Linear Algebra
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Lecture – 49
Canonical cycle decomposition of permutations

Come back to this Linear Algebra course. In the last lecture I have defined the permutations and we have been seen the permutation group.

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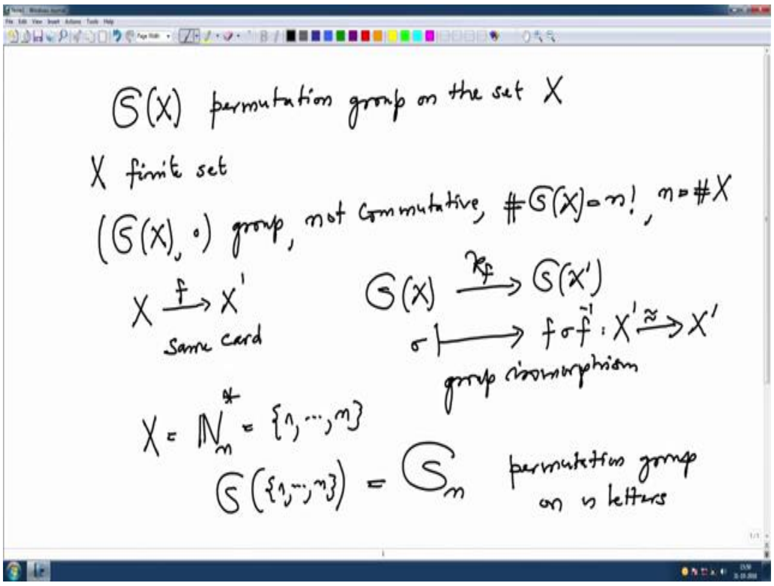

We are studying the group: S X permutation group on the set x and we will assume this x is a finite set. And what we have seen is; let me summarize what we have seen so far. We have seen that this S X group under composition. This is not commutative group. Also we know the cardinality of order of this groups; that is cardinality of S X is n factorial, where n is the cardinality of x.

We have also seen that this group does not depend on x, but it depends only on the cardinality x. That means, if x and x prime are two sets with the same cardinality; that means there is a bijective map f from x to x prime. Then this map f induces an isomorphism from S X to S X prime. This is I have denoted by kappa f. That means, any sigma goes to f inverse sigma f, this is map from x prime to x prime. And if sigma f is bijective, this is also bijective so that means it is indeed in element here. And this kappa map is a group homomorphism.

So, it is a group isomorphism. So to study this permutation group without loss we can assume that x is our standard set which has N elements that is this 1 to n set. And in this case I will denote s of 1 to n by shorted notation s suffix n. And this is called permutation group on n symbols; on n letters or n symbols. And we want to study this group and last in the last time I have motivated you that if I want to study finite group theory it is enough to study this group in a very detailed way.

So, if you have a good knowledge of the permutation groups then you will get knowledge about arbitrary finite group. Because, any arbitrary finite group is a sub group of this is S n for some n that was k (Refer Time: 03:42) theorem.

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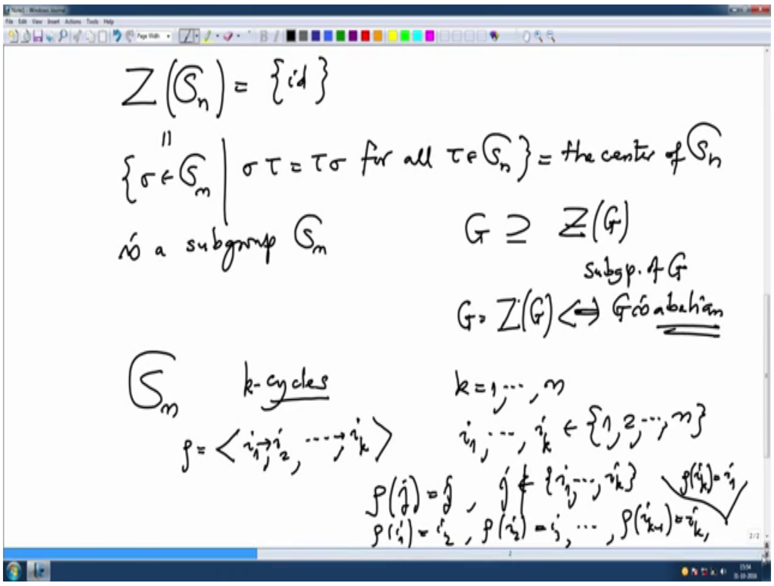

And also we have noted that the centre of this group is trivial; centre of S n is only identity. Centre means all those elements sigma in S n all those permutations which commutative to every other permutation, sigma tau equal to tau sigma for all tau in S n. This is called the centre of S n. This is clearly a sub group; is a sub group of S n. In general if you have a group G, then the centre is set of all those elements of G which commutative with every other element of G; this is a sub group of G.

