Linear Algebra

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Lecture – 46
LR decomposition

Welcome to these lectures on linear algebra, in the last lecture we have seen the use of elementary matrices to reduce arbitrary matrix of rank r into a diagonal matrix. So, let me repeat a statement what we proved last time and then we will improve this. So, theorem what we proved last time.

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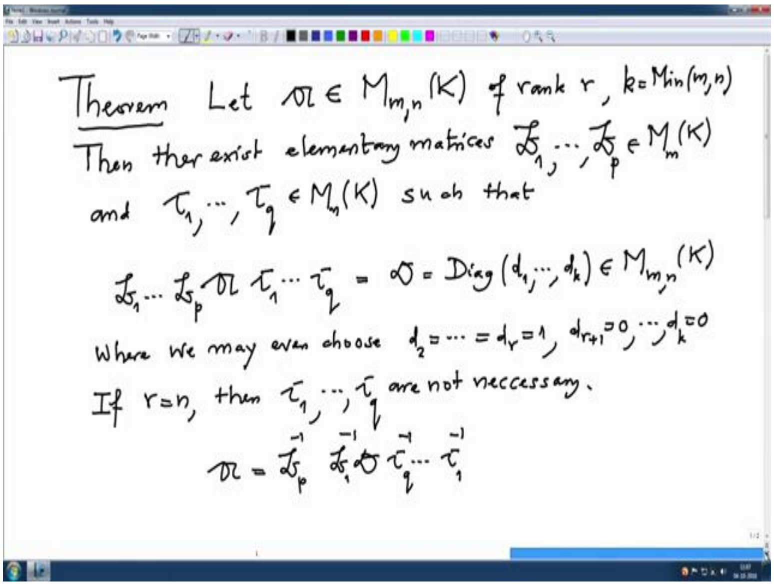

So, A be a matrix with entries in a field M cross n matrix of rank r, we have; let me remind also we have proved that the rank of matrix is the dimension of the column space of the matrix also equal to the dimension of the rule space of a matrix rank r and let us put K to be the minimum of m and n then there exist elementary matrices b 1 to b p and this will be of order M square matrices elementary matrices always square matrices.

Elementary matrices b 1 to b p of square matrices of order M and C 1 to C q of square matrices of order n such that when you multiply a by pre multiplying by b 1 to b p. This means we are making the row operations on the matrix A corresponding to these elementary matrices and when I multiply by post by C; C 1 to C q; that means, I am making a elementary column operations on the matrix a according to these elementary matrices and then you get a diagonal matrix D, this is the diagonal matrix diag a 1 to d 1 to d k which is in M m n K, this notation is that d 1 to d k is on the main diagonal of the matrix and the remaining integers 0 and also we can also where we can even choose we may even choose d 2 onwards till r to be one and remaining diagonal serves 0.

And in the case in the case of a square matrix, so if field rank is that is r equal to n case r equal to n case, this C 1 to C q are not necessary then C 1 to C q are not necessary this is what we proved the last time and this d one is that it actually will be the it is related to determinant when it comes I will say that time now I want to do it little bit even better. So, the next thing I want to prove is the following.

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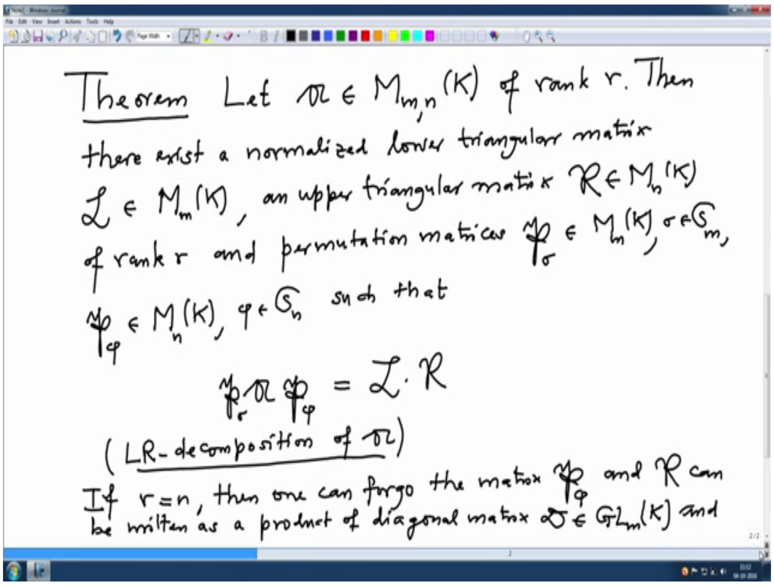

