Linear Algebra
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Lecture – 43
Computation of the rank of a matrix

Now, come back to this another half and then let us talk about rank just now I approved that row rank of a matrix is also column rank and this common rank I will keep calling a rank, and as we have seen the proof was computational and it involves coordinates and so on, but now I want to give second proof which will be less computation and with this one should realize that it is better to study linear map then the matrices.

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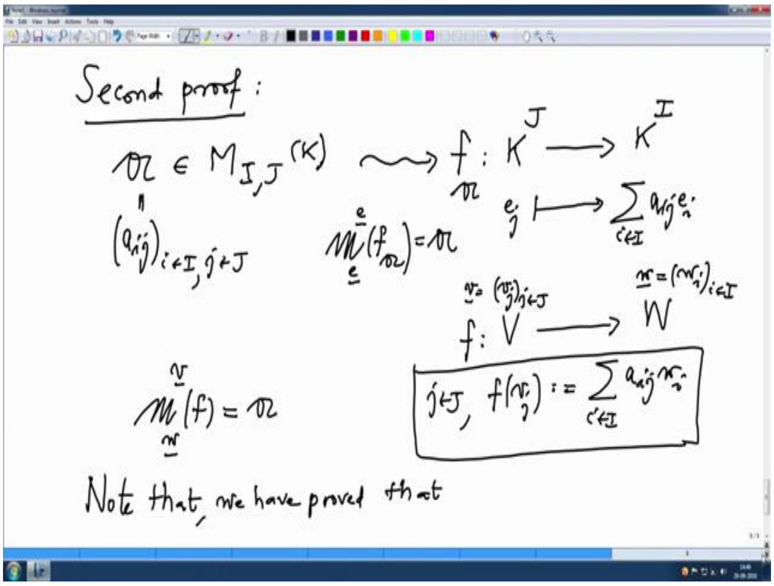

So, second proof. So, this we have given this matrix a which is imagine this is a i j, i in I j in J, but 1 should think really this is a linear map f, f a which is from the vector space K power J to K power I namely I should tell you where the basis goes.

So, the basis e J the standard basis e J here is map to the element summation a i j e i and note that this this e i on the right side is basis standard basis of K power i. So, this is a linear map and whose matrix with respect to the standard basis will be then the given matrix a just also to a word this confusion between the standard basis here and here you can also think directly this is f is the map a comes from a map f vector space V to W and this vector space V is with a basis v i v J numbered by the same indices have the columns are indices column index and this is V and this is a W is the vector space with basis w and basis of w numbered by the set i.

And if you want the linear map we just have to give what is f of v j for all J, but well defined f of v j by this formula and this way we do not have confusion with this standard basis and coordinates and so on. So, therefore, this matrix a you can think a linear map f and again if you take this matrix f the matrix of f with respect to this basis this is precisely the given matrix a. So, if I want to prove the rank of row rank of a equal to column rank of a, I will have to prove that prove the relation a row rank is nothing, but row rank is nothing, but the dimension of the image that is rank of f sorry column rank is nothing, but the dimension of the image face of f, and row rank will be nothing, but the dimension of the the image of f star.

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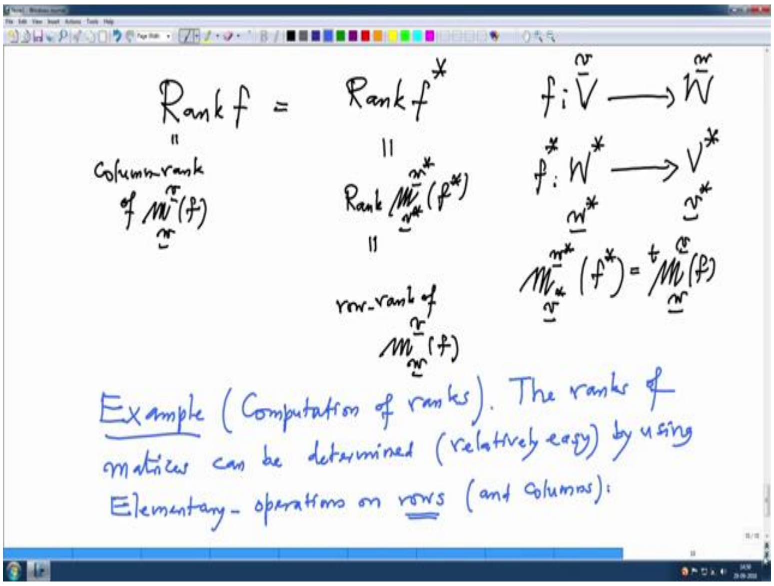

