Linear Algebra
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Lecture – 50
Signature of a permutation

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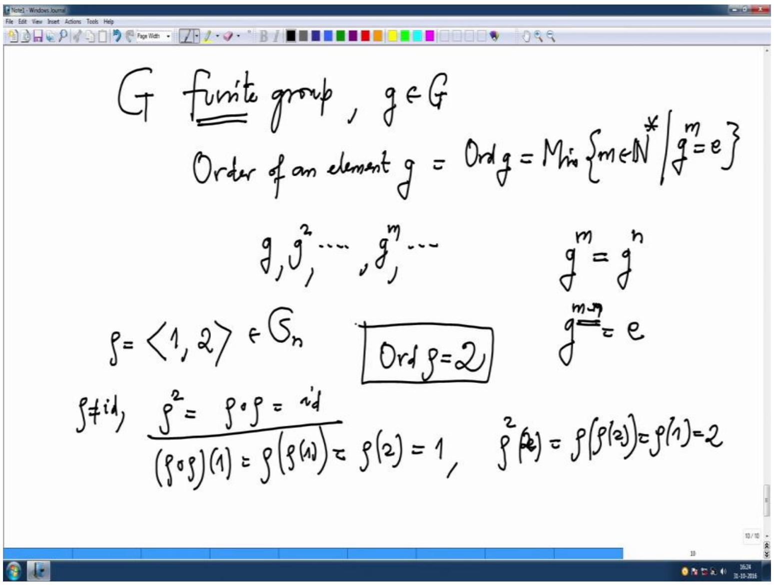

We will compute order of some element see in any finite group G, if G is the finite group and g is an element in g it is most important to decide what is an order of g order of an element g which I will denote odd g, this is the smallest this is the minimum. This is the minimum of m in N star such that g power m is identity the first time when it becomes identity; obviously, such as minimum exists because; obviously, if I start with g; g square and go on all this elements cannot be different because g is the finite group. So, they will be 2 of them at least 2 powers becomes equal. So, g m will be equal to g power n and then whichever is less power I will multiply by the inverse of that then I will get g m minus n is definitely identity.

So, there exists at least 1 non natural number for which the power becomes identity and such a power the minimum is called the order of an element g. So, for example, let us compute some order of some permutation. So, for example, if I will take for transposition 1, 2 in s n this mean only 1 and 2 are moving remaining integers are fixed if I if I call this a s rho what is rho square rho square is by definition on rho compose rho; that means, I have to compute these have to know what is value on 1 and 2 because remaining elements are fix under rho. So, I do not have to worry about other than 1 and 2 and 1 where it is going? So, rho compose rho at 1 is by definition rho of rho 1, but rho 1 is 2. So, these rhos have 2 which is 1.

Similarly, rho square of 2 is by definition rho of rho 2 which is rho of 1 because 2 goes to 1 and rho 1 is 2. So, rho square of 2 is also 2; that means, no one is identity on 2 is also same thing; that means, we have approved rho square is equal to identity map because all other elements are fixed under all. So, therefore, the 2 cycle as order what we have checked is rho is not identity and the next power is identity so; that means, what we have to checked is order of rho in the group permutation group is 2 and nothing special about 2 cycle even if you do 3 cycle.

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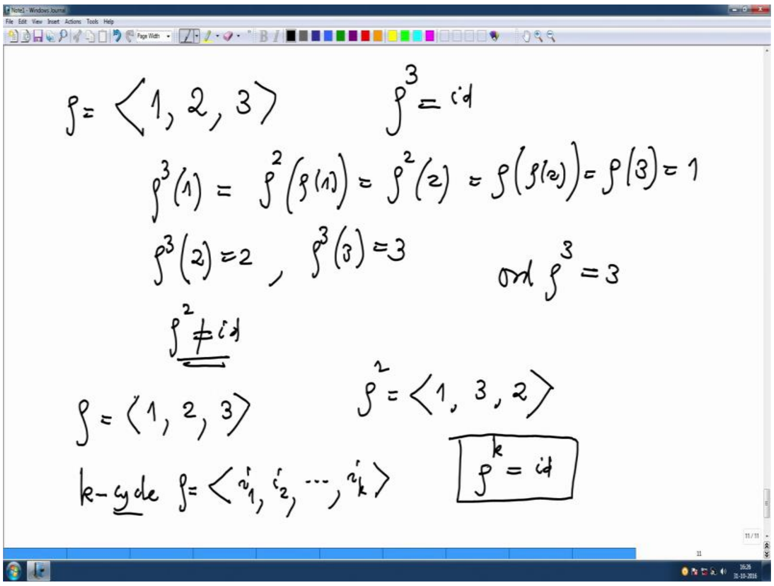

