Linear Algebra
Prof. Dilip P Patil
Department of Mathematics
Indian Institute of Science, Bangalore

Lecture – 44
Elementary matrices

So, let us start this lecture with new section on matrices. In this section I want to establish many assertions on the representing a matrix by as a product of simpler matrices, and also use this representation to compute inverse of a matrix in a simpler way over a field and then I will also at the last I will also say something about the matrices with integer entries, and use this to in a theory of abelian groups to establish a structure theorem for finitely generated abelian groups.

So, this may happen in this lecture and next lecture also. So, let us start defining. So, this is a section on elementary matrices.

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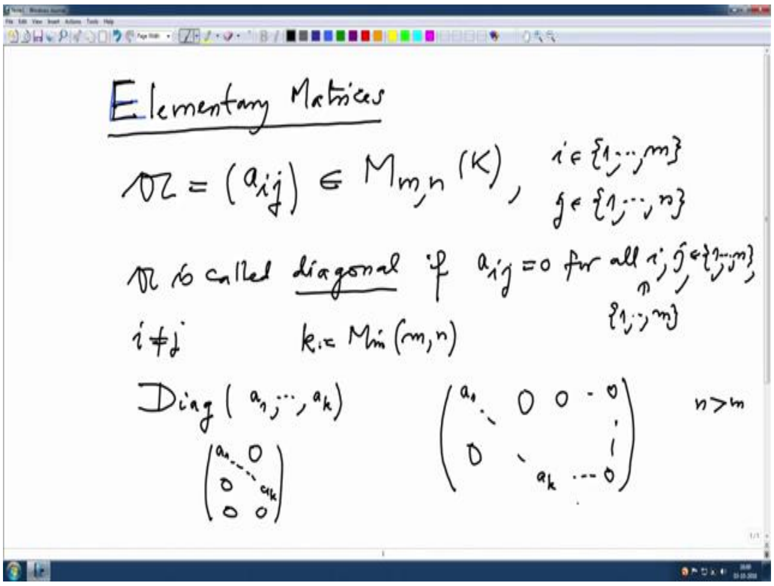

So, let us first recalling or I want to set up some notation which I will use it again and again on the time. So, suppose we have a matrix a numbered rows are numbered by i columns or numbered by j and this is a matrix M n k. So, I is varying in the said 1 to m and j is varying in the said 1 to n. We will say it a diagonal matrix is called diagonal, if all indices a i j are 0 for all i j in I in the set 1 to m, j in the set 1 to n and i not equal to j it may not be square matrix.

So, for example, and this I will denote it by and if you put k is the minimum m and n, this matrix I will denote this diagonal matrix I will denote by D i a g Diag that is a short form for the diagonal, and a 1 to a k. So, it will look like this it will depend whether the m is smaller or m is bigger. So, a 1 to a k and the remaining a's are 0. The in this case the number of columns are more this is the case where n is bigger than m, similarly you can write the other case it will be like this then there will be more 0 0. So, such a matrix is called a diagonal matrix, square matrix is diagonal if we it is the num a m equal to n and all other entries are. So, this is called a diagonal matrix.

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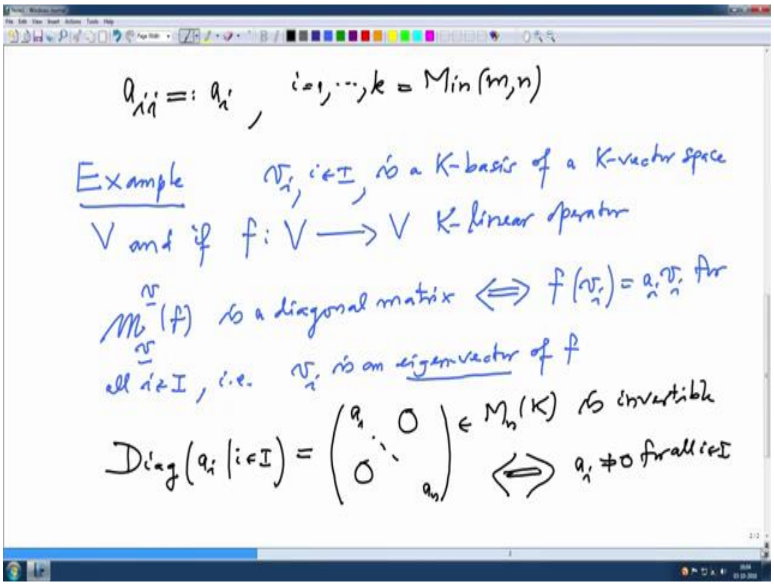