This theorem is also known as some people call it L R decomposition. So, this is before i go on i just wanted to say that the earlier theorem this will also allow us to compute the if one want to compute the determine inverse from here that we also allow us to compute to inverse because we know inverse of a diagonal matrix we also know inverse of a elementary matrices. So, this when you rewrite it, this will go A equal to I will multiply from this side by this inverse is of b 1 b 2 b p 1 by 1.

So, this will write as d here and then you multiply by 1 by 1 b 1 inverse first then b 2 inverse etcetera b p inverse and this side is first C q inverse C 1 inverse. So, a will have a representation between representation as a product of elementary and diagonal matrix and this will also allow us to compute the inverse of A; if A is a square matrix then you can just compute the inverse on the right side and that is the easy way to compute the inverse not only that this also proves in A square matrices case that these elementary matrices along with the diagonal matrix they generate all matrices. So, these I will say do when I talk about group theoretic properties of the G l n.

So, the next theorem now again let a belonging to M m n K rank of rank r then there exist normalize a normalized lower triangular matrix L in M m k recall normalize lower triangular matrix so; that means, only the below the main diagonal entries can be 0 above main diagonal entries are all 0 and normalize means on the main diagonal the entries are all one and an upper triangular matrix normalize again do not need upper triangular to be normalized upper triangular matrix what do we call it r which is in m n K of rank r and permutation matrices remember the notation for the permutation matrices as p suffix sigma where sigma is a permutation and this is in p m n k sigma should be a permutation on m letters.

And p 5 a permutation matrix of order n there 5 is a permutation on any letters such that the given matrix a when I multiply p by p sigma which make sense because order is correct here note that orders are correct. So, that product make sense similarly I can multiply on the post by p phi this is equal to L times R. So, some people some others also call this as a L R decomposition of a which is also related to popularly known as singular value decomposition matrix I will when I have a enough vocabulary to deal with that I will say it, but this is essentially for that.

So, if r equal to n; that means, rank is full rank then one can forgo the matrix p as in the earlier theorem case see you do not need this matrix then because you can manage with alone row operations this means and r can be written as a product of diagonal matrix in d in G l m K.

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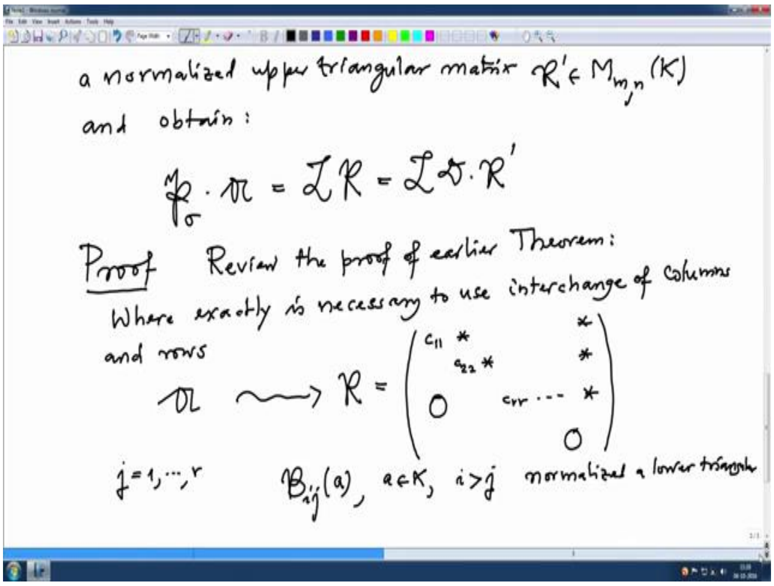

And normalize a normalized upper triangular matrix R prime in M m n K and obtain p sigma times a equal to l and then R, the original of the R that we have R, we have written it as diagonal d times R prime, this R prime is upper triangular matrix. So, therefore, the statement is from a given matrix A by applying or permuting the rows we can arrive at lower triangular normalize matrix d diagonal matrix and R normalize the upper triangle matrix and all this product make sense the orders correctly.

