Linear Algebra
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Lecture – 52
Multilinear maps (continued)

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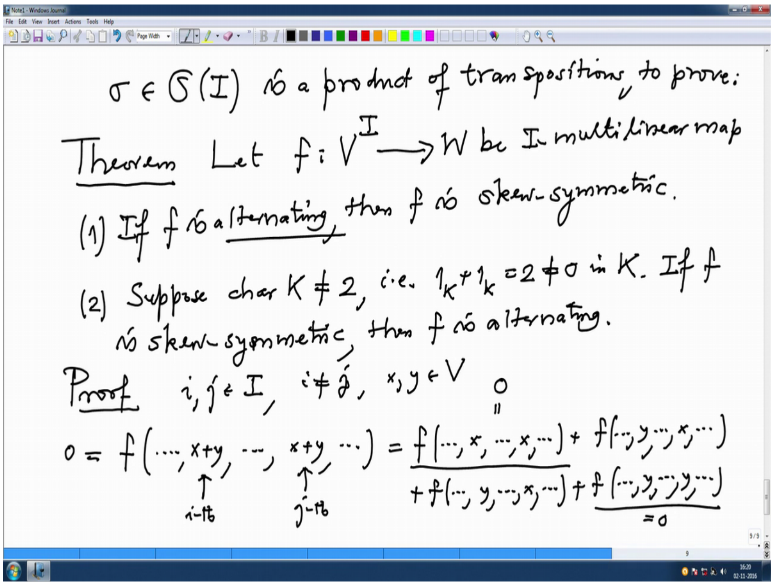

Use a fact that every permutation sigma is a product of transpositions, to prove the following theorem. So, its let us write it as a theorem. So, theorem let f from V power I to w be I multilinear map one if f is alternating, then f is skew symmetric the converse is not true as you can see after this small example, but the converse is true when you assume suppose characteristic of the field K is not two. So, this means one K plus 1 K which is 2 is not 0 in the field, typically this happens for a finite field which is a field extension of the field z mod 2 which is use in communication engineering and so on. If f is skew symmetric then f is alternating proof we want to show that f is skew symmetric.

So, let us take two coordinates to two elements I and j in the set I so, if that I is not equal to j and let us compute. So, recall the definition that it is alt if alternating means if two coordinates are equal then the values are zero. So, if I take the f is given to be alternating. So, if you look at f of keep the coordinate same other than ith and j th coordinate, ith coordinate you put x plus y plus j th coordinate to put x plus y again and whenever I do not write anything those coordinates are fixed x and y are arbitrary elements in the vector space V this is 0 because this is ith coordinate, this is j th coordinate and they have the same component therefore, this is 0 because f is alternating.

On the other hand it is multilinear. So, I will expand it the remaining dot dot coordinates are same. So, f of dot dot dot x, dot dot dot x, dot dot dot. So, I have taken this and this plus f y x. So, I have taken y and this x two more thing plus f dot dot dot, now y dot dot dot, x dot dot dot plus f, y dot dot dot y dot dot dot. If you simply because of I use a multilinearity here and multilinearity here this will give two terms this will give two terms all together four terms and now f is alternating. So, therefore, this has two equal coordinate because x and x both remains at I and j th position.

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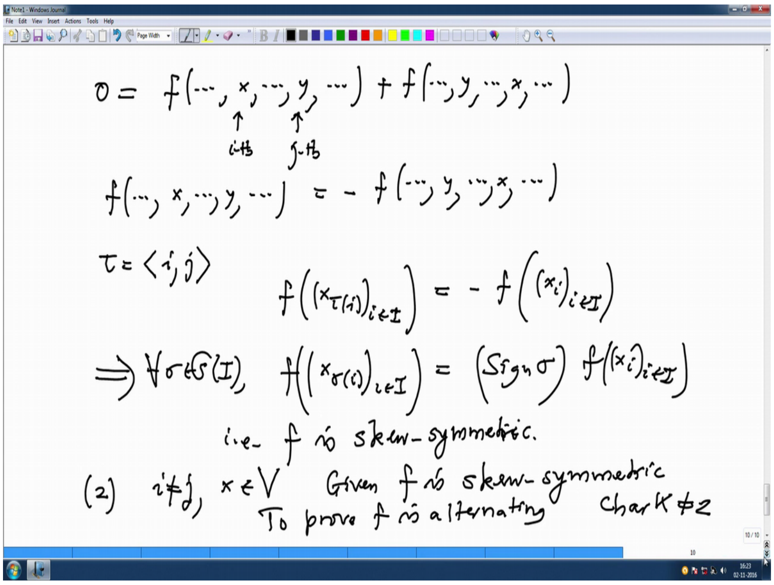