And it is the equality G equal to Z G if and only if G is abelian. So, in some sense it measures the commutativity of a group in general this is a sub group. If it is equality that equivalent say G is abelian; abelian means any two elements commute. Now in the next few minutes we are going to study some elements of this group S n, namely the cycles.

Cycles are special permutations. So, what is a cycle? It is a k-cycle, where k is in between k is from 1 to n.

So, k-cycles means the special permutation. That means, this is usually denoted by this notation i 1, i 2, i k, where you fix elements i 1 to i k in between 1 to n. And this permutation means, whatever the letters are mentioned here only they move under this one let us call this as a row. That means, think row as a permutation which maps first of all any j to j, where j is different from any one of this, where j is not in the subset i 1 to i k. So, all the j is outside these are fixed. And sigma of i 1 is i 2, the next one, row of i 2 is i 3 and so on; row of i k minus 1 is i k. So, this goes under this under row up to here and the last one goes back to the first one. And row of i k is i 1. This clearly a bijective map and it is a permutation, this permutation is called a cycle; k-cycle.

And why it called k-cycle? You can also diagram wise you can write it.

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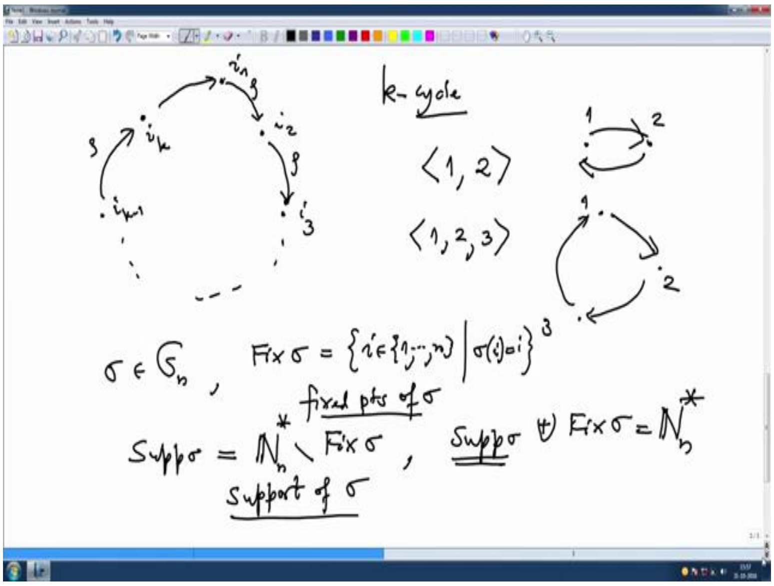

So, i 1 is here, i 1 goes to i 2 this is i 2; i 2 goes to i 3, this is i 3, this is row and keep doing like that i k minus 1 is map to i k and i k map back to i 0. So, it looks like a cycle. That is why it is called a k-cycle and k because the number of elements which are moving is k. For example 1, 2: 1, 2 is a cycle which maps 1 to 2 and 2 back to 1. This is cycle of length 2. If we have 1 to 3, then we have 1, 2, 3. So, 1 goes to 2, 2 goes to 3 and 3 goes to 1; so it is a three cycle and so on. So, these are very important in the description of arbitrary permutation.

So, for example, I want to introduce now notation: given arbitrary permutations sigma in S n I want to call all those elements i in 1 to n which then move by sigma that is sigma of i is i these are fixed points of sigma. As a map it does not do anything, it keeps them to themselves. So, the real sigma can be understood outside this set what happens. So, the support of sigma is by definition compliment of these fixed points of sigma in 1 to n. So, N star n minus fix points. This is called the support of sigma, and these are called fix point of sigma.

And there union is obviously; support sigma union fixed points, this first of all disjoints union because nobody in common and this you get the whole. So, to understand a permutation one has to understand its support.