So, note that let me write it note that we have proved earlier that rank of f equal to rank of f star f star; where f star is a map if f is a map linear map from V to W, f star is a map in the other direction and if you take a basis V here W here the corresponding duel basis w star and v star and we know the relation between the matrices also that is the matrix of the basis matrix of the linear map f star with respect to the basis w star and v star this matrix is the transpose of the matrix of f with respect to the basis v and w; this we have proved last time and this rank of this matrix that is column rank of this if therefore, the row rank of this.

So, this is rank of m w star v star f star, but this is nothing, but the row rank of m v w f and this is a column rank. So, we altogether we approved column rank equal to rho rank. So, now let us come to the computational part. So, now, I want to address. So, let me write this as an example computation of ranks, I am not going to discuss numerical example. So, three by three, four by four matrices which whose entries are actually numerals I am not going to discuss them, but I want to do it little bit more theoretical way.

So, how does one compute the rank of a matrix more effectively or relatively easily. So, the rank of matrices can be determined relatively easy by using so called elementary operations on rows and columns, but I will strict to rows first. So, what does 1 mean by this? So, let us.

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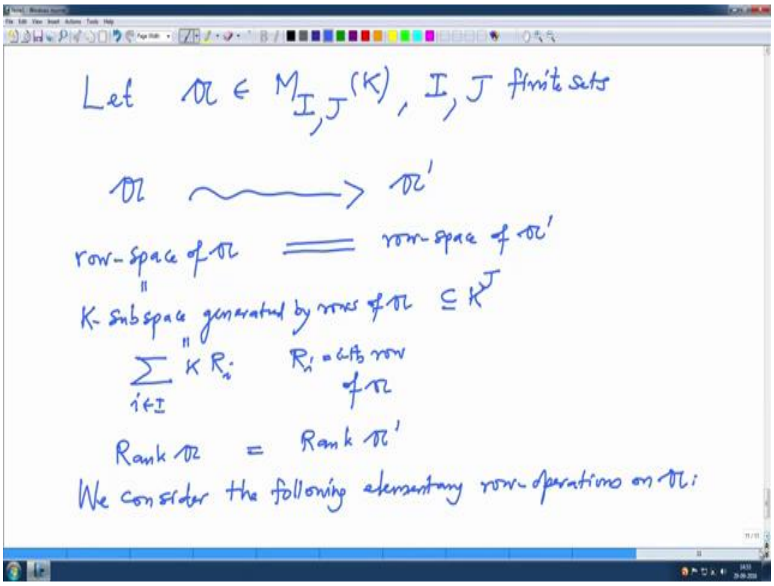

So, let a b a matrix whose rows and numbered by I, a columns are numbered by J entries in the field K and I J are finite sets, and I want to now change this a to some other matrix a prime without changing the row space. So, when I say row space of a; that means, a subspace generated by the rows K subspace generated by rows of a.

So, this is same as summation in our notation summation i in I K R i where R i is R i is a row of ith row of v and this is a subspace K subspace of K power J. So, I want to get a new matrix a prime from a by doing some operations without changing the row space. So, that the row space do not change, but a prime become simpler in some sense so; that means, I want to get more a zeros in the matrix entries of a prime. So, if the row space does not change then the ranks will be equal and then go and get and good simper that is the strategy; and what are the row operations we are considering. So, we consider the following elementary row operations on a there are three types.