Let say 1, 2, 3, if this was rho I will ask you to check rho cube is identity that is very we have check you just have to check on 1, 2, 3, because remaining integers are fixed. So, we have to check rho 3 of 1 rho 3 of 1 is rho square of rho 1, but that is rho square of 2 that is rho of rho 2, but that is rho of 3 that is 1. So, it is fixing under rho 3 rho; rho power 3.

Similarly, check that rho cube on 2 is 2 and rho cube on 3 is 3. So, therefore, the order of rho cube is 3 you have to check that rho square is not identity, but that is very easy actually when you when I given a cycle like this what do you is want to write rho square you jump. So, 1 has to go to 3, 3 has to 3 is going to 1, but you have jumped 1 stage. So, 3 goes to 2 and 2 goes to 3 what you jump one step. So, this is the cycle this is rho square and rho cube you will order like that jump. So, similar thing, so I will not let me not calculate things. So, that is how one calculates the powers quickly.

So, in general if i k cycle i 1, i 2, i k, k cycle means the cardinality of this support is k, if this is rho then rho power k is this is in general we have to check, how do you check this you just check that on all each i 1, i 2, i k are identity they are fix and other which do not that we there is automatically fix under rho therefore, they fix under rho power k.

So, k cycle will have a order of k cycle is k order of a k cycle rho is equal to k that is the number of elements in this is number of cardinality of the support of rho.

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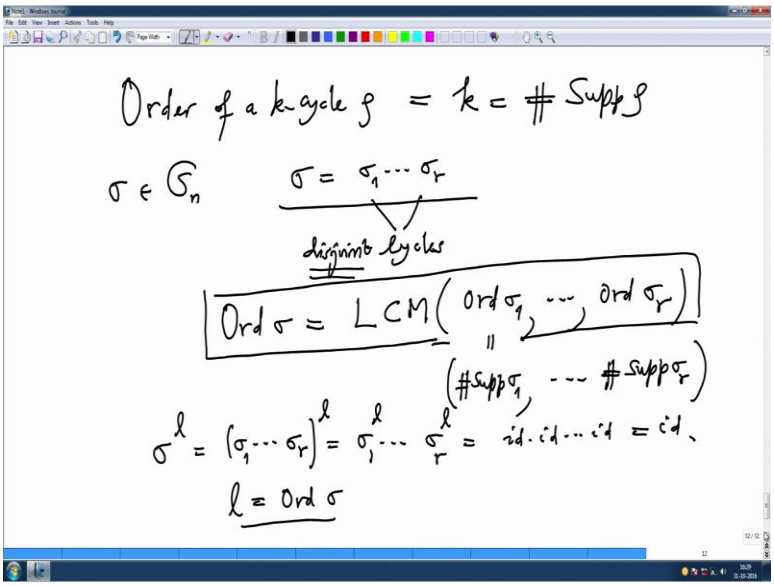

So, once we have this then given any permutation sigma in s n we have canonically decompose sigma into r factors sigma 1 to sigma r where this guys are cycles disjoint cycles and use of disjoint is because they commute each other. So, therefore, if I raise it to power; so I want to claim that order of sigma is nothing, but LCM of the orders of sigma 1 to sigma r and I know this order because it is a cycle, this is equal to cardinality of the support.

So, that is the formula for the order and how do you check this equality here because if you have a LCM then you and you raise to this equation to power l that is sigma 1 to sigma r power l, but because the compute I will take this element inside this is sigma 1 power l sigma r power l, but this l is the multiple of the order therefore, already that will become id this is become id this will become id and so on this will become id. So, this is id.

So, that shows that the LCM is and smaller will not work because. So, l is the order of sigma. So, nice formula order is computed by using the LCM of course, we one needs to calculate l c m. So, LCM is computed by using the prime decomposition which I will not address here of course.