And in this case also I will here I should have put a 1 1 etcetera, but in the case diagonal case only a i i entries are nonzero possibly nonzero, and these entries are usually denoted by a i, this is i is from 1 to k; where k is the minimum all right. Let us see an one example which will be useful when we do linear operators example for example, if V i is a basis V i i in I is a K basis of a K vector space K vector space V then and f is a linear operator, f is a linear operator from V to V, K linear operator then the matrix of f with respect to the basis v this is square matrix is a diagonal matrix, if and only if f of V i is a scalar multiplier field a for all.

This means. So, that is each this V i is an eigenvector of f, which keeps this one dimensional subspace band by V i in variant under f and this is what we will need we want to decompose our vector space given a linear operator on a vector space, we want to decompose vector space so that on each piece it will look like this. So, other thing want to mention here if the diagonal matrix when is it vertical, that is diagonal matrix now let us take a square matrix diagonal matrix a i, i in I this means diagonal on the diagonal entries are given a i. So, a 1 to a n remaining entries are 0 this square matrix is invertible if and only if all a either nonzero.

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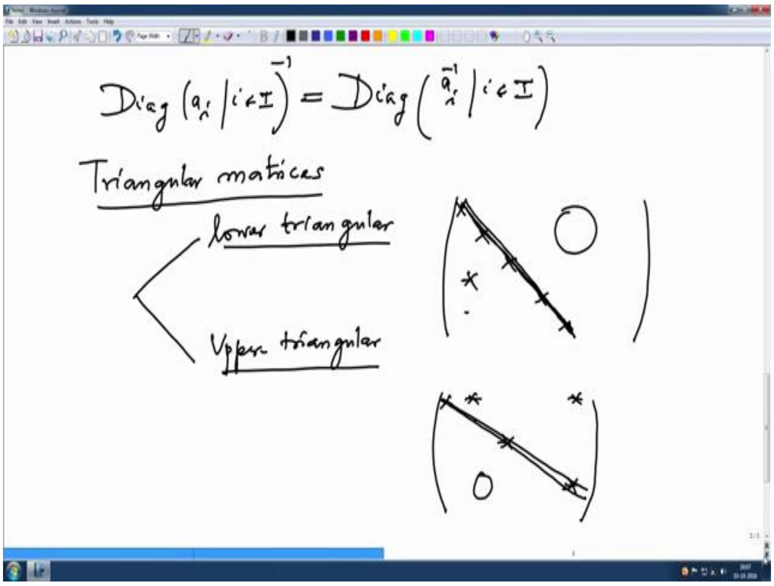

And in this case the inverse is very easy to write down without any calculation Diag a i, i in I inverse of this is also diagonal matrix with the diagonal entries inverses of the elements a i this is very easy.

Similarly, I want to define triangular matrices; so triangular matrices are two type lower triangular upper triangular, and given a matrix there is a main diagonal there is a main diagonal that can either be like this when the number of columns are more or it can be also like this; see this is a rows are more this is columns are more. So, this is called a main diagonal of a matrix this can be either these or these depending on who is more in the case of equal that is it is a usual diagonal, and lower triangular means this upper part is 0 and down is some stars this elements are some elements in the field, upper part is 0 above the main diagonal and upper means upper part is arbitrary elements could be 0 may not be 0 and the down elements are 0, this is upper triangular this is lower triangular.

In case of a matrix with a square matrix triangular matrix very easy to find whether it is invertible or not we would have seen in earlier assignments also, that if the main diagonal entries are nonzero then it is invertible and inverse we need to compute. So, we will address this question of computing inverse little soon. So, that is again I want to first write one example in terms of the matrix of a linear operator, and in addition to these matrices diagonal triangular and I will define another special type of a matrices and our aim is to prove that every matrix can be written as a product of these matrices. So, let us write one example.