So, this is 0 similarly y remains ith and j th coordinate. So, this is also 0. So, the result is together what you get is f of. So, 0 equal to f of dot dot dot dot, x, dot dot dot dot, y dot dot dot dot, plus f of dot dot dot dot, y dot dot dot, x dot dot this. So, here note the the coordinates are interchange ith position this is was ith this was j th and this they are switched so; that means, f of this equal to minus f of; that means, you have proved that if I take any tuple and apply the transposition tau equal to ij then you get the formula f of any tuple x tau I, i in I this is same as minus f of x i. So, we have proved that for any transposition f of the permuted tuple is minus of the original tuple.

So, therefore, it will follow because every permutation is a product of transpositions you follow that f of x sigma i, i in I this is seeming as sign of sigma f of original tuple x i;

that means, f is skew symmetric is it true for every sigma. Here actually we use every permutation is a product of transpositions as many transpositions as either even or odd depending on the sign is one or minus one now proof of two suppose i is if not equal to j and x is any vector and we want to prove that we have given that given f is skew symmetric and remember we want to prove that to prove that f is alternative and remember we will have to use the condition characteristic of K is not equal to two.

So; that means, I want to show that f on any tuple which has at least two different components equal then f has to be 0 on such tuples that is the meaning of alternating.

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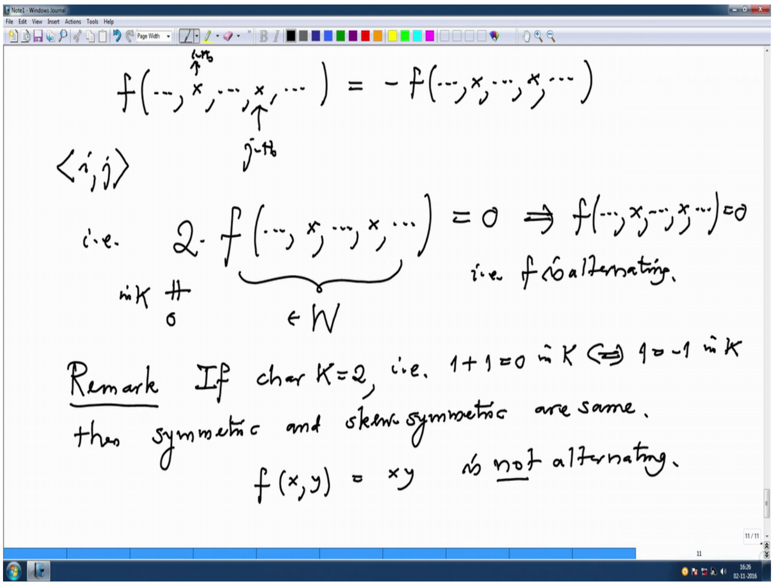

So; that means, I need to prove that if I put x at ith position and also x at the j th position then such values are zero, but I definitely know if I switch; that means, if I apply a transposition ij then this change at the sign this is a f of dot dot dot and these are change that you cannot see it, we have change the coordinate, but they are equal coordinates. So, you cannot see it. So, this is minus of them so that means, two times bring this to the side two times f of x dot dot dot dot x this is zero, but this is an element in the vector space w this is 2 is not 0 in the field in K.

So, I can multiply by two inverse because two in multiple in the field. So, that you can conclude f of this is zero. So, that is f is alternative. So, the just wanted to remark that remark what happens in the characteristic 2. If characteristic K is 2 that means, 1 plus 1 is 0 in K or equivalently one equal to minus 1 in K then symmetric is an alternating and skew symmetric; these two concepts are same and there is no reason for symmetric to be alternating. So, for example, you could take these by linear form x x y equal to x times y. So, this is not alternative ok.