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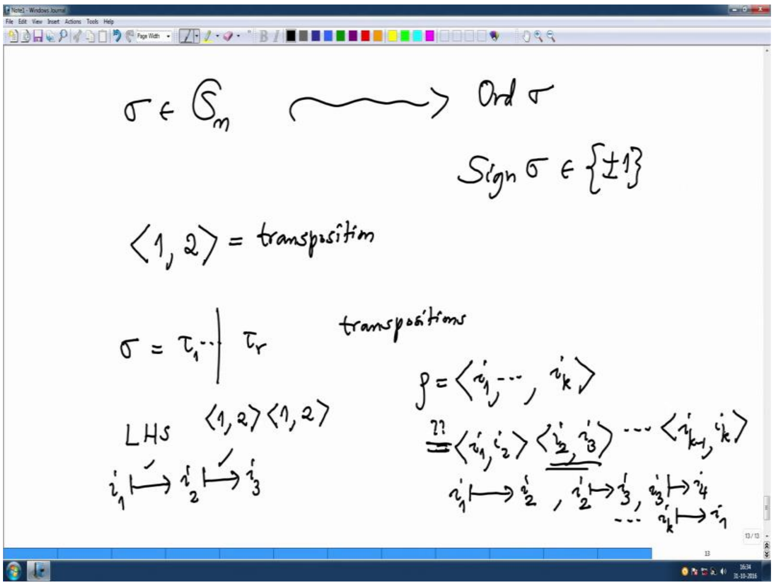

So, we know, so therefore, to each permutation sigma we have attached an integer which is the order sigma, now I also want to attached another I want to attach a sign; sign sigma this is a this should be an element in plus minus 1 and how it is attached.

So, first I will write down for a cycles what happens to cycles. So, for example, if you have 1 2, this is the transposition. So, this should be the number. So, given any permutation sigma I want to decompose this into cycle’s transposition. So, tau 1 to tau r transposition and count how many are they are transpositions; obviously, this decomposition is not unique any more like canonical decomposition because what my happen is like somebody might insert here 1, 2 and 1, 2 again this is identity actually. So, I have not; I mean certain identity, but it is this. So, I make club. So, this decomposition is not unique the number is not unique, but the parity is unique this is I want to proof.

So, to do that sigma is; obviously, product of transposition because for that I just I will write every rho is every cycle i 1 to i k this the product of transposition because if I can do that and already I know sigma product of disjoint cycle and each cycles are I will decompose into transposition. So, all together in each sigma I would have decompose as a transposition, but I cannot claim anymore uniqueness because this when I decompose in the cycle into transposition that the composition is not disjoint not anymore.

For example this 1 is i 1, i 2, i ,2 i 3, we will check this equality and. So, on the last one is i k minus 1 i k. Now to check these are equal this 2 products are equal let us check that. So, I have to check that what happen should images of i 1 to i k under rho on both sides well the left side and know i 1 goes to i 2 this is LHS and let us check now RHS also I want to check that i 1 where do i 1, let me check that where do i 1 go? Now here j remember, we are applying this first and then this then this and last this. So, under this i 1 is fixed because they are different elements.

So, i 1 is not one of them. So, i will be fixed here i 1 is also fix here, here also fix only i 1 is moving under this first transposition and they are i 1 goes to i 2. So, there is here is same. Now let us check where i 2 goes here i 2 goes to i 3 on the left hand side on the right hand side see as long as you do not reach here all this keep i 2 fix the first time i 2 move is this transposition and then i 2 move to i 3 and here i 3 is fix. So, i 3 goes to i 2 goes to i 3 so which is also here.

And similarly all other things i 3 will go to through i 4 same reason last 1 i k will go back to i 1. Let me check only the last one. So, where do i k i k goes to i k minus 1 here i k minus go to i k minus 2 and so on, it is goes back ward. So, i 2 will go to i 1. So, i k go to i.