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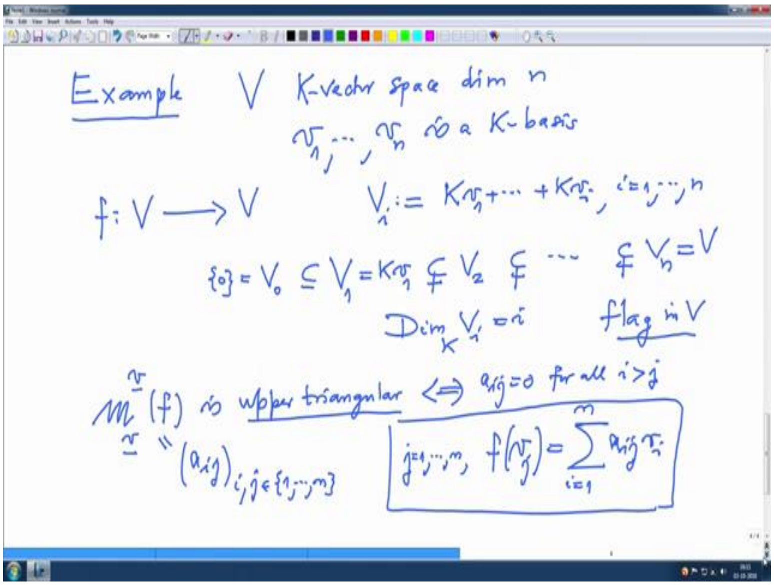

So, suppose we have a v is n dimensional vector space, V is K vector space of dimension n and v 1 to v n is a basis, and suppose we have a linear map f V to w V to V, and let us put capital V i is the subspace spend by V 1 to V i, K V i this is from i equal to 1 to n. So, we have a chain of subspaces V 1 which is generated by V 1then V 2 which is generated by V 1 V 2 and so on V n this is a whole vector space V and at each stage they are proper and before this there is a trivial vector space V 0, which is a single time 0 and each time dimension is up more because this is a part of a basis. So, dimension of the subspace V i is precisely i. In fact, V 1 to V i is a basis of V i.

So, this is a sending chain of subspaces it's also called a flag remember this is a flag in V. So, when we will the question is now when we will the matrix of v matrix of f with respect to the basis v is upper triangular; that means, what do we need we need see this matrix you know this matrix is if you call it a i j, i j are in their in 1 to n this square matrix, and this is defined this a i j's are defined by this equation for any j from 1 to n f of v j is the sum a i j v i, i sum 1to n these are sort of n defining equations of this matrix and this f of v j we are writing it as a j th column and we want upper triangular so; that means, below the diagonal all the entries in the j th column should be 0.

So that means, below means the row number should be more than the column number so; that means, the condition is this is if and only if a i j 0 for all I strictly bigger then j, because when you write it in the column below the main diagonal they will be 0, but this condition you can nicely put it that is equivalent to saying.

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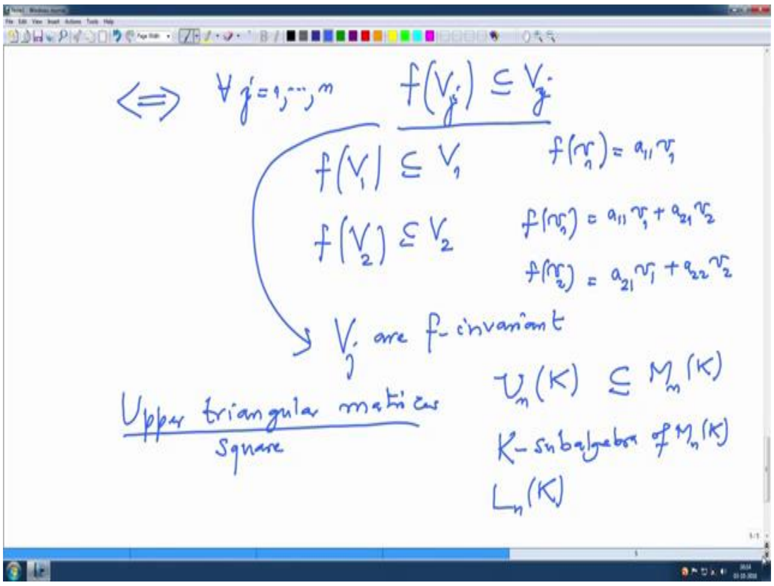

That is equivalent to saying for each I i in 1 to n, the subspace V i is in variant under f because if it is so, then when you write let me write the index j when for example, f 1 f of f of V 1 should be contained in V 1 this means f of small v 1 I should be able to write in terms of V 1 only that is a 1 v 1 remaining coefficients are 0, f of V 2 should be contained in V 2; that means, f of V 1 and f of V 2 both I should be able to write only in terms of V 1 and V 2.

