of finite dimension vector spaces K vector spaces is equal to the rank of the matrix of f with respect to any basis v and w of V and W respectively . So, that is in the symbols rank of f equal to rank of the matrix. So, like you have defined rank of a matrix is strictly defined by using the columns similarly one can also define rank of a matrix by using rows and that should be called row rank.

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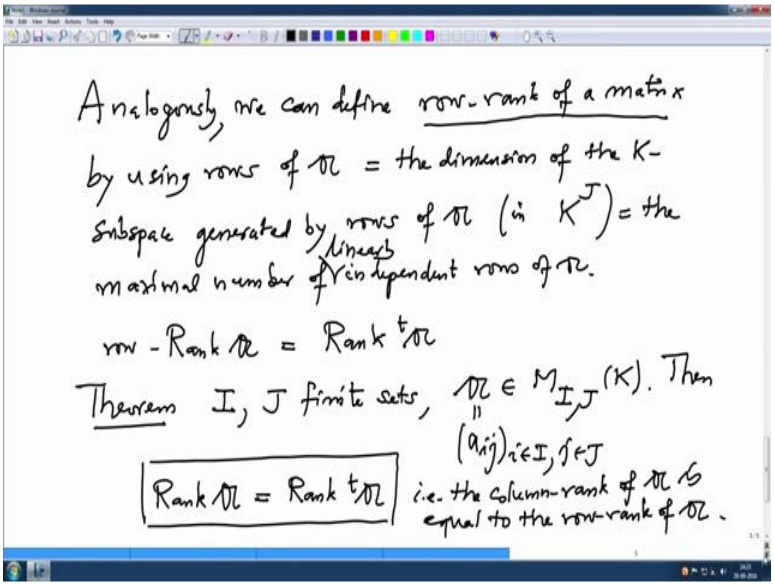

So, it is noted and analogously we can define we can define row rank of a matrix by using rows of A and that is which is equal to the dimension of the subspace K subspace spand by. So, generated by rows of A and where is the subspace in K power J because rows are elements of K power J.

So, and this is; this is also equal to the maximum number of the maximal number of independent rows of a linearly independent linearly A, soon we will prove that these 2 ranks are same, if we want to prove the row rank as same as column rank and therefore, we will use the word rank without any ambiguity its convenient to note that the rank of A; rank of A equal to rank of which transpose row rank.

Let me write row rank of A equal to the rank of the transpose because row rank of A is a rows are used and when I say rank columns are used. So, just columns and rows are interchanged when you make that transpose.

Now, we immediately prove that the row rank and row rank is same as column rank we have to we have to prove that the ranks of these 2 matrices are same A and A transpose. So, that is the content of the next theorem, theorem A is a met I J finite sets and A is a i cross j matrix. This is a i j i in I j in J then the solution is the rank of a equal to rank of A transpose so that is the column rank of A is equal to the row rank of A and this we will keep saying rank. So, I will leave 2 proofs, one is the usual matrix proof, but it will be arranged in a meter way.

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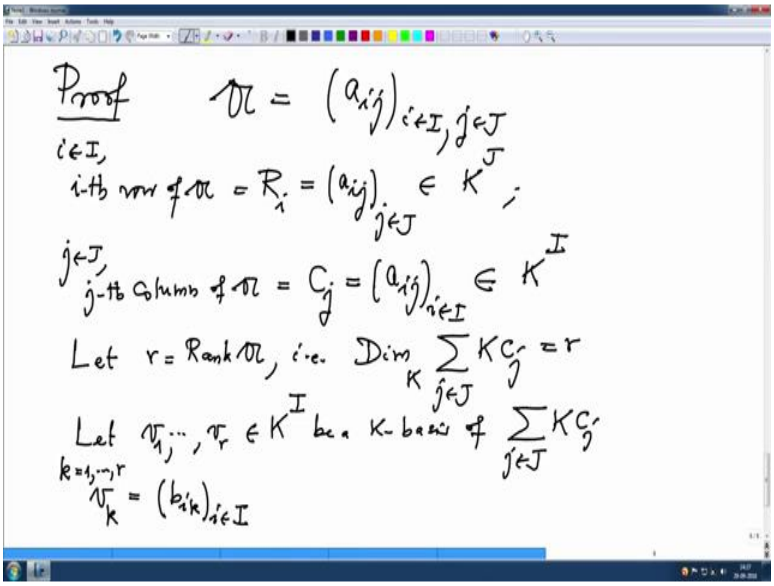