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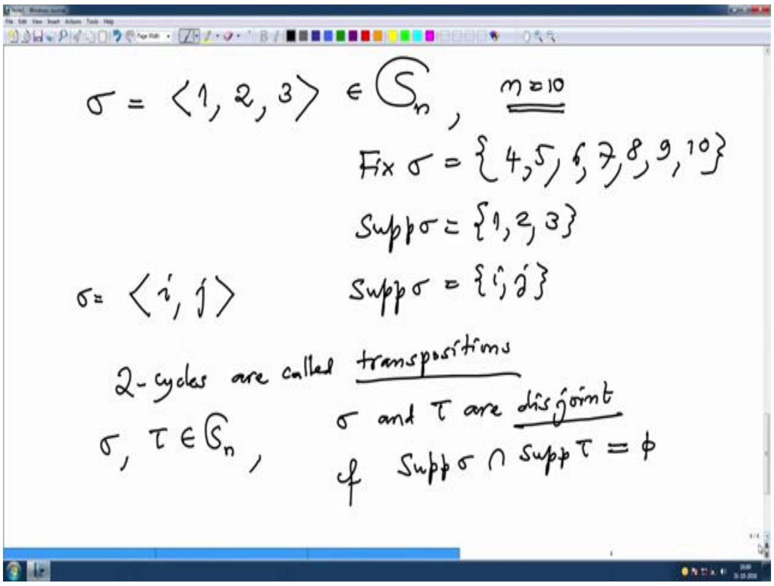

For example: if I write sigma as a cycle let us say 1, 2, 3 and say this is a permutation on n letters, where n could be say 10. This means we are only seeing 1 2 3; that means, how other than 1 2 3 every element is fixed. So obviously, fix points of sigma is which they do not which are not listed here and up to n o so that is 4, 5, 6, 7, 8, 9, and 10; this are all fixed points. They are unnecessary to they are not shown up in a notation because it is unnecessary burden on notion to say them.

And these are the support is the remaining one. So, support is seen to be 1, 2, 3. For example, you may write i comma j which is sigma, then obviously the support is only two; support has only two elements i and j. And the letters which do not appear here they are fixed that is why they are called fixed points. Also the two cycles are called transpositions.

Now our problem is to write every permutation as a product of cycles, and not only arbitrary cycles but the disjoint cycles. So, first let us recall a definition: if I have two cycles sigma and tau i 1 is k-cycle and the other is a l cycle we call them disjoint sigma or this is this definition works for any permutation so I write only first cycle. Take arbitrary permutations. Then we say that sigma and tau are disjoint if they supports are disjoint; support sigma intersection support tau is empty then you call the permutation to be disjoint.

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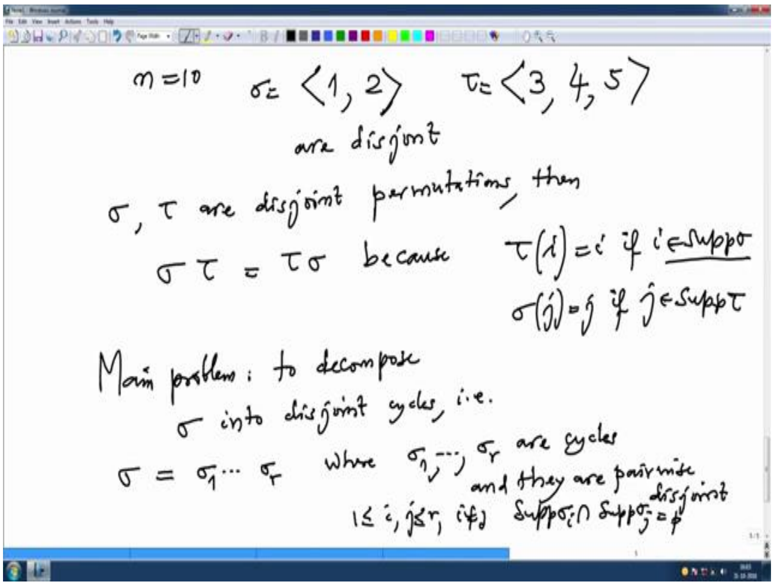

For example if I take a permutation n equal to 10 and suppose I take two cycles 1, 2 and the other is three cycles 3, 4, 5. This is sigma this is tau, then the support here is 1 2 support here is 3, 4, 5 they do not have anything in common, therefore sigma and tau are disjoint. And the advantage of disjointness is if sigma and tau are disjoint cycles disjoint permutations, then they commute; then sigma tau equal to tau sigma. And this is obvious because the elements which are moved by tau, they are not moved by sigma and element which are moved by sigma there are not moved by tau. So, this equality is obvious.

