Linear Algebra
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Lecture – 45
Elementary Operations on Matrices

So, you used a first immediate corollary to the above characterization of nilpotent operators we have proved I will note that immediately in.

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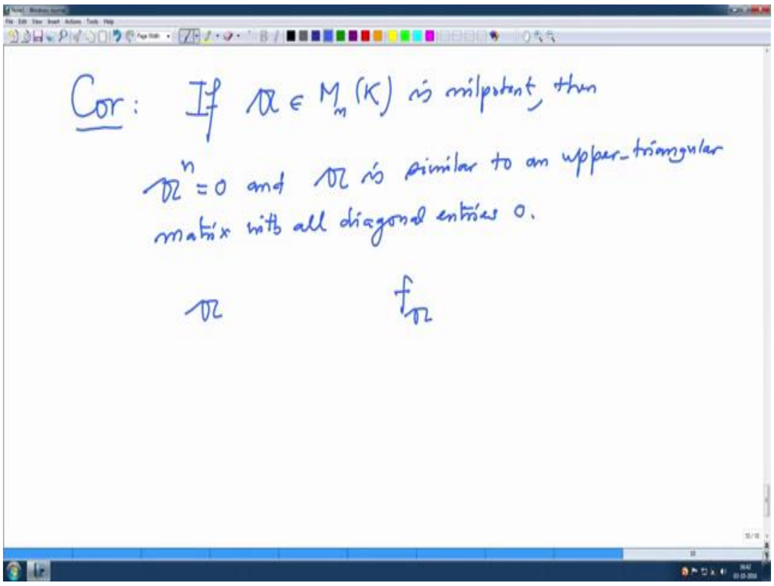

This is now the matrix, if a matrix, square matrix is a square matrix is nilpotent then already A power n is 0 and A is similar to an upper triangular matrix with a diagonal entry 0 with all diagonal entries 0.

This is immediate from the above because matrix a we can think of a linear operator f A and if this is nilpotent the linear operator is nilpotent and then we found a basis whose matrix of this a linear map is an upper triangular matrix with diagonal entry 0, but this was A, this was a matrix of this with a few standard basis I will change a basis. So, the 2 matrices will differ by a similar therefore, this is obvious from the earlier part and a fact that if you have a linear map and if you change a basis of a vector space the 2 matrices are related they are similar nature.

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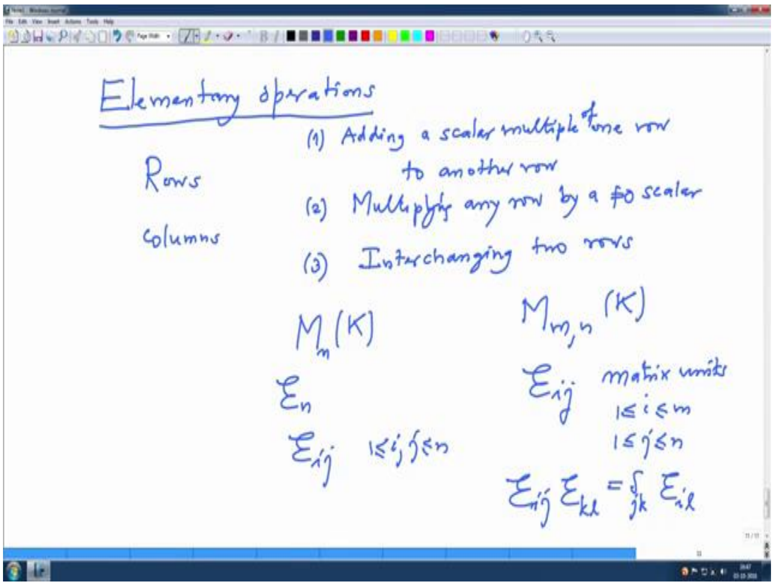

Now let us; now I want to define elementary matrices. So, this elementary operations your remember they were elementary operations which were in volume rows and columns and now on there are 3 types of operations 1 is adding a multiple scalar multiple of 1 row of 1 row to another row 2 is multiplying any row by a non 0 scalar. So, multiplying any row by a non 0 scalar and third was interchanging 2 rows and similarly for the columns and I want to set up a notation for this. So, for that let us; let me remind you that we are now working in M n K or more will be M m n K, so matrices with n rows and n columns.