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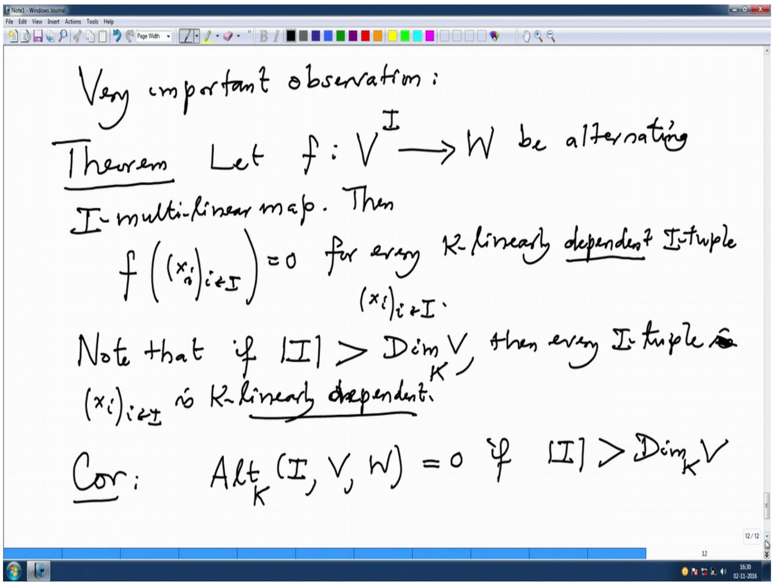

So, all right. So, now, I want to construct no first very important remark I want to make very important observation the following theorem. So, theorem let f from V power I to w be alternating I multilinear map then f of a tuple x, x i this is 0 for every K linearly independent tuple are not independent dependent K linearly dependent I tuple xi note that before I prove this note that if cardinality of this finite set is strictly bigger than the dimension, then every I tuple is linearly dependent every I tuple x i i in I is K linearly independent K linearly dependent that is because we have proved that every linearly independent subset you can exchange it to a bases, and every two bases are the same cardinality of the dimension.

So, definitely every tuple which has more elements more components than the dimension has to be linearly dependent. So, this will mean immediate corollary to be this theorem will be there is no alternating lin alternating multilinear map if this I is more than the dimension cardinality has more than the dimension; that means, I did not introduce in addition, but I will do it now if I write alt K I V W this means all alternating I multilinear maps from V power I to W, then this side will actually with a 0 if cardinality I is more than the dimension because any tuple the value will be 0; that means, f has to be 0 map 0 map is the only alternating multilinear map when I has more elements, and the next case we will study when cardinality is exactly equal to the dimension and w equal to K and those will be called determinant functions before that let us prove this. So, we have to prove that on in linearly independent tuple. So, proof.

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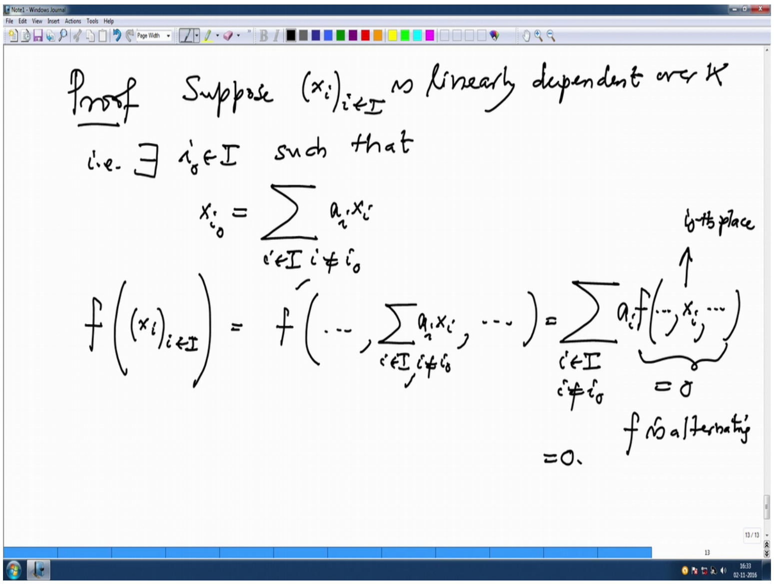