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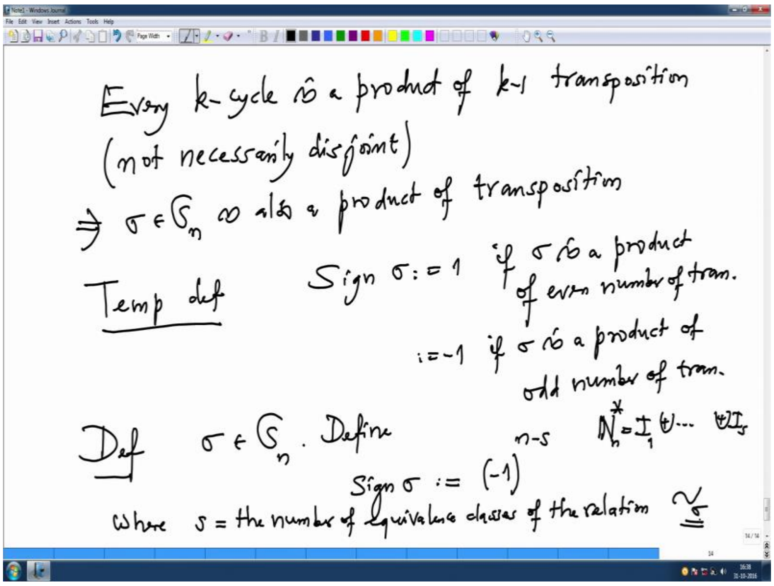

So, therefore, we have to check that every k cycle, actually we can write it as a product of how many transpositions 1 2 k minus 1. So, what do you have proved is every I will write the result every k cycle is a product of k minus 1 transpositions and not necessarily disjoint were not disjoint at all the see here, here this cycle and this cycle they have might to common and next one I think will be common and so on. So, they are not the disjoint and because they are not the disjoint you cannot demand uniqueness.

So, once in every case cycle is a product of the transposition in every sigma is also product of therefore, every sigma is also a product of transpositions because you take sigma write sigma as the product of the disjoint cycles and each cycle you decompose into k minus 1 transposition.

So, each sigma will be product of transposition now our problem is how do you decide how many and the only thing you can decide they either odd or even. So, that is what we want to prove now. So, this is the temporary decision I want I wanted to prove temporary definition I wanted to write sign of sigma is one if sigma is a product of even number of transposition and minus 1 if sigma is a product of odd number of transposition.

Now, we have to check that this definition make since so; that means, what do will have to check we will have to check that that if somebody writes sigma as a product of even number of transposition and somebody else also do it in some other method then the number should not change from even to odd; that means, the parity should be same this is the problem we want to we want to check either do that on then defined the said defined in it some other way and check this. So, I preferred to do the other way I will define sign which will be canonical definition and then we will check that this parity will remain the same.

So, the proposed this is not I do not want to defined this a. So, I proposed definition given sigma I want to define a sign. So, define sign sigma by minus 1 power n minus s where what is s where s is the number of equivalence classes of the relation equivalence in relation this sigma let, what is that? That means, we have decompose the said n star n into exactly is i spaces i 1 union union i's some of the could be single term and some of them are not single and only the non single term will gives you a cycle which is in the product of sigma and this decomposition is very canonical if this on it use a only a permutation sigma and nothing else.

So, let us first analysis this definition, what it is, alright. So, this is called a signature of sigma.

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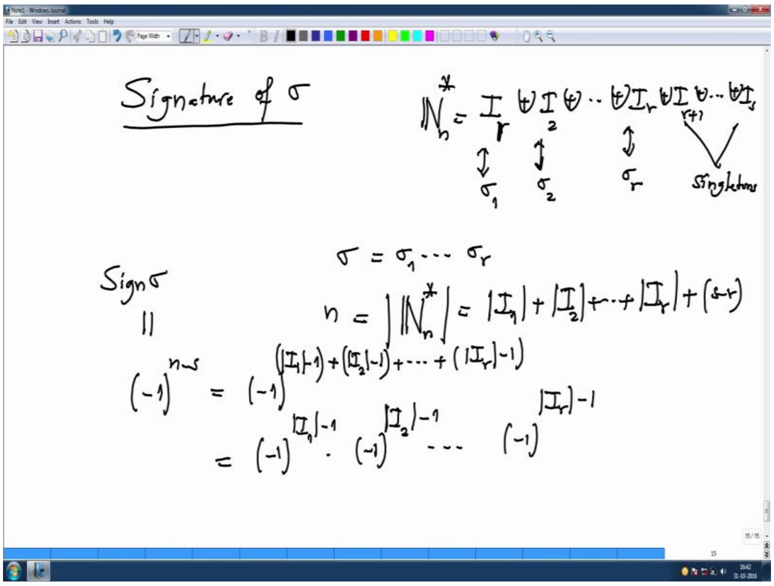