So, let me remind you that here we denote E n is the identity matrix and also here E i j these are so called matrix units that says i j E th entry of the matrix is 1 and all other entries are 0 that makes sense for an arbitrary m cross m matrix and this is a basis for this vector space this is a vector space of dimension M m and this where i varies in between 1 and m and j in between 1 and m. This is a basis of this vector space in this case actually this is algebra. So, these are also the units and this now 1 and j they are varying in between 1 and m i and j varying between 1 and m. So, this is a vector space of dimension n square and all these multiplication is determined by how I multiply E i j with E k l and this we saw last time what it is this is again a matrix unit, this is again E, this is only you know this is delta J k i l this is this.

This is very simple to check we saw it last time also. So, that is how the calculation I will make calculation either here or here. So, the and also when I make a elementary operations on the matrices I will prefer to use 1 maximum, I will avoid using 2 and 3 as far as possible.

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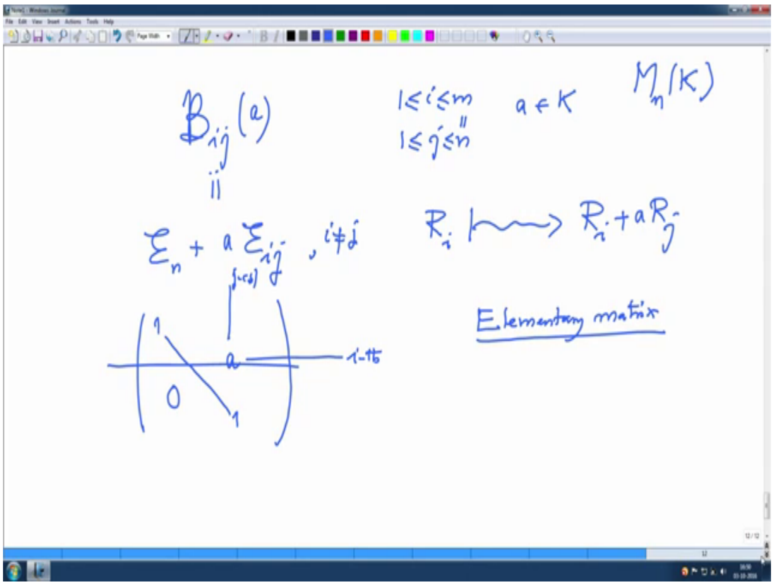

So, for one for the operation 1 what we are doing is. So, the matrix corresponding to that will be; I will denote B i j inside the bracket a. So, this is defined for 1 less equal to i less equal to m and 1 less equal to j less equal to n and arbitrary scalar a and what are we doing in this if you want to write in terms of row operations i eth row is replaced by r 2 r i you add j timeth a times r j th row that is how this matrix is this matrix is by definition identity matrix E n plus a times E i j this is the definition of elementary matrix.

So, let us do it for square matrices first. So, I am assuming m equal to m. So, these are done only for m equal to m is that clear, sorry to make a confusion I am making; I am working in the square matrices for there any 2 indices i and j and any scalar i will define this matrix to be identity matrix plus E i j; that means, only 1. This is identity matrix once on the diagonal and at somewhere some place where i is not j there is a entry here and everywhere else it is j this is this entry is at i eth row and j eth column, this is called elementary matrix .

So first thing to note is when I; let me write because I want running indices.

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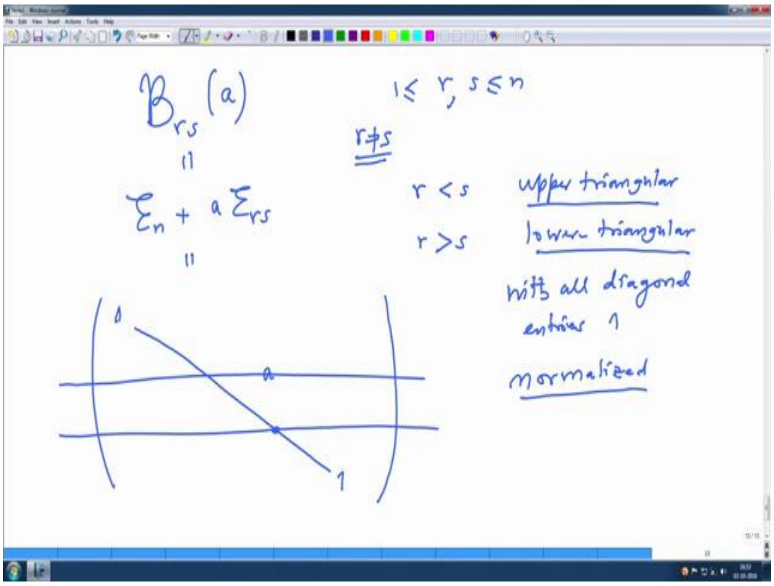