So, suppose x i i in I is linearly dependent, dependent over K that mean they were linear dependent dependence relation so; that means, there exist definitely one index i naught in I such that x i naught is a combination of the remaining one a i x i i arrange in I and i is not i naught you know you can always arrive at this equation we got dependence relation we will have at least one nonzero coefficient, and we have we are over a field. So, I will make that coefficient to be one and then shift to the other side. So, definitely such a dependent relation we have and now we want to compute what is f of xi well in this I am substituting I not coordinate. So, it is I not coordinate this summation ai xi i in I and i is not i naught this is the I naught rate coordinate remaining coordinates are same as xi when I use a multilinearity at this i naught coordinate these will come out.

So, this is same as summation this summation is running over this i in I i not equal to I naught and this ai is will come out, and remaining will be f of and i naught position xi this is I naught position, but remember this is i naught position and there are the remaining positions also be the same there is somewhere else in this ith position is somewhere else that different from I naught position. So, this has to be 0 because f is alternating, for each I different from I naught this xi is sitting at ith position as well as I naught position and f is alternating. So, this sum is 0 therefore, everything is zero. So, that is what we proved it f vanishes on a linearly dependent tuples, alternating map vanishes on the linearly dependent tuples now I want to give a canonical process.

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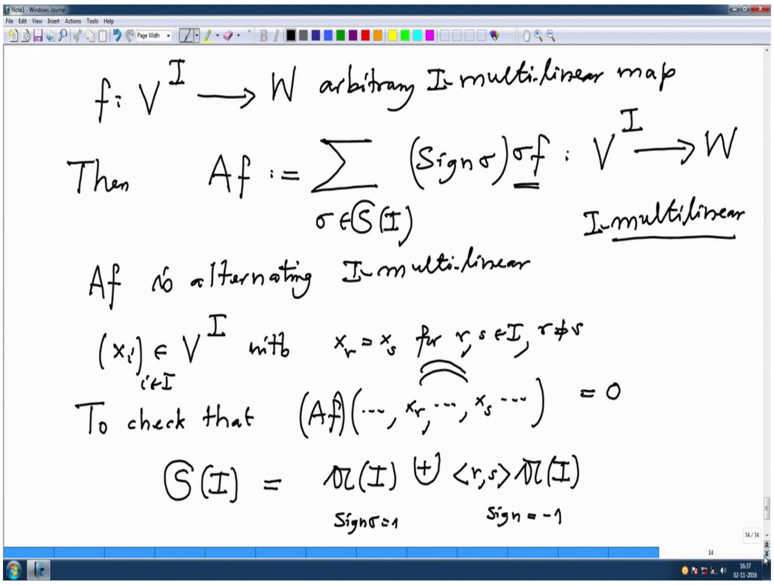

So, that given an alternating map given a multilinear map f, f is from V power I to W arbitrary I a multilinear map, I want to give a standard recipe to make this map f to be alternating not the map f I will alter a map f and arrive at a alternating map. So, what am I saying then if I defined Af by definition, this is summation sigma summation is running over all permutations sign of sigma times sigma f. So, I summed it up over all permutations. So, that signs multiply sigma f by the sign sigma and I want to check that these Af clearly this Af is a map from V power I to W, clearly it is I a multilinear because we have check that I multilinear maps form a subspace; that means, if I take different multilinear maps and add them with some scalars then also I get multilinear map.

So, this is a sum finite sum these are multilinear, these are some scalars one or minus one and therefore, these sum is definitely I a multilinear. I want to check that this is alternating f is Af is alternating I multilinear. So, I should check that if I take a tuple whose two coordinates at different positions are equal then the value on those tuples of A f is 0. So, I have to check that if I take any tuple x i in V power I with r th coordinate of this tuple is xr plus and s th coordinate is x s they are equal for r s in I not equal. Then I want to check that then to prove or to check that Af on these tuple xr, xs this is 0 I want to check that these are equal that is given to us. For this I will use the fact that the symmetry group and remember this is the alternating group ai these are permutations whose sign is one and if I multiply the permutations whose sign is one by a transposition rs, then these are all odd permutation these are all even permutations these are all sign is because sign changes sign is product preserving this is minus 1 this is all sign here are 1. So, the signs all signs r minus ones here.