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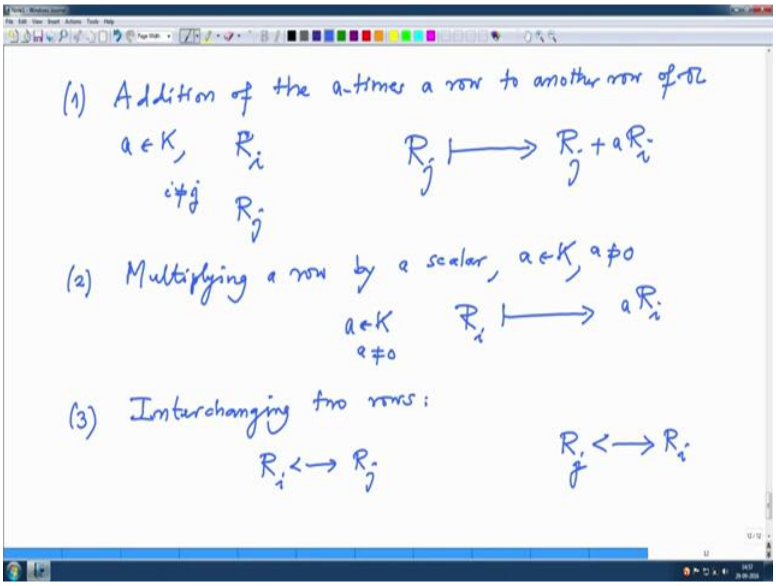

The first type that is addition of the scalar times a times row to another row of a. So, that I want to write this in the notation. So, a is a scalar arbitrary scalar i is a row. So, i th row to another row and j is r j is another row. So, i not equal to j and I am replacing R j by r j plus a R i. So, the new matrix a prime and a they will only have all the rows are equal only jth row will be replaced by this new jth row of a prime, this is one operation and; obviously, here you see raise space is not changing because R j is replaced by this. So, I can always recover R j from this by using R i. So, in this case rank of a prime is not change the second type of operation is multiplying row by a non-zero scalar by a scalar which is non-zero a in K, a non-zero in notation; that means, we have a scalar a, a is non-zero is very important field is also very important and the i th row is now replaced by a times i th row.

So, the new matrix will have only the i th row is change, and that will be multiple of scalar multiple of the i th. So, it this is also clear that the row spaces not changing. So, a and a prime will have the same row space. Third one interchanging 2 rows; that means, I have 2 rows R i and R j in the original matrix and now you only I will positions are changing. So, R j will become R i th position R j will become R i th position and this will become j th position they are only interchanging. So, now, the row space is also clearly not changing. So, this I will going to get a new matrix use rank will be same and I will acquire more zeros I will give still more concurrent example, but also I want to remark here that operations 2 and 3 and I am not going to use column operation then also I am going to use very sparingly operation two.

Reason being because in the next few lectures I also want to address for on the problem for computing the rank of not only a matrices with entries in a field, but matrices with a entries in the ring like integers, and for that I have I will not allow to you we are not allow to use by non zero scalars because in that case scalars are integers though integers are non zero they are not invertible in the ring so ok.

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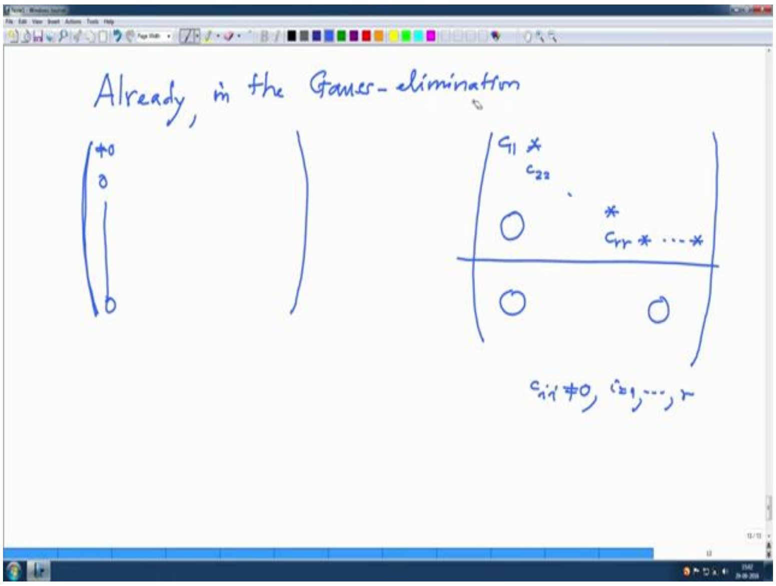

