Linear Algebra
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Lecture - 51
Introduction to multilinear maps

In today's lecture we are going to study Multilinear Maps.

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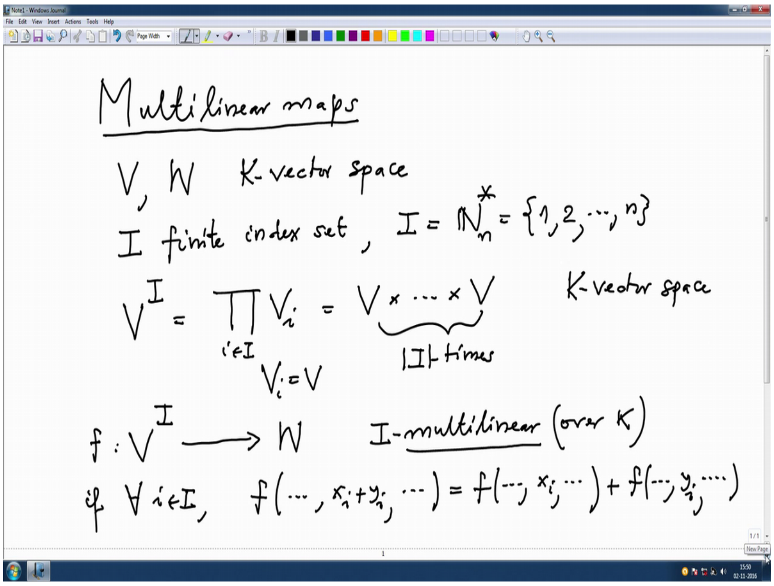

As you can see from the title it is we have seen linear maps and for multi variables. That means the following. So, as usual we will denote V and W K-vector spaces, and we take any finite set I the finite index set. Normally one can take I to be standard N star n usual 1 to n and when you take V power I; that means copy the V index by I this is the product V i- i in I and each V i is V. That means, we are digging V cross V cross V finally many times as many times as the cardinality I and this one is indexed by I and so on.

So, this is also vector space these also K-vector space component wise structures; component wise addition and component wise scalar multiplication. And we are considering in maps from V power I to W, and we will call it I multilinear. Actually K should also be in the notation I multilinear over K, but this is really drop from the notation. That means, when you fix any i-th coordinate so that means call this method f is called multilinear if for each V in i f of i-th coordinate vary like this x i plus y i and the remaining coordinates the same; this equal to f x i plus f y i.

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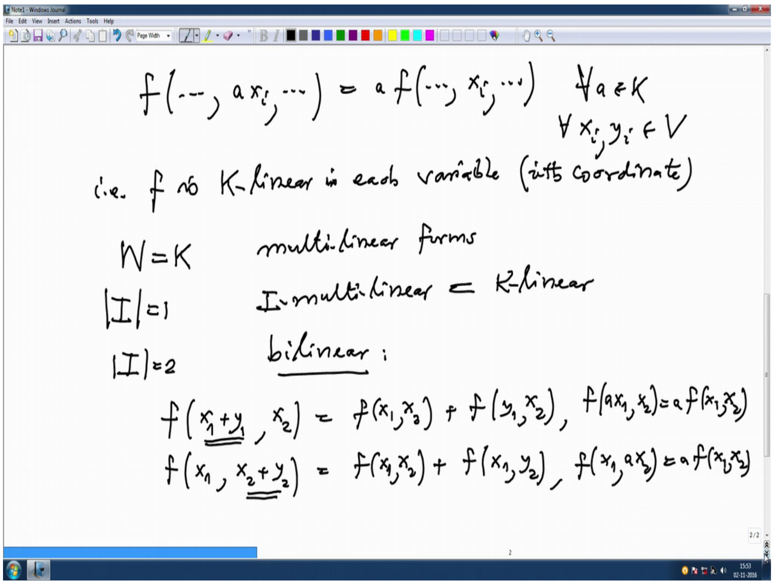

Similarly, for the scalar multiplication f of dot, dot, dot, dot, a x i this is a f dot, dot, dot, x i. This is true for all scalars and; obviously, earlier one all x i y i in V. Then you call it multilinear. This simply means if you look at the map, if you fix all variable you can late x i vary then it is linear in that variable.