So, they do not intersect and if you count them this is one factorial by two, this is also one factorial by two. So, they will cover all the permutations. So, this means this permutation group is decomposed as its partition of even permutation and odd permutation which are multiplied by this particular transposition to all even permutation. So, I will use this side. So, now, let us use the definition of a when compute this left hand side remember definition of Af is summation running over the permutation all permutation, this summation I will split into two parts one is here and the other is here.

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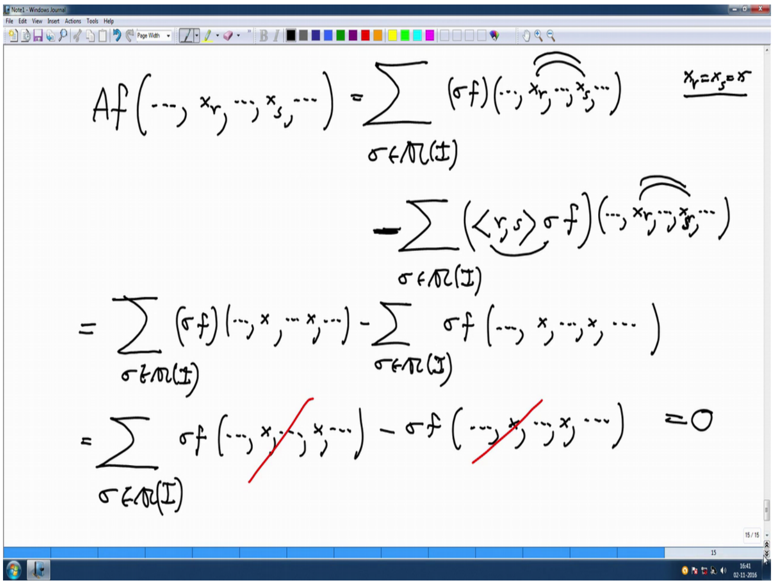

So, therefore, Af at this tuple xr, xs this tuple is by definition summation now running in sigma in ai alternating and in this case I do not need a sign because sign is one in this case.

So, that is just sigma f evaluated on this tuple; then the other sum that will come with minus sign summation sigma running in I again and I will I had to apply permutation the

the transposition which changes r to s sigma and this f applied to this tuple and of course, the sign which I have taken minus out, this is because of the sign of this permutation is minus i, but now when you do this remember these coordinates are saying these two coordinates are saying that is our assumption.

So, when I apply sigma f then the still these two coordinates will be equal and when I now interchange r and s the difference will not be seen. So, let us apply this, this is the first sum is summation sigma f applied to this let me write both are x let me write xr equal to xs equal to x then this will be x this is also x is with the minus sign sigma in a I this sum is also sigma in ai and then this will be same as sigma f evaluated on x and x because when you interchange r th than x th coordinate, it does not change this will be f of x this will be f of x 2. So, it does not change and I could now club these two sums together summation running sigma in a I sigma a sigma f x x this minus sigma f x x this and this is equal to this. So, it gets cancelled. So, all together it is 0 this will get cancelled this all the sum gets cancelled. So, it is zero.

So, that proved that f is alternative, similar construction you can also do it to construct a symmetric operator see remember from multilinear operator we have constructed alternating operator.

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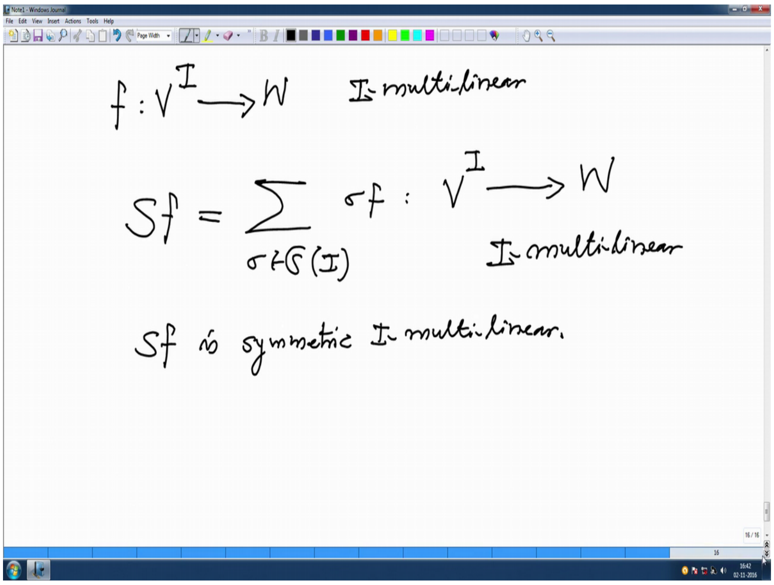