This is a 2 1 v1 plus a 2 2 v 2 and so on. So, each v j should be invariant under f. So, this is this condition we save as v j are f invariant. So, it is in general it is an interesting problem given a linear operator how do you decide there is a invariant subspace, and what is the dimension and so on this we will do it. So, the upper triangular matrices upper triangular set of all upper triangular matrices, I will denote by U n K this is square matrices only. So, this is upper triangular square matrices. So, this is a subset of M and K and not only subset note that if you add the two upper triangular you get the upper triangular if you make a scalar multiple it is a upper triangular again, also if you multiplied two upper triangular matrices you get upper triangular matrices.

So, that simply mean that this U n K is a K sub algebra of M n K. So, also we would look for an operator how do we decide the matrix of a linear operator is upper triangular in general, and once we said upper triangular similar thing you can do for low triangular. So, L n K this is low triangular matrices that is also sub algebra of M n K ok.

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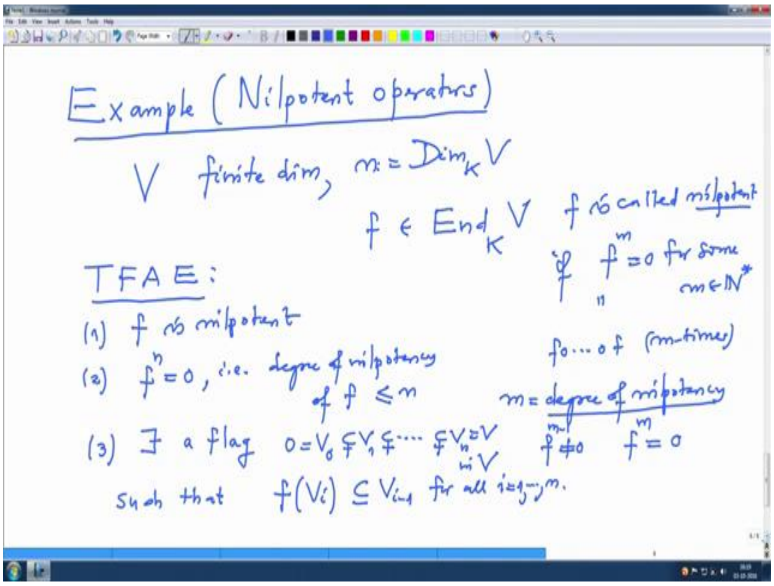

So, continuing this example little bit let me study now what are so called nilpotent operators. So, this next example nilpotent operators, so V is finite dimensional we will put n the dimension of V and in this I want to characterize nilpotent operator.

So, what does when mean by nilpotent operator. So, if f is an operator; that means, in element of this K algebra we will say it f is called nilpotent, if f power some power f m is 0 for some m. So, this is a compose of f m times this is f compose f compose f m times. So, in I will give example if we will clear many examples after the characterization, this concept of nilpotent elements is make sense in any K algebra. If you have a K algebra or further arrangement if you have any ring, then an element is called nilpotent if some power of that element is 0; and smallest power where it is vanishing that is called the degree of nil potency degree of nil potency; that means, f power m is 0, but not the earlier one f power m minus 1 is not 0, then m is called a degree of nil potency ok.

So, now I want to write the equivalent statement so that we can so this the following are equivalent, this is easy or different way to test when operator is nilpotent one f is nilpotent. 2: f power n is 0 this is n is the dimension of v. So, that is the degree of nil potency of f is smaller equal to n could be 0 earlier, but definitely at n it become 0. Third there exist flag mean 0 equal to V 0, V 1 this V n equal to V, in V such that f of V i is not only containing V i it is even contained in the earlier one this is containing V i minus I for all I from 1 to n this is a third condition and fourth condition.