So, I will write B r s. So, r and s; they are in between this, this is E n plus a E r s, r sth entry is a and everywhere it is a default. So, first thing to note is if r is less than s, what did we do? Here this is ones on the diagonal and a somewhere a can be a or here. So, if the row s is bigger than it is upper triangular because this is r th row this is r th row, this is s th row then we are adding this 1 in the s th row, this picture is not very good. So, let me rub and draw it. So, this is the matrix ones on the diagonal and s suppose it is later than r then this is the diagonal entry s if r is here we are not changing s th row, we are changing r th row, this we are adding it here by multiple of s. So, it becomes upper triangular similarly if r is bigger than s then it is lower triangular.

So, these elementary matrices are either low and we never allow r equal to s r is not equal to s r equal to s will mean we do not allow r equal to s. So, these matrices have upper triangular or lower triangular and the diagonal entries are always 1 with diagonal entries with all diagonal entries yeah such triangular matrices are also called normalize normalized triangular matrix means either it is upper triangular or lower triangular and all diagonal entries of that matrix are 1.

Also they are clearly invertible see these matrices are invertible because diagonals are one and we have seen such matrices are invertible and they are also again if originally is upper triangular this is also upper triangular and the; it is a low triangular then it is a lower triangular. So, the inverse is i; in fact, in this case we can write down the inverse very well.

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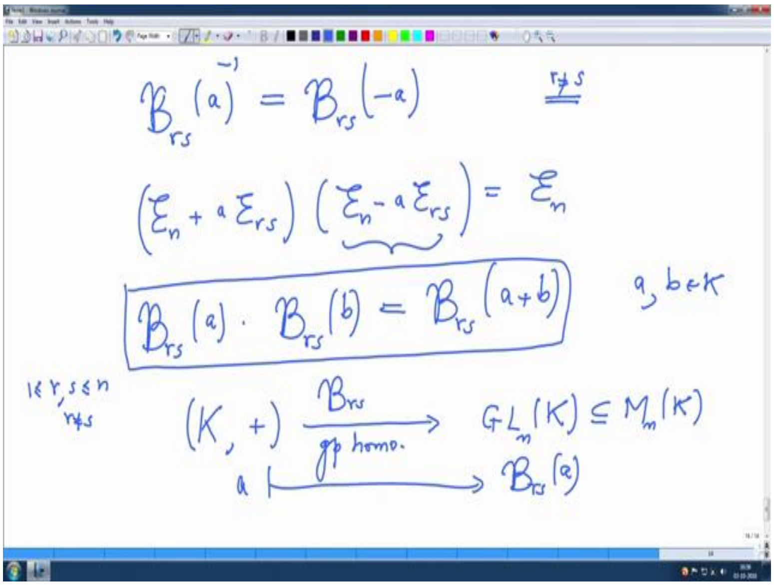

So, let me write it down. So, what is the inverse of B r s this inverse is nothing, but B r s minus a this is the inverse you can you can just compute see for example, if you want to prove this the way to prove this is look at E n plus a E r s this and multiply E n minus a E r s and compute this we will get that data; that means, this guy is the inverse of this and this computation you do it like what I said that is it.

More generally you can also write down a formula like this B r s a times B r s B this will be nothing, but B r s a plus B for any scalars remember r s r is not equal to s.

So, this means what this means this means these matrices you can think this means it is a group homomorphism this means a following though I want to put in it little bit fancy language let us take the additive group of the field K, K plus and then take g l n k this is the group of invertible matrices of order n this is the units in this row and we have a national map for a fixed r s; r s are fixed in between 1 and m are not equal to s then we have a map B r here B r s here in a goes to B r s a we have seen these matrixes in multiple and these what we wrote here this equation say that this is a group (Refer Time: 14:50) because addition here will go to the product here this is a group homomorphism .