So, let us write, so A is; so proof A is our this matrix a i j and let us write i th row; i th row of a which I am going to denote by R i and this is the column vector a i i is fixed for this row and j is varying. So, this is j is varying, this is a limit in K power J and I is varying. So, for each i in I where this is the i th row, similarly for each j in J, j th column, this i denote by C i C j, j is fixed. So, it is a i j i in I, this is an element in K power I and one is the again thinking as a column we have decided vectors in this coordinated vector spaces we had always thinking them as columns.

Now, we want to prove that row rank is column rank. So, let us call r to the rank of A; that means, the subspace generated by these columns has dimension r. So, and I want to choose so that is dimension of the subspace generated by all these columns this is R. So, I want to choose the basis for this subspace.

So, let they will r elements, v 1, v r be in; this is the subspace of K power I. So, K power I be a K basis of this subspace and therefore, each V K is a; is the vector in K power I. So, I want write that so; that means, V K for each K from 1 to r is of the form. It is numbered it is a vector in K power I. So, and it depends on this index. So, I am denoting b i k i in k i in I, alright.

So, because it is a basis of this subspace generated by columns each column will be linear combination of these vectors. So, write it.

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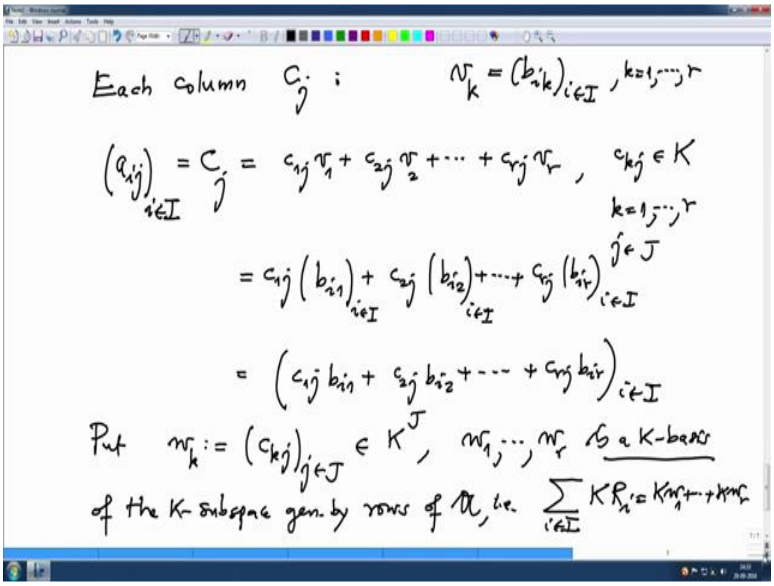

So, each column C j, we can write it like this C j equal to C 1 small c, this is small c, c 1; it will depend on the js. So, c 1, j v 1 plus c 2 j, v 2 plus, plus, plus, plus, plus c r j v r because it is a basis and this small c is c k j, they are elements in the field K; scalars K is running from 1 to r and j is running in j, but the c j is we have written c j is a i j, it is a column of jth column of the matrix a. So, this is this column J is fixed and I is varying. So, this is element in K power I.

So, we have this, but we have we have remember we have written we have V K is V K is b i K i in I and K is 1 to r. So, just substitute the coordinates of v's so, this is nothing, but c 1 j b i k b 1 k b no; not 1. So, I am writing b 1. So, K is 1. So, b i 1 plus this is running, i is running the next is c 2 j b i 2 and so on.