So, the proof is again reviewing the proof of the earlier theorem. So, all that one as to do is one has to be only careful about what operations we are using. So, proof I only want to review the operations on the matrix say we want to do either row or column when do they occur with the permutations and not the elementary matrices of type one row adding it to the multiple of one row adding it to the other row these are the most important for us.

So, review the proof of earlier theorem. So, what do you want to review where I want to review where exactly is necessary to use interchange of columns and rows? So, we remember when we in the proof of the earlier theorem in we have matrix say we have changed it to the matrix r and r has the form I will write that form r has the form something in the diagonal C 1 1, C 2 2, C r r below everything is 0 this part is also 0 below this everything is 0 and these are some entries these are some could be non 0 integral in the field this we can make it only with the row operations let us see, how did we do it?

So, first we make the top entry to be non 0 that that is where you will need possibility to interchange the rows and columns and then one we have done that then all below that you can make 0 similarly C 2 and so on. So, to understand exactly; that means, I have to do this for j equal to 1 to r. So, suppose we have done it for up to some j and what we do after that so; that means, if you have done up to here then after that below that we have want to make all the entries 0 once you bring one entry to be non 0 and thereafter that we have to bring all the entries to be 0 and that for doing that we are going to multiply only the row operations; that means, we are going to pre multiply the a with the matrix b i js b j.

So, we are applying this B i j a, these matrices where a is a scalar and i is lateral row. So, i is bigger than j. So, therefore, these matrices are already normalized and low triangular these are normalized and lower triangular we have to multiply by a on this side and may be the permutation matrix, but what I want to analyze, now I want to interchange these operations so; that means, whenever i need a permutation matrix after this; that means, if I have to multiply this operation further by interchanging the rows.

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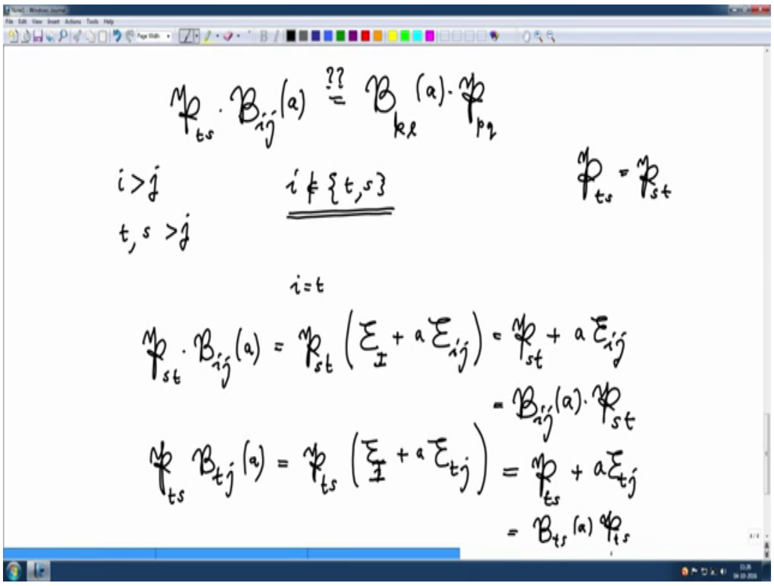

So, 2 rows P t s and times B i j a, this we have to multiply from the left to the a if I could interchange this, if I manage that this I interchange these operations then what will happen then this A, these altogether I will absorb those permutation matrices corresponding to the transpositions in the big permutation matrix and the before these elementary matrix will become, but because they are normalized lower triangular I will take their inverses and get here. So, that will become l. So, we need to only to analyze that what happen to this can I write this as this as the product the other way.

So, B is somebody B, what will I call it? B t naught t, I need now many letters B k l a times some other permutation matrix p is something p q, if I can manage this I have to analyze this, if we manage then I will; this l will get the inverse of these all these product elementary matrices and the universe of a lower triangular is also lower triangular normalize also it will remain and I will shift this to the other side and will get l that way and this p these are the corresponding to the transposition they will all collect together and that will become the permutation matrix. So, I only have to note this.