So, this means that is f is K linear in each variable or in each i-th coordinate, then you call such a map to be multilinear mapping. When W equal to K then I will keep saying multilinear forms. When cardinality I equal to 1, then I multilinear is same as linear this is same as K linear. When cardinality I equal to 2, then we will use the word bilinear. And bilinear simply means if I take f of the first coordinate x 1 x 2 or x 1 or to be in const consistent with the earlier notation x 1 y 1 x 2, this is f of x 1 x 2 plus f of y 1 x 2. And similarly f of x 1 comma x 2 plus y 2 same as f of x 1 x 2 plus f of x 1 y 2. And also the scalar multiplication and later for this f of a x 1 x 2 is same as a f of x 1 x 2, and for this in the second I will f of x 1 a x 2 is a f of x 1 x 2.

This is a bilinearity condition; this means linear in this variable bilinear in this variable there are only two variables.

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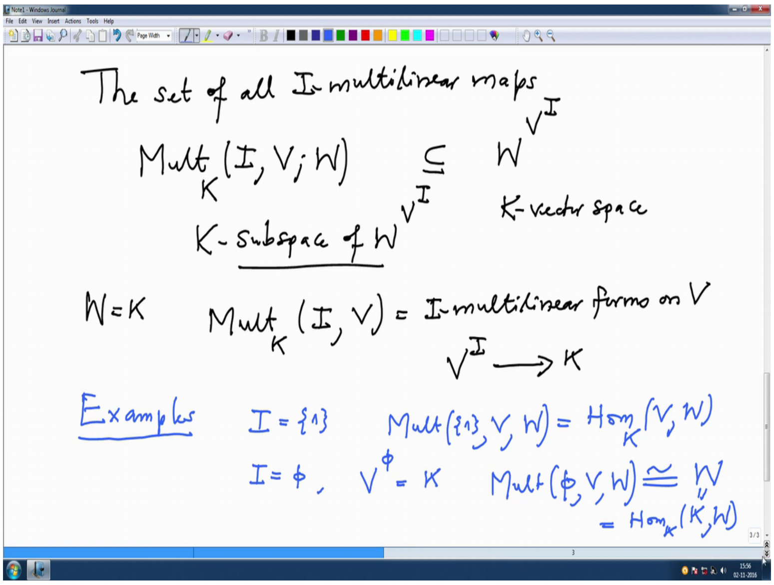

So, the set of multilinear maps: the set of all I multilinear maps I will denote by mult. So, we create remember K linearity I is this index set, V is a vector space, and W. This is the subset of W power V power I, and we know this is a vector space. And it will take one minute to check that if I take two multilinear maps their addition make sense and addition is also again component wise which is coming from here. Similarly, scalar multiplication so that will form a K subspace of this W V power I, this is where if you this if you have verified several times such. I think so this is easy.

Now when W equal to K these are linear forms so that the multilinear forms, that I will simplify the notation and write mul t K I V; these are multilinear I multilinear forms on V forms on V. So, they are maps from V power I to K multilinear in each variable. So, for example, let us see some examples; some standard examples.

Examples: so as I said earlier multilinear forms are when I singleton 1then mult 1 V K V W this is precisely all linear we have so it is hom K V W. And note very important case very important extreme case if I is empty set then what should be V power empty set this should be K and then mult empty set V W this we can identify this with W, because K to W map we just they are identified with W. This is actually one should say this is home K K W which is this. So, this is extreme case, but useful for guessing results.