So, this is because tau of any element i is tau if i is in the support of sigma; i in the support of sigma so it cannot be in the support tau therefore tau i is to be i. And similarly sigma j equal to j if j is not in the support of tau; j is in this not in the support. If j in the support of tau that means, j is moved by sigma then j cannot be moved by sigma. Therefore, it is fixed. Now, we want to decompose.

The main problem is to decompose sigma into disjoint cycles. What are that mean? That is in the notion; I want to write sigma as sigma 1 product, product sigma r, where sigma 1 to sigma r are cycles and are disjoint cycles and they are disjoint; they are pairwise disjoint. That means, for any i and j in between 1 and r i not equal to j support of sigma i and support of sigma j they are empty.

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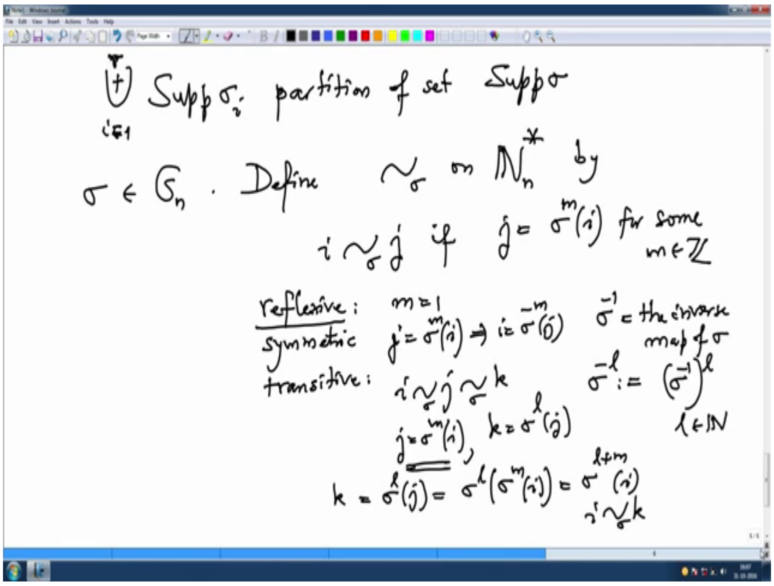

Or in other words, this support sigma i this union is disjoint i is from 1 to r; this is a partition of the set support sigma. And because they are disjoint I can reorder also them that the product will not change. And this is very easy. So I have to give a partition of the set. So, this is very easy. So, only thing we have given is a sigma; sigma is a permutation given. Now the sigma defines equivalence relations; define I i would denote tilde sigma on the set 1 to n by what definition i is related to tilde sigma j if j equal to sigma power m of i, for some m in integers. Remember when I write a negative integers sigma minus inverse is sigma minus 1 is for example the inverse map of sigma; sigma is a bijective map so inverse make sense. So, sigma inverse is that.

And sigma minus power any l is by definition sigma inverse power l, so where l is natural number. So, that will define all powers of sigma. All integer powers of sigma are define. Now let us check that this is an equivalents relation. So, what we have to check?

We have to check three things: reflexive, symmetric, and transitive. So, for to check reflexivity we have to check i is related to i, what well I will take m equal to 1. So, in this case take m equal to 1.

Symmetric means: if j is sigma power m i then I will be equal to sigma power minus m i. If I equal to sigma power m j, j equal to sigma power m i then I will be equal to sigma power minus m j. Just apply sigma power minus m on both sides and then you get sigma power minus m; sigma power m times sigma power minus m is identity map (Refer Time: 19:17). Transitivity: if i is related to j under sigma and j is related to k under sigma this means j equal to sigma m i and this means k equal to sigma l j. Then you apply sigma l to this equation, you will get sigma l j which is sigma l of sigma m i which is sigma l plus m of i. On the other hand this is k. So, k equal to this so that show that i is related to k. So, we have checked that this is an equivalents relation on the sets 1 to n.

So, the equivalence classes are subsets. And equivalent classes will form a partition.