So, what are the assumptions on indices a and j here let us review that mean this index i is bigger than j first and these 2 indices t and s t and s both are bigger than j because only later stage i will interchange. So, these are the assumptions now we want to therefore, analyze this product. So, now, there can be 2 cases this I; either belong to t coma s not belong to or it is one of them or without loss I will assume i equal to t because if it is one of them then this matrix I, can B p t s is also P s t. So, I can without loss, I can assume it is the first index t or first index t.

So, in this case, what happen to this product? Let us compute, so P s t times B i j a just put the definition of B i j a that is P s t times identity matrix plus a times the matrix units e i j and now you just multiply it out this is P s t and I am taking this case. So, this i has nothing to do with t and s and also j i have nothing to do t and s because both indices are bigger than j. So, this interchanging will not disturb this. So, this is same as a e i j because multiplying this we like interchanging the s th and t th row.

So, this is same as saying this is same as saying B i j a times P s t they use little bit thing to check here this equality, but there is more or less obvious. So, this means directly this matrix p r s will commute with this in this case. So, therefore, I will just interchange their order and then gone in this case; in this case P s t times B i is now t. So, I will write t here t j a, but this is same as P s t into again E I E capital I, this is identity matrix plus a E i is t; t j where this is equal to P s t plus now a times a times this now t is a common. So, this will become no i wrote the other way this is i wrote sorry, sorry, this I wrote t s, this is t s of course, is this s t also, but I want to write t s.

This is t s and then plus a times E t j, but this is same as B, I am getting confused. So, this is not B t, this is B s because s and i change, no, this is correct b b t j a times p t s. So, it also commutes this is b t s this is s. So, they commute. So, i will make bring them on one side and then we are done because then we can write all collectively the permutation matrices absorb here and the multiply by the inverse of those elementary matrices and get this lower triangular matrix so on. So, that is this is r and then you are multiplying this side by this. So, now, after this we have to make the diagonal and so on. So, that was same procedure I have earlier.

And similarly for r equal to n k s, we can we can carry on the same. So, all these details see its bit computational to do it, but one has to do it and all these I will write precisely in the note. So, that even if there are some little errors here we can check there alright now I want to discuss what is known as elementary divisor theorem.

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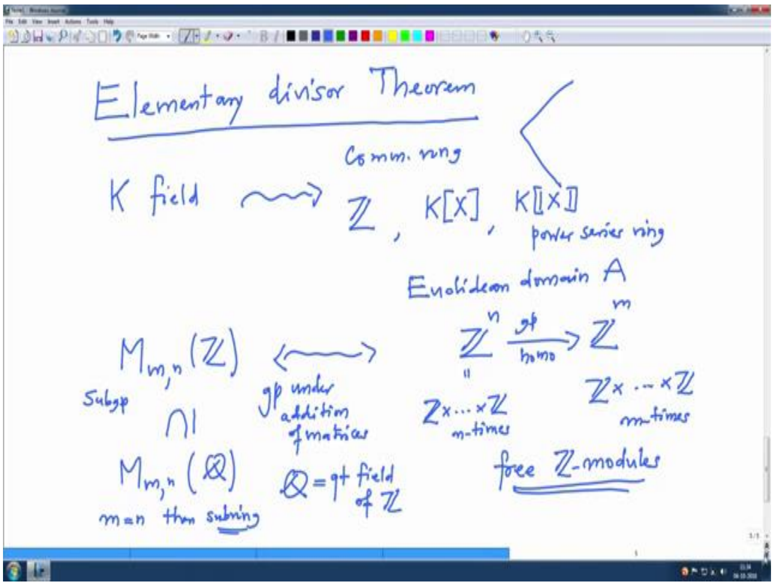

This is very important elementary divisor theorem and I will give 2 applications of this one in a theory of finitely generated Abelian groups and the other will be in number series solving systems of linear Diophantine equations both are of very important for many kinds of many scientist and many engineers especially solving Diophantine equations.