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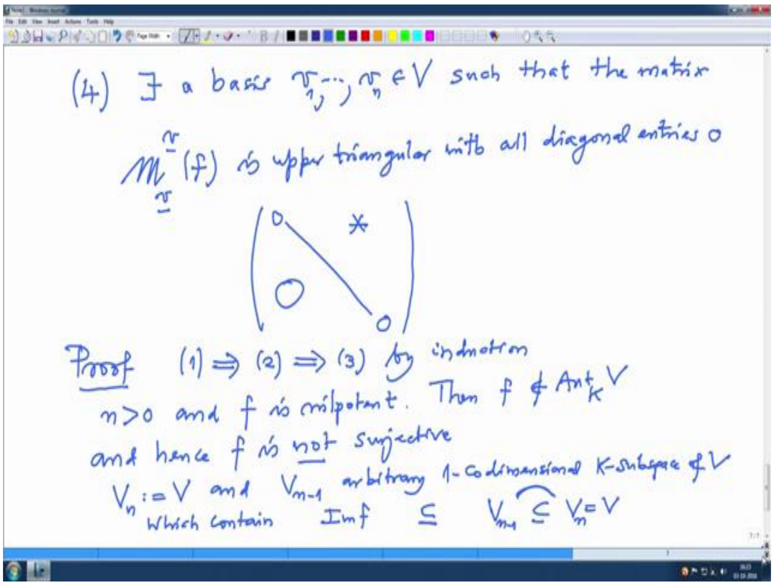

I will have to write in the next page unfortunately; fourth condition there exist basis V 1 to v 1 in v such that the matrix m v v f is upper triangular with diagonally with all diagonal entries 0 that mean it looks like this 0 on the diagonal below also 0 and somebody other.

So, proof 1 implies 2 implies 3 by induction. So, let me indicate the proof and then I will indicate three implies four; first let us prove this. So, let us assume dimension is positive because if dimension is 0 there is nothing to prove. And assume f assume one that is f is nilpotent one is f is nilpotent, then first note that f cannot be automorphism f is not automorphism; because f is automorphism and f is nilpotent I will keep canceling 1 by 1 f. So, f cannot be an automorphism; that means, and we are in a finite dimensional case.

So, therefore, and hence f cannot be a surjective, because on a finite dimensional surjective linear operator is also bijection, so f is not even surjective.

Now let us and what you want to prove you want to prove two. So, let me show with the two, two is the degree of nil potency is less then equal to n and 3 is there is a flag. So, I am trying to construct a flag. So, V n you put V n equal to V and V n minus 1. Arbitrary one co dimensional subspace K subspace of V, which contain image of f; see image of f is a subspace of V equal to V n this is not full, so the dimension because it is not surjetcive. So, the dimension of this should be small. So, I am saying choose V n minus 1 so that the dimension this is one and which contain this possible; this may have dimension n minus 2 in that case I will enlarge it by one more dimension and so on.

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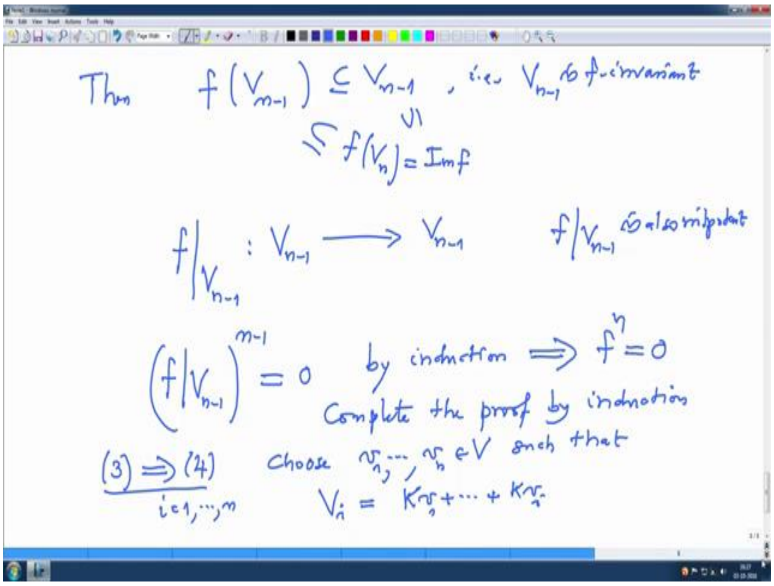