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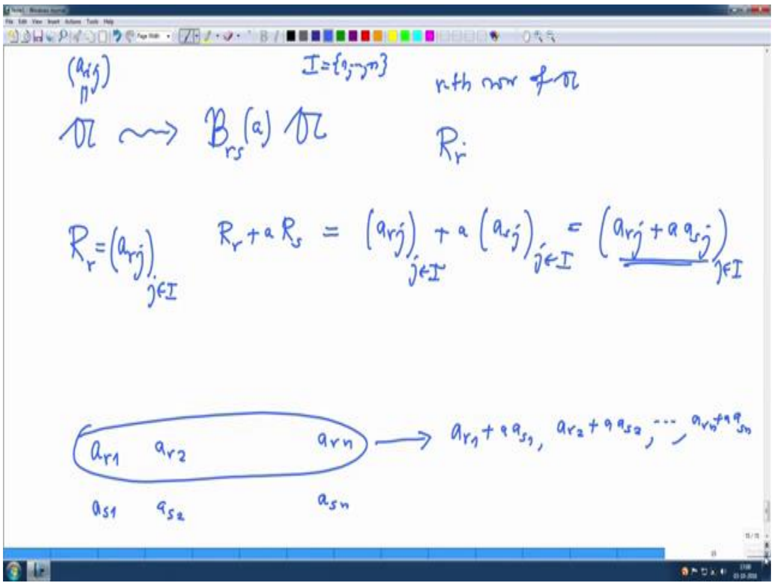

Now I want to check the following this is at least once we have to check if I multiply a given matrix a by B r s a on the left side it is like obtaining from a when you go here that is like changing the rth row of the matrix. So, rth row rth row of a, this rth row that is r let me write it rth row here R r and here. The new rth row will be rth row of a plus a times sth row of a this is what will happen to this new matrix and let us check this rth row is the vector if you call this matrix as a i j, I am checking once this is a r is fixed. So, r j this is the rth row j is 1, let me call j n i if you like i is our; so 1 to m, this is the rth row and what is this row? Write down this first rth row is a r j j in I and this 1 is plus a r s that is a s j j in I, but the we are added like this a r j plus a is component wise a s j, this is the new row of this new matrix and what did we do from the rth row. So, rth row is here a r 1, a r 2, a r n and s a through is I am assuming s is bigger for simplicity a s 1, a s 2, a s n and we have multiplied this sth row by a and added here. So, this row will now change it to this row this row will change it to this plus a times this. So, a r 1, plus a a s 1 comma a r 2 plus a a s 2 and. So, on a r n plus a a s n. So, this are this is surely this row watch it j is varying here.

So, multiplying a a by pre multiplying by this elementary matrix is equivalent to making a row operation similarly I will say that if I want to multiply from the right side that is like a column operation and the proof is similar with this. So, all these operations what do I want to do? I want to prove the following and also the permutation matrices.

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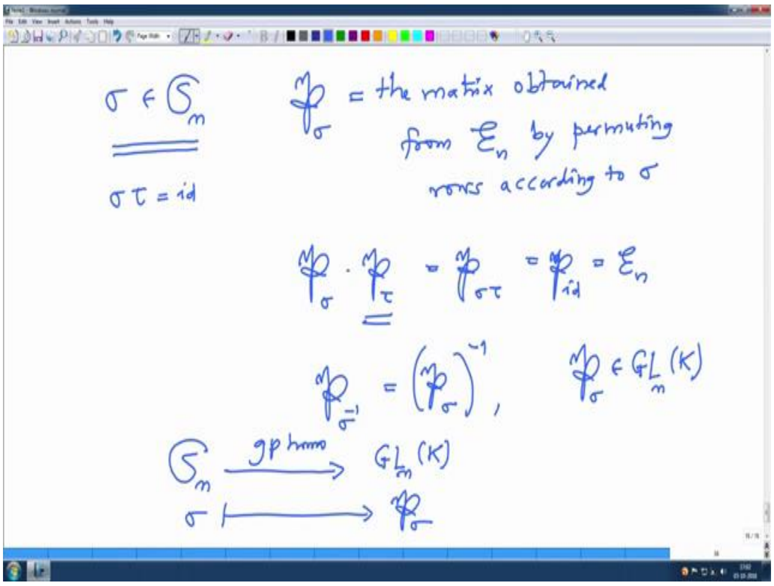

So, to define a permutation matrix if you have a permutation of the letters 1 to n then I denote p sigma p sigma is a matrix. So, this is the matrix obtained from the identity matrix by permuting the rows according to a permutation sigma and we want to say that this matrix is actually invertible, but that is again very simple because what you do is because permutations are in a group. So, there is a inverse to permutation. So, there is a permutation. So, that sigma tau is identity permutation.