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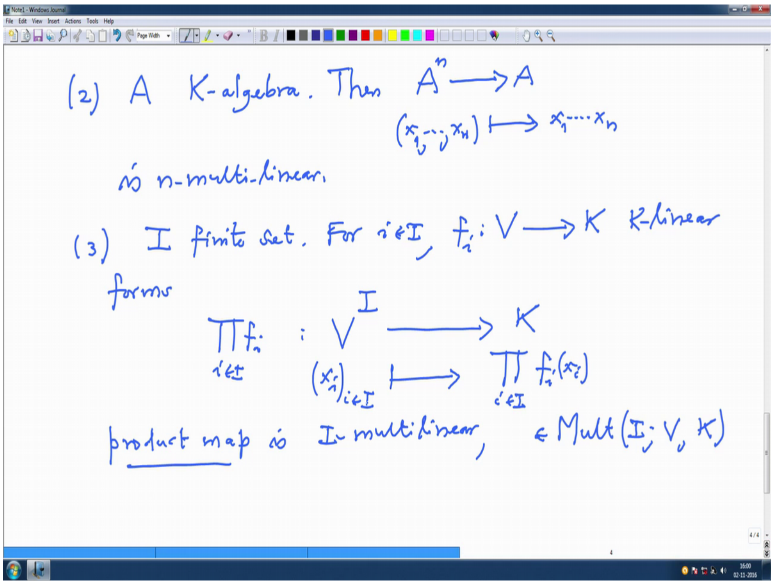

The second example: suppose A is K algebra, recall that K algebra means which is K-vector space and a K is a ring and the ring structure ring multiplication and the scalar multiplication they are compatible with each other. So, look at the map, then a power n to a if I take a tuple x 1 to x n is mapped to the product x 1 to x n this is n-multilinear. So, these are precisely the distributive properties in the algebra, when I change one of the variable to be the sum or scalar multiplication then this we use the distributive or compatibility properties in algebra to check that this is multilinear.

Third one: if I have linear forms. So, I is any finite set and for each I, suppose you are given linear maps f i linear forms V to K K linear form, then obviously I have a product map from the V power I to K. Namely if you have a tuple x i map it to: first you apply if I to each x i. So, these are elements in K and now take their product in the field K this makes sense, this is the standard notation for this is product map. This product map is called a product map is I multilinear. That is this is an element in mult I V K, it is a multilinear form only. This is obvious to check because you when you change x i to the sum or scalar multiply it go changes here and then the distributive property in the field K addition with multiplication and so on so we can spell it out and then it is multilinear. That is all one checks is.

This is very important I will use this map to construct determinant functions. I am doing these multilinear maps in order to construct determinant functions. And the determinant functions are very important to study determinants abstractly.

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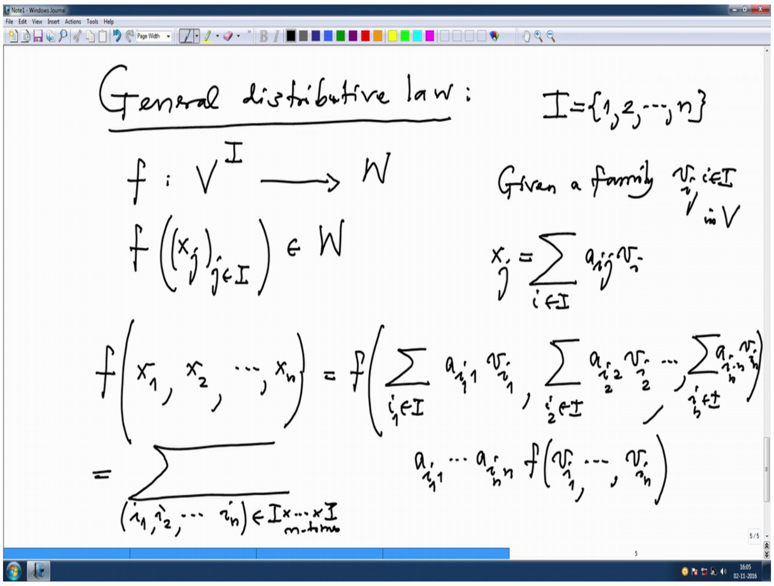

Now, just one thing I just want to note that a general distributive law; that mean the following. If you have multilinear map f from V power I to W, that means I have a definition I have given that for each tuple x; I want to use the same notation x j j in j j in I I mean I apply f I get an element in W and it is multilinearly each variable.

So, I want to find out the formula for when I want to when I want to substitute this is x j; suppose I have given a family V i i in I in V. And then and suppose I have written this x j as the sum of this family a ij V i i in I; this is typically will be the case when this V i is the basis of V. When arbitrary x j I can write in terms of this. And then I want to you evaluate I want to substitute x j equal to this and find out this. So, I want to expand it so that means what for f of and here to get a better understanding let us assume I is 1 to n.