Let us write more precisely. So, what do we have done we have this n star n we have decompose to let me now write I r union i 1, I 1 disjoint in I 2 this disjoint mean I r, this is k cycle sigma 1, this gives a cycles sigma 2 this give cycle sigma r they not single tense and then the start single tense union I r plus 1 disjoint with always union I s and this guys are single tense. So, that do not contribute to this cycle decomposition cycle the canonicals are cycle decomposition with this sigma 1 to sigma r, we do not have appear because they corresponds to that identity and then what you know from this because this Cartesian this said as partisan into several pieces. So, definitely we know cardinality of n star n is cardinality which is n is broken into. So, many cardinality I 1 plus cardinality I 2 plus, plus, plus, plus, plus cardinality I r plus now, 111, how many times plus r minus s s minus r s minus r that is how the cardinality n broken into the pieces.

Therefore what is minus 1 because of this what is minus 1 power n minus s this is equal to minus 1. Now n; I will put it from here and the; this s is getting canceled with this. So, and then these r I will distribute one among each one of them this is minus 1 power cardinality I 1 minus 1 plus cardinality I 2 minus 1 I have just broken I am just writing n minus s plus cardinality I r minus 1, this is same as this, this, this plus and this minus 1 come in term exactly r times that is and this s is are shifted to this side. So, this is same as minus 1 power cardinality I 1 minus 1 minus 1 power cardinality I 2 minus 1 cardinality and so on minus 1 power cardinality I r minus 1 therefore, I would therefore, then this was our this one definition of sign of sigma.

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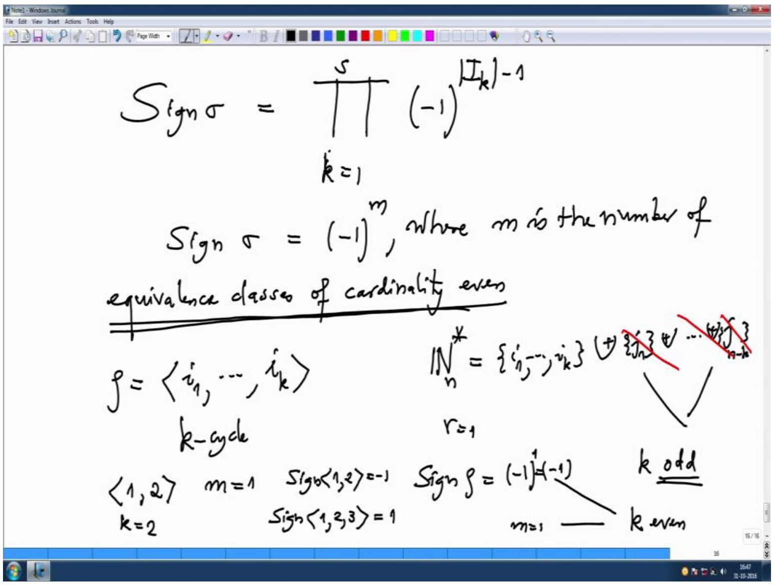

So, therefore, we get a formula. So, sign of sigma equal to product I is from a k is from 1 to s minus 1 power cardinality I k minus 1 if you note here when you go from the signal term this is does not contributed everything. So, therefore, what is this therefore, signature is the number orbits of even cardinality. So, sign of sign of sigma equal to minus 1 power m where m is the number of equivalence classes of cardinality even because if we equivalence class as cardinality odd this is becomes even and then it goes it does not contribute anything.

So, that is the formula now, if you if you compute lets a rho if you rho cycle I 1 to I k is k cycle k cycle then what do we Cartesian corresponding to rho there is only one equivalence class which is the whole thing and the remaining either single terms because they are fix. So, therefore, this has been becomes decompose n star n into only one in this case r equal to 1 and that it is I 1 to I k this is an union they complement of this single tense. So, this goes then union single tense j 1 union disjoint union disjoint union j how much j n minus k they are all single tense. So, they do not contributed nothing to the canonical decomposition only this is computed in the canonical decomposition.