So, therefore, we need to check the formula p sigma times p tau equal to p sigma tau which is identity which is identity the p identity p identity is identity matrix so; that means, this is a inverse of this so; that means, we have proved that p sigma p suffix sigma involves is p sigma and then the inverse of the matrix. So, all these matrices are invertible so; that means, p sigmas, p sigmas are actually invertible matrices not only that we have a nice group homomorphism s n to g n g l n k sigma is mapped to p sigma this is a group homomorphism where operation it needs composition operation here is the matrix multiplication.

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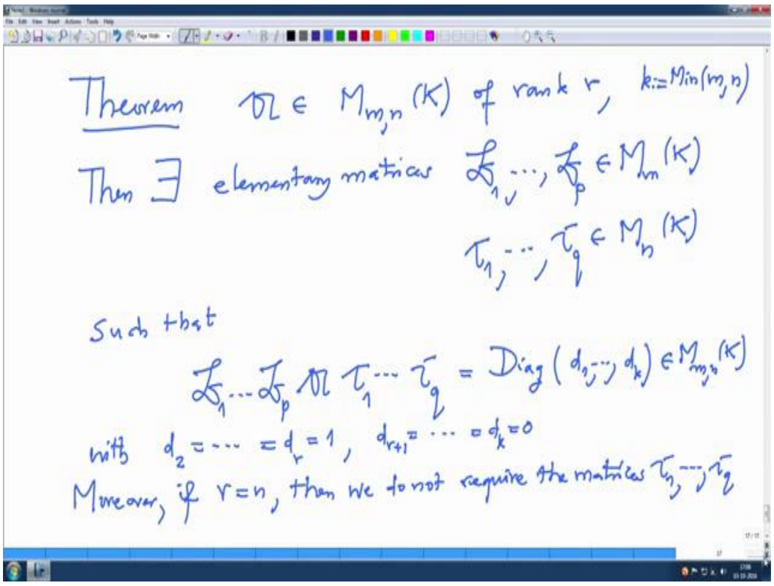

So let me come to the main theorem, which I would like to prove, I think I will see how much we can prove today. So, this is the theorem, I want to prove a is a matrix M m n K of rank R and let us put minimum is equal to K then there exist elementary matrices many of them. So, B 1 to B p elementary matrices are always square matrices. So, they are in M m k and also c 1 to c q, they are in M m k such that if I multiply a by this pre multiply a by this Bs. So, B 1 B p times a and from this side, I multiply by cs c 1 c q these product is a diagonal matrix d 1 to d k which is this matrix is this I have written diagonal, but it is the matrix here so; that means, depending on whether m is smaller or m is smaller than this diagonal matrix will look like.

Where with I can also arrange that d 2 onwards d 2 to d k they are all 1 d 2 to d r sorry, r is a rank T 2 to d r is 1 and d r plus onwards up to d k they are 0 if moreover if the rank is full and it is equal to n then we do not need cs we do not require the matrices c 1 to c q this is very very important theorem and I will show you many many applications of this.