Let us further continue, now I want to recall already when we did gauss elimination in the gauss elimination, we demonstrated that row operations of a matrix we can transfer we can find a new matrix. So, that the rank actually you can see it what it is from the new matrix and. So, that will be like this.

So, let me recall it. So, I will write the final matrix here what we have got from the original matrix. So, C 1 1, C 2 2, C r r and then below that everybody is 0, here also 0 and here there may be some entries here also there may be some entries and how did you get this. So, we had original matrix here where c i i are non-zero for i is 1 to r. So, what did we do first we first found a top entry which is non-zero for that note that we might have had to use interchange the columns because all the columns will not have ha 0 entries on 0 entries. So, we could have interchange the 2 columns also and then once we have a non-zero entry here we use this to use row operation true make the entries below that zero.

And repeat this process. So, this way we have got on the diagonal you see it is not a square matrix. So, when I say diagonal it is a main diagonal. So, they will be may be rows or more columns are plus and so on. So, this is the main diagonal. So, some entries non zero and after that it become 0 0 0 above that we do not know what it is. So, and because. So, here only to note that we have we have allowed also to interchange columns, but we did not make the other column operations, and because we are in a field we could have made actually them 1 11 1 if you are working over a field, but if you are not working over a field still we have we have manage this for integers also, and here it is clear that the number of non-zero entries on the main diagonal is precisely the rank.

So, so that is that was what the gauss elimination also I have the similar motivation and. So, another remark I want to make is in general the computing a rank of a matrices on the by using the computer software is can be difficult and can be more time consuming just because and also it may not compute the especially for real and complex numbers because the computer software’s will approximate the real numbers to some decimals, and when the entries become smaller computer will it ignore it and instead of that will become 0 entry and this will miss lead the calculation. So, whereas, the maximal rank one can compute the matrices which has maximal rank that you can do little bit better that I will come back to when I do it for integers ok.

So, also next one is I also want to show that this row operations on the matrices also will allow us to compute inverses of matrices rather easily. Compute not only compute inverses also to decide whether a matrix is invertible or not and in that case what is a inverse both this we can do it simultaneously.

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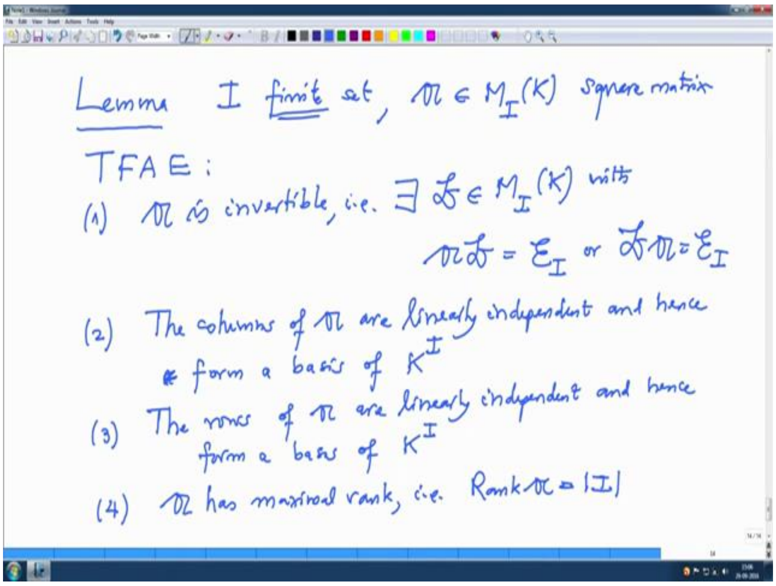