Similarly from a multulinear operator V power I to W, I a multilinear if I want to construct the symmetric operator multilinear symmetric then simply I could define sf equal to summation sigma in si I do not take sign sigma f, this is again a operator is again a multilinear map from VI to W, I multilinear is obvious because sum of multilinear maps is multilinear and the same proof what we have check this will make f sf is symmetric I multilinear; with this now we are ready to study what is called determinant functions.

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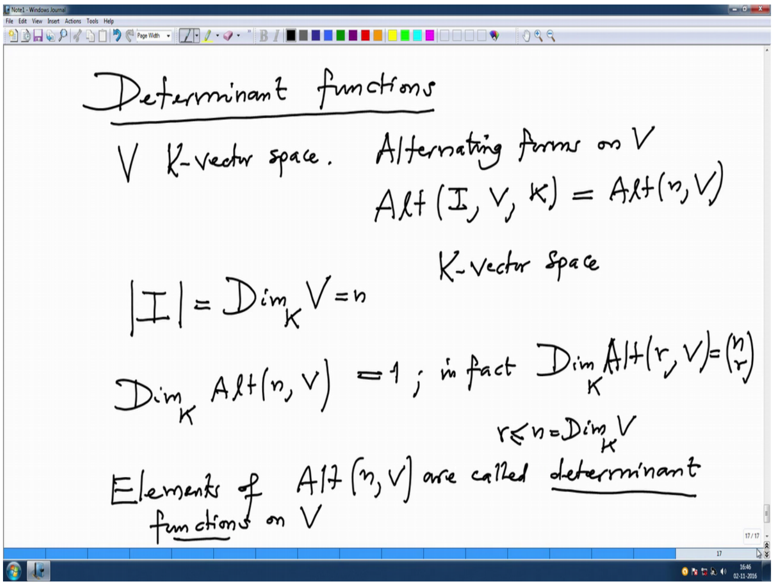

So, the next topic I will start is determinant functions. So, in this section I will prove the existence of determinant functions and actually what I want to do is the following. Suppose we have a V is the K vector space and now I want to study when I say functions they are means their forms, I want to study alternating meeting alternating forms on V; that means, I want to study these object I I V K we have seen it is a vector space this is a K vector space, and now we are a special case what we are doing is we want to study it when cardinality of I is exactly equal to the dimension of V, because we have seen when cardinalities more then these vectors which is actually 0 vector space when it is less it is little bit more complicated to study especially I want to compute what is the dimension of this vector space.

So, this also I will if you put this as n, these also I will just simply I will write by sign alt nV alternating n linear forms on V where n is the dimension and what I want to prove is dimension of this vector space is one it is one dimensional vector space this is what we will prove. In fact, more generally we can prove this I will not prove in this course, but maybe the next course I will prove that dimension of a alt r V, where r is smaller equal to dimension when r is more than dimension then we know it is 0 vector space in this case it is nothing, but n choose r where r equal to n case is what we are going to do that is n choose n is one this is what we want to prove and actually we want to construct a canonical basis for this vector space. Given a basis of V we want to construct a canonical alternating form which will be a basis of this actually ok.

So, this is what the content of this next lecture is I will. So, elements of alt n V are called determinant functions n V. So, let us recall what I stated some minutes back the distributive law and I will use that distributive law to expand this, but that I will do it next time and I will prove next time the dimension of the this vector space is one and we will construct a canonical or alternating maps on V by using the basis tuples f v.