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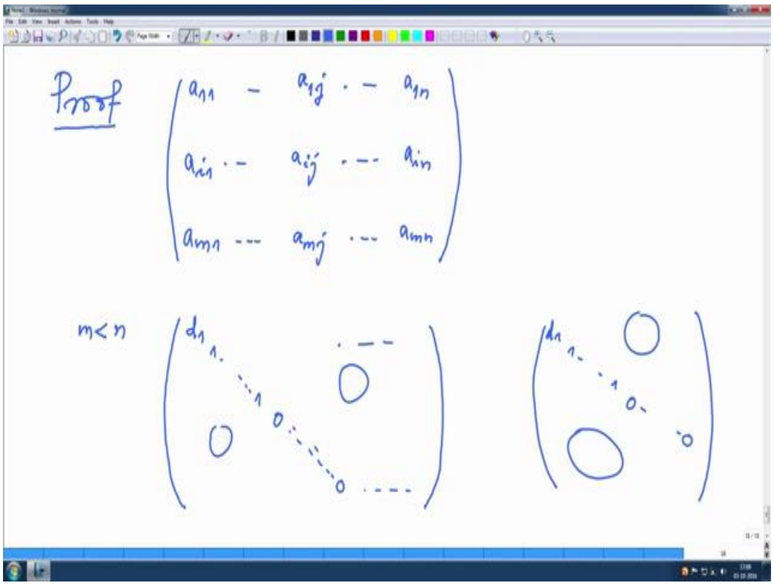

So, let us start proving it. So, proof first let us see what we want what we want. So, we have given a big matrix a 1 1 a 1 n a m 1 a m n and in between also. So, a i 1 a i n a 1 j a i j and a m j and how what do you want to bring you want to multiply from this side and that side. So, that the matrix becomes like this somebody here that is d 1 then 1, 1, 1, 1, 1, 1, 1. So, many places total r this is this is r cross r matrix then 0es and depending on where your m is smaller or m is smaller this is the main diagonal right. So, this is in the the picture I draw there the columns are more. So, in this m is smaller than n otherwise it will look like. So, now, in the other case it will look like B 1, 1, 1, 1, 1, 1 and then it will hit the other side faster and then they still. So, there is 0 here 0 here a 0 here 0 here.

So, that is what we want to achieve and what are we allowed we are allowed to make row operations and column operations and whenever we make a row operation we multiply from the left whenever we make a row operation a column operation we are multiplying from this or also here also allowed to change the rows and columns right, but I do not want to use that multiplying a matrix by a non 0 with a row by a non 0 scalar or a column by a non 0 scalar these operation I do not want to use only one and 3s I want to use. So, already in Gauss elimination what did we do. So, by changing the rows and columns interchanging we have brought this element non 0 and then we have used that to kill down everybody and. So, on right and then we did this.

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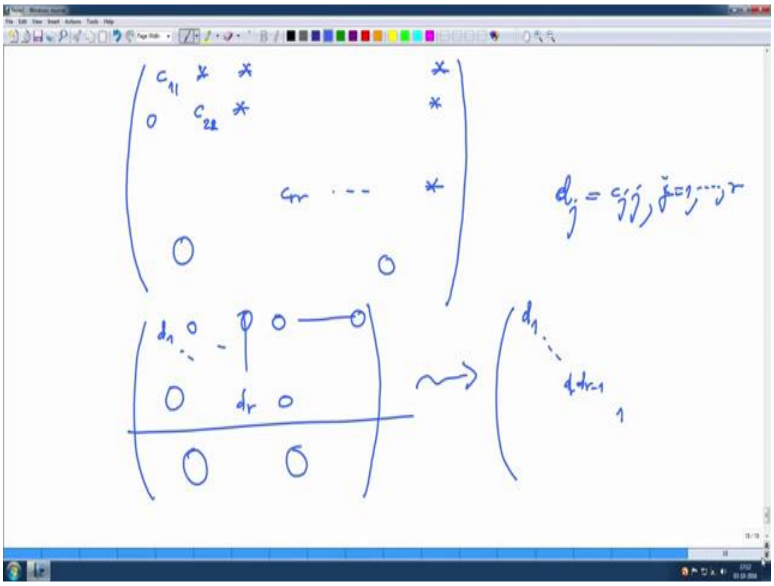

So and already then if we use those operations then we already get a matrix in this format c 1 c 2 c r r and 0s below everybody here 0 everywhere 0 here also 0 and there are some stars somebody here some stars are big these are this format we always bring it like a gauss elimination we make this we bring this entry non 0 as far as possible we use a rows and then kill the entries below that by using that row and elementary operations similarly this and once you have done for this block then already we know the rank of these matrixes are and these rows are linearly independent and therefore, columns are also linearly independent and therefore, all these guys the below here cannot be non 0 because if it is non 0 the column rank will go up, but we already proved column rank is row rank and so on. So, already this we can achieve where c j is c; c let me call c 11 c 2 2 c r r this. So, d j i am calling c j j from a j is from 1 to r.