So, let me write one lemma for that. So, because we are dealing with a inverses of a matrix I want to start with a square matrix. So, I finite set and we have a squire matrix a is in M I K this is square matrix. Now rows and numbers are numbered by a the same set i then the following are equivalent 1 a is invertible that is there exist a matrix b with a b equal to identity, this is enough or b a equal to identity this last time we saw it one of them will imply the other because finite dimensional i is finite that is very important.

Two: the columns of a are linearly independent and one they are linearly independent they are correct in number and hence basis form a basis of K power I. Three once you say something about whenever you have matrix and say something about columns one need to say something about rows also. So, the rows of a are linearly independent and hence form a basis of K power I. Fourth a as a maximal rank that is rank of a in this case equal to cardinality of I. So, let us see the proof again proof I always want to see a proof by using a linear map.

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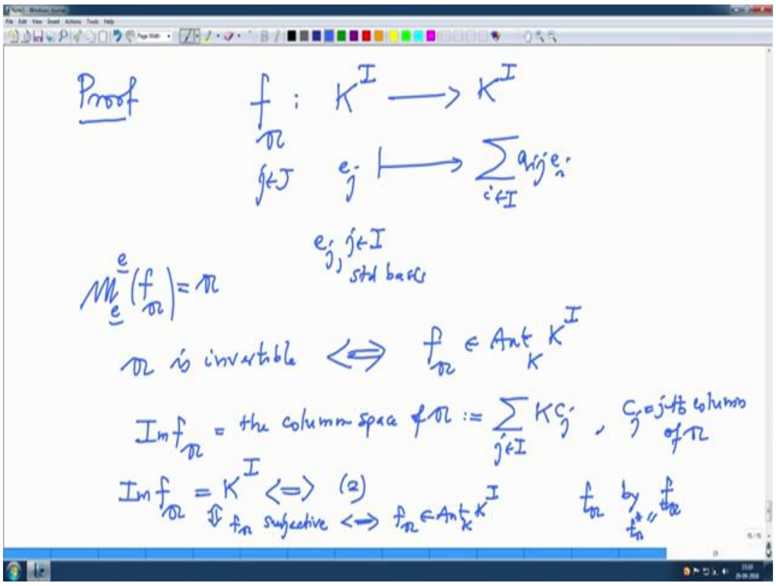

So, let us proof lets go to linear map remember our matrix will give a linear map f a which is from K power I to K power I; e j th vector here e j th standard basis vector here will go to summation a i j, e i or every j in J this e j j in I standard basis and in this case there is no confusion because the same set.

Now, the matrix of matrix of f with f a with respect to the standard basis this is the given matrix a; and we know we have proved matrix is invertible if and only with the linear map is an automorphism. So, a is invertible if and only if f a is an automorphism K automorphism; and what is the image of f a image of f a is the column space of a by definition column space is of a, matrix is the subspace generated by the columns C js are columns j th column. So, the column, the column what is the statement here I just want to show you this one, second one columns are linearly independent that is equivalent to saying it is a basis, if it is a basis; that means, f of image of f of a must be the whole. So, image of f of f suffix a is the whole K I if and only if 2.

A first was invertible, but the 2 if and only if this means f is surjective. So, this means f a is surjective, but once f a is surjective that is if and only if f a is bijetcive, that is it is an automorphism. So, that proves one if and only if let us one if and only if 2 it proves, and similarly 1 if and only if 3 be then instead of that; then I will apply I will apply the same remark to the not to f, but the f star and rows and columns will get interchange and so on. So, replace f a by either if you want in a matrix it is f a transpose or f a star whichever f a start is nothing, but this. So, then as a remaining are equivalent that is clear. So, that proves the lemma all right.