So, f of I am writing the first coordinate that is x 1 x 2 x n, but this is same as I want to plug this value; f of x 1 is summation a i. And this i is varying so i want to vary these I want to give with different names what for different a i 1 V i 1- this i 1 is running in i. X 2 is summation i 2 in I a i 2 to V i 2 and so on. The n-th coordinate will be summation over anymore I n in I a is n n V i n. And now when I use multilinearity this a i's will come out from here, from here, and from here. So, this will be the sum, it is a big sum running over i 1 i 2 etcetera I n they are in the set I cross I cross I m times. And then this will come from here this will come. So, it is the product i in I a i 1; a i n I do not need a product here that these products times f of f of V i 1 V i n.

So, this is the distributive law: I have not done anything I have just taken out this sum out, this sum out and this sum out and then this is some running or so many indices and this guy will come out as scalars and then what remains is this. So, this is very useful when one wants to compute values on the tuples of multilinear maps.

Now, I want to define something when multilinear map is symmetric or anti symmetric or alternative. So, for that I will use the notation remember in the last lectures I have spent some time about the permutation.

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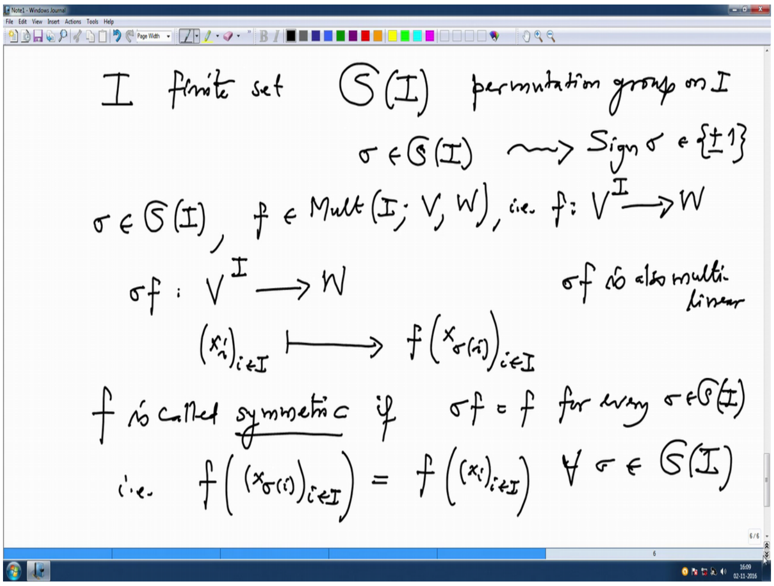

So, when I is a finite set we have these set of permutations on I is denoted by this; this is the permutation group- group on I. And remember for each permutation we have assigned signature of a permutation. And the signature is either plus 1 or minus 1 depending on when we write sigma as a product of transpositions even number of transposition then the sign is one when we will when sigma is the product of odd number of transposition then the sign is minus 1. And we have also check that this even or odd does not depend on the number is not well defined, but the parity is well defined. This is what we have checked in the last lecture

So, for each sigma a permutation and for each multilinear map f you can define sigma f. Sigma f is again a map from V power I to W. And what is the definition the f is a map from? That is f is a map from multilinear map from V power I to W and then I want to use this f and this permutation to define a new map. And this new map is nothing but if I take x i i in I map it to first variable the f x first permit the variables and then apply f. So, this is f of sigma x sigma i i in I. So, I get a new map which is called as sigma f f.

And note that if f is multilinear then this map is also multilinear; this sigma f is also multilinear. So, we will call f is called symmetric; if sigma f equal to f for every sigma. That means, f does not change after permuting the coordinates. So, this is spelled out that is f of x sigma i value on this tuple, value on the value of f on the permuted tuple is same thing as value of f without permuting for every sig minus I, then we call f to be symmetric multilinear map.