And we want to compute how many equivalence classes are of cardinality even well this guys are single term. So, they are cardinality odd. So, these do not compute anything to m and this one depending on n is even or odd it will they. So, this the sign of this rho will dependent minus 1 power cardinality if this guy if k is also odd then this is the also odd cardinality. this is also contribute because minus 1 power this will its own contribute. So, only contribute when k is even and then here it is a minus 1 power the cardinality here that is k minus 1 minus 1 power 1 is actually by k minus 1 minus sorry minus 1 minus 1 if k is even its k is even; that means, this cardinality is even and there is only 1. So, m is 1 in this case.

So, this case m equal to 1. So, 1, this is minus 1. So, what do you have noted is if this k were even then the sign is minus 1 for example, tested for rho the transposition 1 2 k is 1 in this case k is 2 and the what do you wanted you wanted to know how many equivalences of cardinality even only 1. So, m is 1. So, sign of a transposition is minus 1 sign away 3 cycle this is k is odd. So, the equivalence class minus 1 the even that may be. So, this is 1. So, this is very very easy to compute we just have to see what are the equivalent classes what are they cardinalities and if cardinality is odd get them if cardinality is even count them how many equivalence classes of cardinality is even and that number is m then put minus 1 power m that is the sign.

And now the biggest thing the most important thing we need to prove is this will not change. So, this is the theorem we need to prove.

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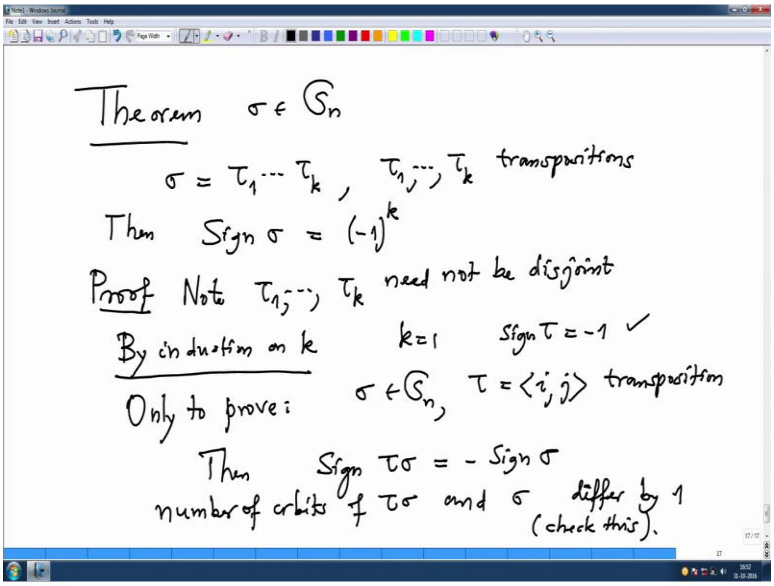

Theorem we need to prove is if sigma is permutation and it suppose somebody writes sigma as a product of k transposition then I want to prove that where tau 1 to tau k are transpositions may not be disjoint then I want to prove this k is nothing, but not k we prove that sign of sigma is minus 1 power k this what I need to prove.

So, if k is even this will be 1 if k is odd this will be odd minus 1 this is what I need to prove or equivalently. So, I we will prove this first let us prove this first. So, proof and then the proof this is as it is see remember what is most important to note is this tau into tau make cannot be disjoint because if they are disjoint is nothing to prove. So, note tau 1 to tau k need not be disjoint and that makes more complicated to compute and this formula I want to check by induction on k on k. So, start when k equal to 1 sigma is a transposition and then we have check the transposition as sign of transposition is minus 1.

So, induction starts therefore, to simplify the notation I only I have to check only to prove if I have arbitrary permutation and have if I have one transposition tau is the transposition that is which maps i to j and j to i transposition then I have to prove a formula sign of sigma tau sigma equal to minus sign sigma. So, one at a time will remove. So, this is what I have to prove; that means, I want to choose show what how do I show this; that means, I want to show that the number of orbits number of orbits of tau sigma and sigma they differ by 1.

So, I will compute the orbits under sigma and orbits under tau sigma and check that they differ by one and therefore, the sign will differ by minus 1 power 1 which is minus 1. So, I will leave it for you to check that this is very easy. So, we stop this and next time we will introduce multi linear maps and alternative maps.