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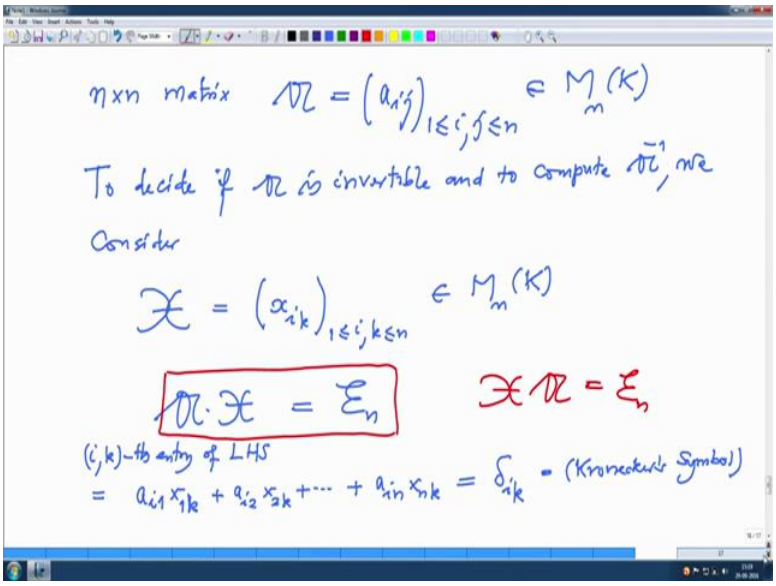

Now, about finding the inverse, now, let us take arbitrary n cross n matrix n cross n matrix a which is a i j, 1 less equal to i j and this is a matrix square matrix of order n and we are looking for inverse, we want to decide to decide if a is invertible and to compute a inverse, if it exist we consider a matrix x, here x is a matrix, but the entries are variables x i K let me call it small j small small x i k; k is a j k is are 1 less equal to i k less equal to n. So, it is also n cross m matrix, but I want to determine exercise that is why call them variables and then I will come back we will see what we want. So, what we are looking for we are looking for such a matrix x, so that a times x becomes identity matrix.

So, I want to compute this x i k. So, that this this equation, but what are this equation means let us write down this means what is the i K th entry here; i K th entry on the left side i K th entry of LHS equal to that will be the i th row of a times j th column of x, not j th K th column i th row of a times K th i th row of a times K ith column of x. So, that is a i 1, x 1 k, plus a i 2 times x 2 k and so on plus a i n, x n k correct this is equal to i K th entry on the right side, but i K th entry delta i K this is kronecker symbol.

So, it is a linear equation a nominee variable will be a linear equations in these variables x 1 K, x 2 K, x n K. Now, therefore, I will linear system of equations and this linear system the coefficient matrix.

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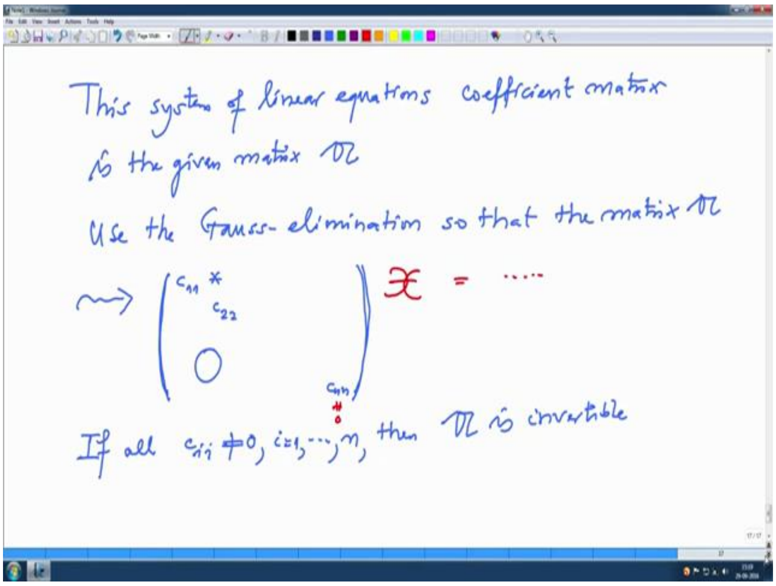