C r j b i r, but the way the scalar multiplication is defined, I can push it in and I can add component wise. So, this is same as c 1 j times b i 1 plus c 2 j b i 2, etcetera, etcetera, plus c r j b i r i r. So, these 2 tuples are equal, they are i tuples and they are equal; that means, the corresponding entries are equal, but then I want to define now vectors.

So for so let us put w k by definition we made a j tuple now. So, w j w k is c k j. Now here what is varying is K is fixed and j is varying. So, this is in K power J and I claim that this w 1 to w r generate or the basis is a k basis of the subspace K subspace generated by rows that is rows of a that is what I am saying is that is summation K r i i in

I, this subspace is generated by w 1 to w r, it is a basis. So, in particular this subspace will have dimension R and that is the row rank and we would have finished our proof.

So, just to note that this is the K basis, but to say it is a K basis I have to prove 2 things, I have to prove that these w's are w's generate and they are linearly independent.

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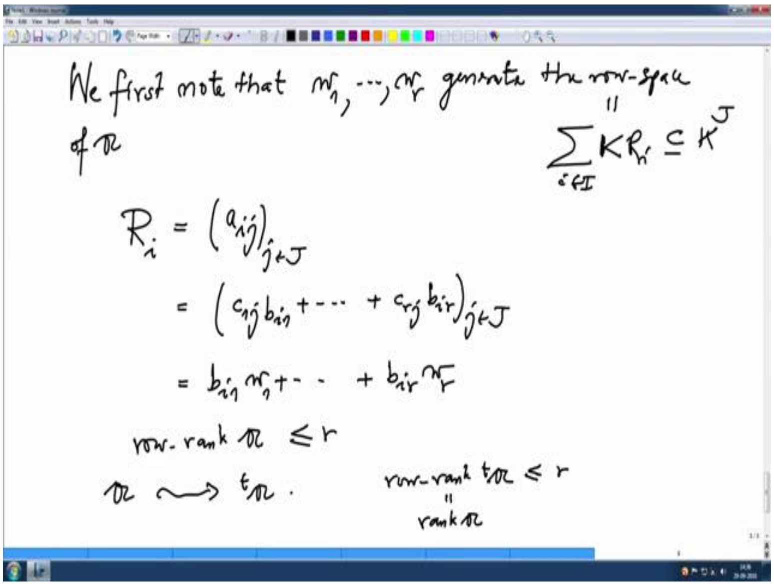

So, it is I will prove that W generate. So, we will prove first note that w 1 to w r generate the row space, let me call it a row space generate the row space row space of a row space is by definition the subspace generated by the rows this is the subspace generated by rows which is the subspace of K power j.

So, I need to write every row R i as a linear combination of w 2, w 1 to w r so; that means, R i is by definition ith row. So, that is a i j j is varying in j, but we have noted that this is nothing, but c 1 j b i 1 etcetera, etcetera, c r j b i r j is varying in j, but this is nothing, but I will take b i 1 out, see earlier I push c 1 j inside. Now I am putting b i 1 out. So, b i 1 and then this is w 1 plus, plus, plus, plus b i r w r, this is that precisely the definition of w r is made like that. So, therefore, what we check that is the row space is contained in the subspace generated by w 1 to w r.

Therefore row space that is the row rank of a will have dimension less equal to R that is because it generated by R elements, but now if I interchange, if I use the same argument when I replace a by a transpose, I will get this is less equal to rank of A So, that proves equal So, this proves no, I should write little bit better, when I replace A by a transpose I would have proved that the row rank of the transpose is less equal to r, but row rank is the rank.

So, that proves R equal to both are equal R, R equal to both column rank as well as row rank. So, that proves another way to I will come back after the break and even we will; I will tell you another way to prove this. So, we let us break here.