So, you choose that then I want to then V n minus 1 within variant under f because f of V n minus 1 is actually contained in f of V n which is the image of f; and we have chosen this so that this is contained there. So, this is, so to l. So, that is V n minus 1 is f in variant. Now because its invariant I can look at the restriction of f to V n minus 1, this is an operator from V n minus 1 to the vector space V n minus 1, and the dimension drop by one and the nil potency is still variant this is nilpotent f restricted to V n minus 1 is also nilpotent.

So, therefore, by induction this is an n minus 1 dimensional vector space, this is linear operator nilpotent. So, the two for this will be this power n minus 1 is already 0 by induction and from here I say from here it follows that f power n is 0, and then you keep doing by induction. So, this is a little thing to check that if this is 0 then this is also 0. So, now, I then complete the proof by induction. So, this proved 2 and then three also we wanted to prove, but 3 also I was trying to prove similar simultaneously. I contributed when in minus 1 now I will forget original v and original f and work with this and keep doing the construction of the flag downwards.

Now, 3 implies 4; 3 says there is a flag of invariant subspaces, and 4 says there is a basis which the matrix of f with respect to that basis is upper triangular with diagonal entries 0. So, we have given a flag. So, I choose a basis for the flag so; that means, I choose v 1 to v n in V such that for each i 1 to n V i this has this basis K v 1 plus plus plus plus K v i this must be the basis of V i and because.

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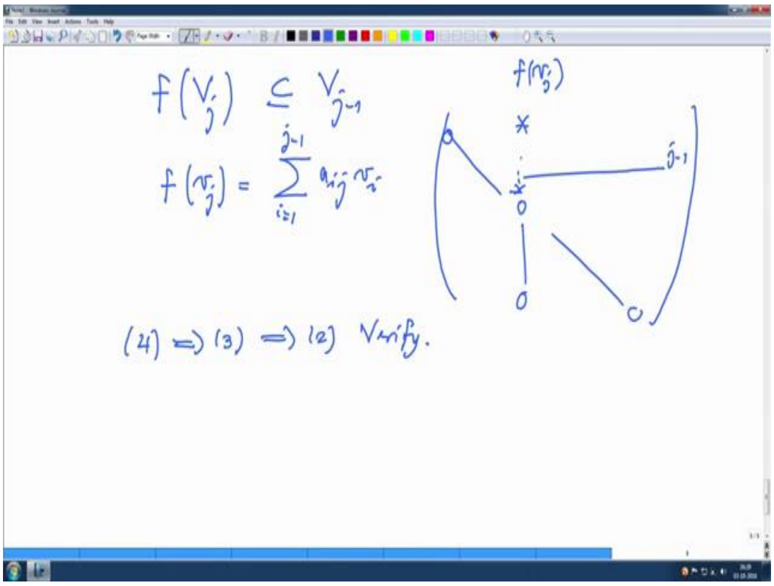

Because f of V j is contained in V j minus 1 that is given with the, this is a property and 3 this is given.

So, therefore, f of v j in particular should be an element here so; that means, we should be able to write it as this is a summation a i j and I will run only from 1 to j minus 1 v i so; that means, the j th column of a matrix with respect to the basis v will not contain including the main diagonal nobody down. So, this is the j th column f of v j. So, here entries will come up to this is the j minus 1 th row, and this is j th row which is 0 there is no coefficient of v j in this and below also nobody. So, this is a. So, and then something might be here so; that means, the matrix of f will look like this diagonals is 0 and somebody up. So, that is its low triangular upper triangular matrix with main diagonal 0 that is precisely was a 4 ok.

So, now we can also prove four implies three that is no so difficult and 3 implies 2 also you can prove these are all simple so just verify them. So, that gives a big motivation for studying a triangular matrices upper lower also the diagonal matrices, and in addition to that now I want to introduce a matrices which will come from elementary row and column operations this we do it after the half.