So, this system of linear equations coefficient matrix if the given matrix a and we want to we want to solve it simultaneously for that one user the gauss elimination. So, use the gauss elimination. So, that the matrix. So, that the matrix a will get change to the new matrix like this diagonal C 1 1 C 2 2 which is square matrix C n n below that 0 above that somebody.

Now, it is clear that if all these cs are non-zero if all C i i non zero for i equal to 1 to n then then there is a solution a gauss elimination says now and then a is invertible. So, remember what we have done is a following, let me just go back and show you see we wanted to solve this equation this this equation and we are making row operations in an gauss elimination on the matrix a. So, this is the same operations you are making on the identity matrix. So, here when you make the gauss elementary operations we have made the matrix a like this. So, the equation became this this times x this times x equal to this matrix as change and now first we decided that it is invertible or not how to test it only the diagonal entry should be non zero.

And now in this case what we do one this is all these are non-zero this is non-zero in particular now I start making row operations from below. So, then there will they will get transformed to this again this identity matrix will get transformed. So, that will become that will solve this x because this this matrix will become identity matrix and then x will become matrix so; that means, I have solved for x also. So, this way we can find the inverse. So, if you do it the other way I just want (Refer Time: 37:07) if you want to solve instead of this if you want to solve this kind of this that also one can do it, but then instead of row operation you have to use column operations well this is also equally simple (Refer Time: 37:24) same.

So, with this I will not like to do a numerical example what you can just one remark and then we can stop for today.

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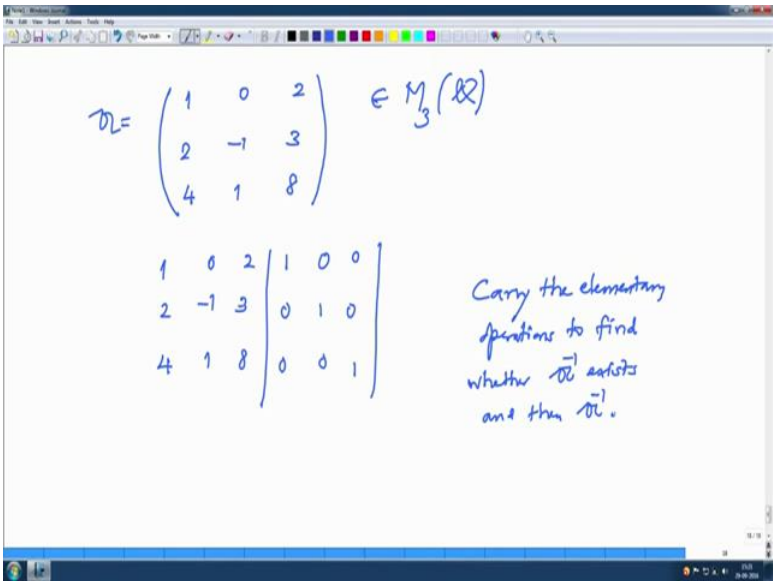

Suppose I take such a matrix suppose I take a matrix like this a three by three case 1, 2, 4 0 minus 1, 1 and 2, 3, 8 this is three by three matrix with rational entries, and suppose I want to decide whether it has inverse and inverse exist not etcetera. So, what usually we does it you on this side you write this matrix 1, 2, 4, 0 minus 1, 1, 2, 3, 8 and here you write identity matrix and start making operations here and then you do the corresponding operations here and then this becomes the identity and then becomes the inverse that is what, but I you carry out this carry the elementary operations to find whether a as a inverse exist and then a inverse. So, we will stop them.