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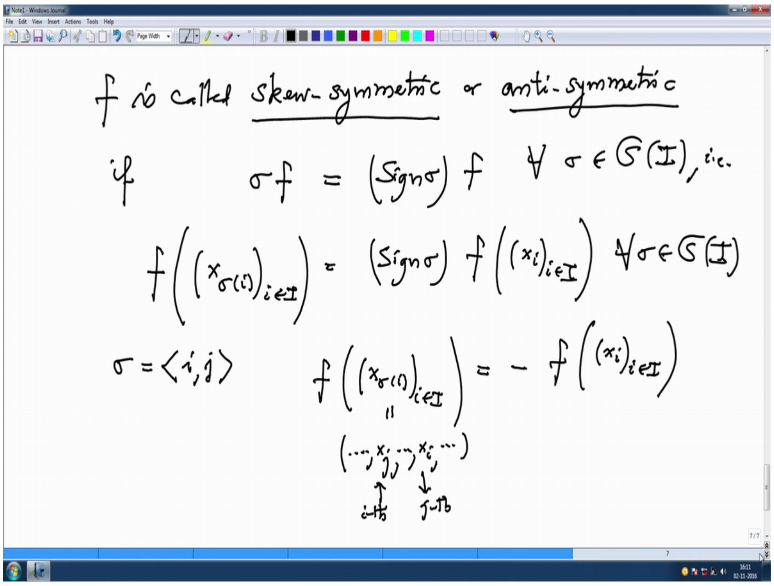

Similarly, we can call f to be skew symmetric: f is called skew symmetric or anti symmetric if sigma of f is not f but up to a sign, sign sigma times f for every permutation sig minus i. So, this means that is if I take f and value on the permuted tuple is not the value on f value of f on the original tuple, but these multiplied by sign of sigma for every sigma. Then we call it skew symmetric or anti symmetric.

So, in particular when you take sigma for example, sigma is a transposition i j. This is a transposition that means i is mapped to j, j is map to i and all other elements in i are fixed then f of this permuted tuple is nothing but minus f of original tuple. And these tuple you know only the i-th and j-th coordinate will change and this tuple will looks like all coordinate are same i-th coordinate will become j-th coordinate; and j-th coordinate will become i-th coordinate. So, this is i-th coordinate and this is j-th coordinate. So, if the two coordinates are switched when the value changes with the minus 1.

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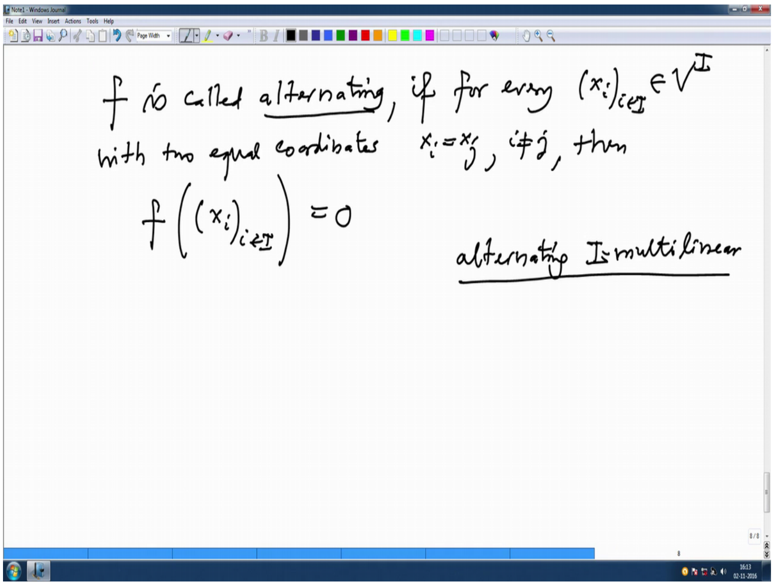

One more definition: f is called alternating if f of; if for every tuple x i in V power I which has two equal coordinates, with two equal coordinates. That means, there is x i these I and j so that x i equal to x j for the coordinates i-th and j-th different coordinates the value of the tuple at the coordinates are equal then f of this tuple should be 0, then we call it an alternative mapping. So, alternating multilinear; I multilinear. That means, whenever you have a tuple with two equal coordinates at different positions then the value of f is 0, then you call it alternative.

Now, after the break I will study the relation between symmetric, skew symmetric and alternative maps.