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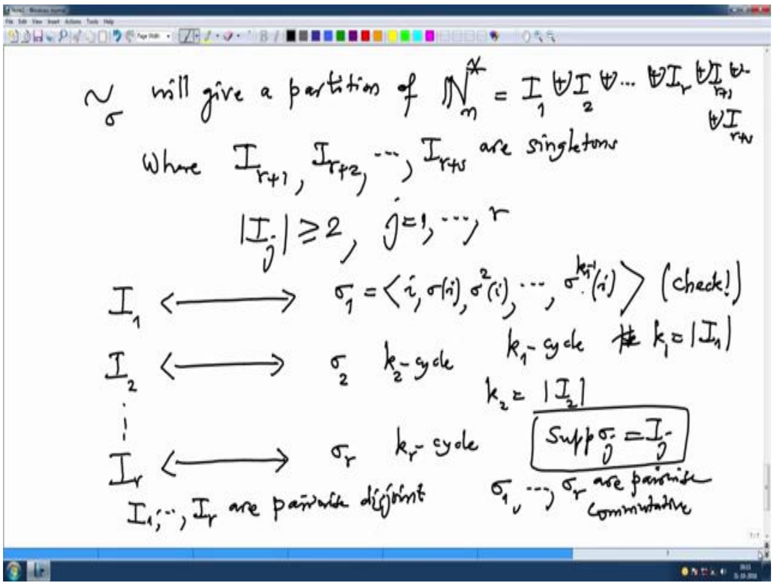

So, this relation will give a partition of the set N star n. In these partitions some element could be single turn some element may not be single turn. So, in this we have written this as I 1 disjoint I 2 disjoint disjoint I r. And the remaining ones I will show that I r plus 1 union union I r plus s; where I r plus 1 is single turn, this I r plus 1 I do not need to write what it is, I r plus 2 I r plus s are single turns. And the I 1, I 2, I r will have cardinality. All I j will have cardinality bigger equal to 2 for j equal to 1 to r.

And each one of them I 1 will corresponds to sigma 1; sigma 1 will be the cycle. What cycle? Now you star with any element of I 1, I then apply sigma to that that is sigma of i then sigma square of i and so on. And then there will be a time where you will have to come back to i so that is sigma power k 1 minus 1 i and then you get back to i. Because, when you want to check this you start writing distinct one and the first time it is not distinct it has to go back here, that is very easy to check. I will live it for you to check that.

So, I 2 will give. So, this I 1 will give and all elements of I 1 are exerted. So, I 1 will give k 1 cycle, where k 1 is the cardinality of I 1. So, I 2 will give sigma 2 which will be k 2 cycle same argument. So, where k 2 is cardinality of i 2 and so on; i r will give sigma r which will be k r cycle. And note that, because this I 1, I 2, I r are different these are commutative; this sigma's will commute with each other. Since, i 1 to i r are disjoint are are pairwise disjoint this permutation this cycles k 1 to k r sigma 1 to sigma r are pairwise commutative because support of sigma j is precisely I j because of this the support do not inter set therefore, those cycle will commute.

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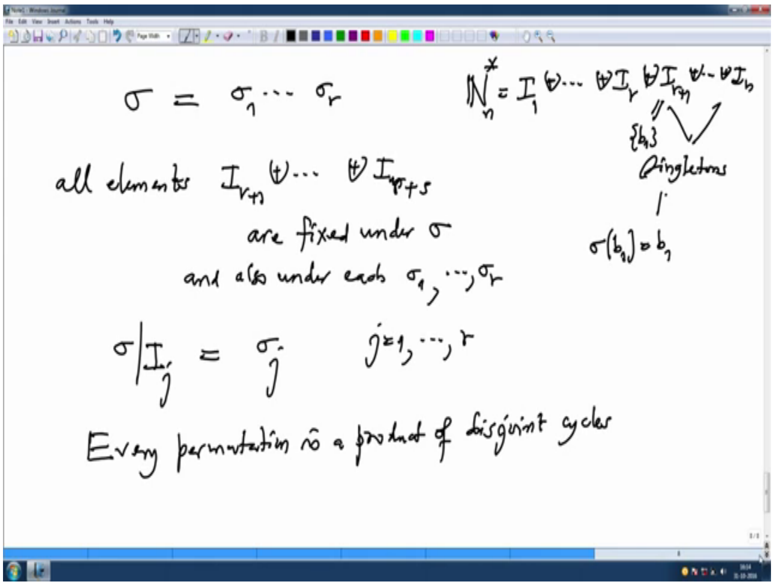

Therefore, we only have to check this equality, sigma equal to sigma 1 to sigma r. To check this equality remember we have decompose this set into a disjoint union I 1 I 2 I r. And the remaining guys are single turns. These are single turns means; that means, sigma of them, so when I apply sigma form example if I call I want to call this element as b 1 then sigma of b 1 has to be b 1, because that is equivalence classes b 1. Similarly all this guys are fix under sigma.