So; that means, we are dealing with now not the field. So, I want to replace a field by a nice ring commutative ring. So, Z integers that will give us a applications to the solving linear Diophantine equations or also I will replace it by a polynomial ring over a field in one variable k x, this is the polynomial ring over a field k in one variable x or also I will replace by a power series ring in 1 variable over a field this is a power series ring formal power series ring or more generally you can consider these are very special cases of the class of quantity livings called Euclidean domains a this all we have seen that this ring of integers is Euclidean domain with a. In fact, the usual divisional rhythm will give the Euclidean function here also the same only thing integers are replaced by polynomials and the degree will give the Euclidean and here it is even better. So, this is all these rings are very important this is study for number theory this is study for algebraic geometric or even polynomials in general this is also local analytic geometric will be the powers series ring.

so what are what are we considering we are considering matrices over this for example, i will stick to this and similar thing similar thing you can do it for all other rings replacing divisional algorithm in general by Euclidean functions. So, we are considering a matrices m n Z these matrices also you can think as group homomorphism from the group Z power n to Z power m group homomorphism of additive groups see these are this is a direct sum of or direct product of Z n times this is a direct product of m Z n times n times and this is direct product of Z m times.

And these modules these have the basis these are the standard basis. So, these are actually free Z modules. So, all the time you were using that vector space has a basis that they are given here for such rings. So, either you think this or thing matrices that that is equivalent the same proof each matrix will give you a homomorphism and each homomorphism will give you a matrix the homomorphism from Z power n to any group will be uniquely determined by its values on a standard basis and those you write in terms of the standard basis equation and those coefficients will give you this matrix.

So, this is group this see this is a group under matrix addition under addition of matrices that is in the component way addition and therefore, it is it is a Z module under a component wise scalar multiplication. So, it is not a subring its not a algebra, but this is you can think of this is a sub group of m n q where q is the field of rational numbers which is the quotient field of Z.

So, this is a sub group this one is a sub group and if you take square matrices m equal to n then actually it is a subring then subring because then you can multiply matrices and with that it is a subring; that means, what I am saying is if I have 2 square matrices and if i with integer entries and if I multiply them you get again a square matrix with integer entry; that means, it is a subring and therefore, when I consider a matrices with integer entries I can also consider them as a matrices with rational entries.

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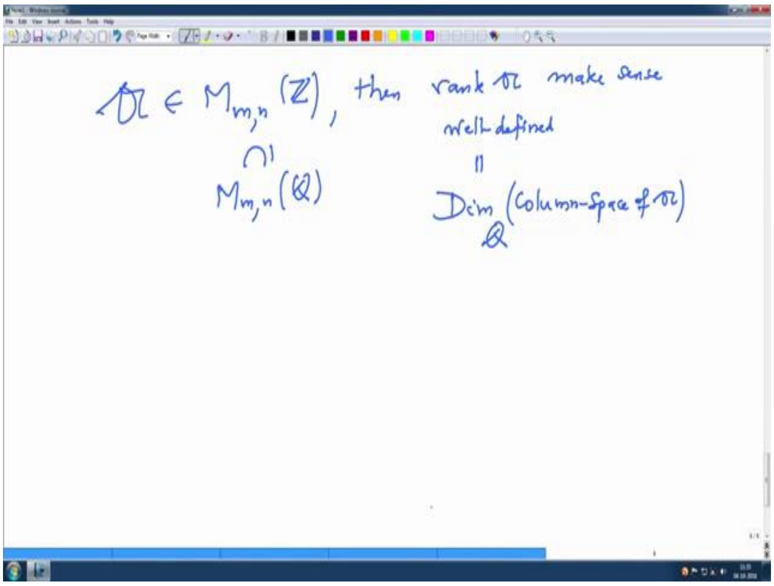

Where now I will apply our theory for the field k because Q is a field and therefore, any what is more important now what remark I want to make is if I have matrix a with integer entries then rank of a make sense when you consider a as actually an element in M m n Q because you are showing it over a field in the rank make sense. So, when I have a matrix here I will think it is a matrix here and rank will make sense and that this rank is well defined it is an integer and so that is what it is that is the number of Q linearly independent rows or also equivalently number of Q linearly independent columns, but it is Z linearly independent and q linearly independent are closely related. So, that is a rank I will keep saying. So, that is this rank is also nothing, but the dimension over q as a column space of which is also same as a row space also. So, we will we will make a break and then we will continue after.

Thank you.