These already achieved now what do you want to check well I wanted to check therefore, therefore, this matrix is I will correct also this is d 1 d r here this is 0 here and somebody here first of all note that now when I use backward row operations I will kill all these guys because these ds are none 0. So, all these I will kill, this is 0 and here now I use start using the columns. So, already the below that was 0es. So, I will start using this and it becomes 0 here all this 0 this everything 0. So, I actually got a diagonal 1, but now only thing only interested thing is I want to push this up, I want to this d r, I want to transform this matrix by again elementary transformation. So, that this becomes d 1 to d r minus 1 and d r and this becomes 1. So, slowly by induction I will push the diagonal entries to the upper diagonal entry this is what with the only non trivial part here.

So, that is I have written down that I have written down that exactly. So, that is very worth noting it this is the most important step which is.

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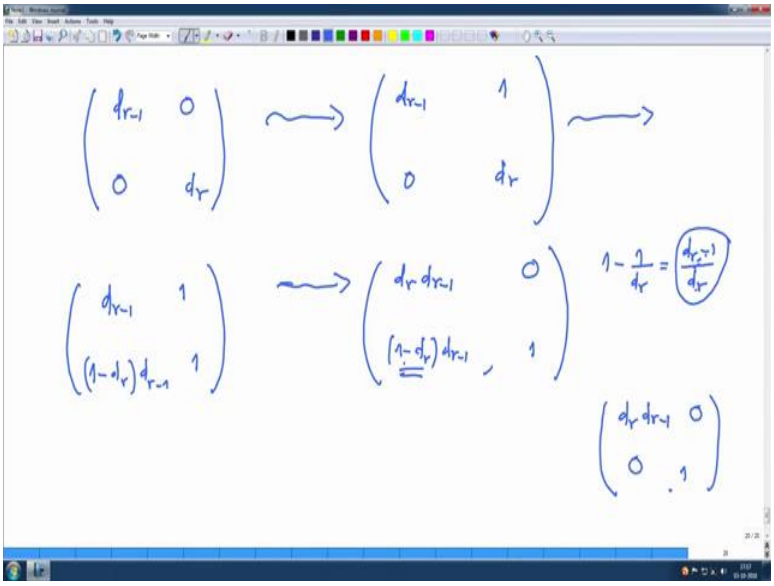

So, it only thing happening at this level, so, I will only write 2 by 2 picture, this is d r minus 1, this is d r, this is 0 and 0 and our problem is to arrive at d r, d r minus 11 and 0 0 this is the problem. So, what do I do? So, this d r is both these entries are non 0s that will this is non 0. So, I am going to multiply this row by 1 o d r and add it here. So, what will I get that way I will get this end this row will not change. So, this will be as it is and this will become this will not change this will become 1 here.

So, from here I want to do I want to kill this guy this d r so; that means, what should i do I should multiply this row by d r and subtract. So, this will not change, but this will change. So, what will it change this change to d r minus 11 this will not change and that will I have to multiply this by d r and subtract? So, this will become yes. So, what will it become 1 minus d r times no I do not want to make it 0 I will first make it 1, I want to make it 1; how do I make it one this, I want to make it 1 how do I make it 1 you multiply this row by 1 minus d r and add it. So, this is what we do I made it 1 here. So, I multiply this row by 1 minus d r and add it here we did this.

Now, we want to remove this one. So, subtract it. So, this will become this will not change now 1 minus 1 minus d r times d r minus 11 and this I wanted to make. So, I just subtract it. So, these I have to I have to subtract. So, this will get cancelled and this remains there with a plus sign see I managed this here now I have to get rid of this. So, that is now, what do we do you multiply this by multiply this by 1 minus 1 o d r. So, that is same as d r minus 1 by d r | multiply this by that and add it here. So, when I add it here this 1 and this one is negative sign. So, this when I multiply this by this, so as the denominator will go numeral will be d r minus 1 and multiplied by this when 1 add it here this is the opposite sign.

So, it will get cancelled. So, you push or elevate this see when I multiply this row by this, first d r will go away then this times d r minus 1 remains this nothing happens here and when I add it here this 1 and this 1 has a different signs. So, they will get cancelled. So, I get this that is it.

So, that is how we get the diagonal matrix of the required formula.