And these are the only elements of equivalence class; that means sigma if you apply both sides they are equal here; sigma of b 1 is b 1 on this side also its b 1. Therefore, all elements of I r plus 1 union this union this r plus s; are fix under sigma. So, there also fix under this and also fix and also under each sigma 1 to sigma r. So, when I evaluate this side on n element from this set it is fixed that it is itself and here also fixed, therefore the equality happens on this set.

And the remaining set; now I only have to check that sigma restricted to I j is same as sigma restricted to j, for every j 1 to r. But that is also clear because, sigma restricted to I j this right had side only sigma j is moving and the other cycle they are keeping it fixed, so therefore this equality.

So, we have proved that every permutation we can write it as a product of disjoint cycles. So, every permutation is a product of disjoint cycles. Let me give some small numerical example so that you feel happy.

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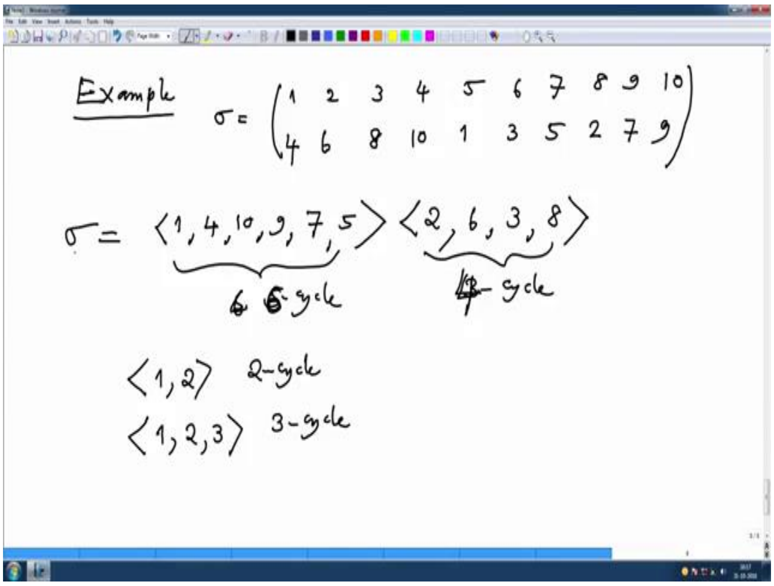

So, some small example let us write it. What is a procedure we will write the recipe. So, sigma is a permutation on 10 letters I am doing it, so 1, 2, 3, 4, 5 6, 7, 8, 9, 10. So, 1 goes to 4, 2 goes to 6, 3 goes to 8, 4 goes to 10, 5 goes to 1, 6 goes to 3, 7 goes to 5, 8 goes to

2, then goes to 9 goes to 7 which remaining now 1 2 3, 4, 5 6, 7, 8, 9. So, this is a clearly a permutation, because has you would have noted that I check that the next row is all distinct elements from 1 to n.

Now, what is our recipe? Our recipe is start with 1, and look at its image under sigma that is 4, write it 4. To get 4 where does it go under sigma 10? So, write the next element 10. Look at 10 it goes to 9, 9 goes to 7, 7 goes to 5, 5 goes to 1, goes back to 1, so complete it. Now look at who is left in this? 2 is not appearing here. So, start with 2: 2 goes to 6, 6 goes to 3, 3 goes to 8, and 8 goes to 2 back. Therefore, sigma is the product of these two cycles; these are cycles this is 1, 2, 3, 4, 5 this is a 6 cycle, no this 5 cycle and this is 3 cycle.

Now is it correct or convention 5 or 6? No, this is 6 cycle that is our convention, this is 4 cycle. Remember the 1 2 is a 2 cycle. The number of elements in that cycles 2 cycle. Number of elements if it is 1 2 3 number of element is 3 so it is a 3 cycle.

Now, the next thing I want to discuses is how do you compute the orders. Next, when we come back after the break we will explain how do you compute the order of a cycle. And therefore, once you compute order of a cycle we will be able to compute order of an arbitrary permutation. So, we will come back after the break.