Linear Algebra
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Lecture - 53
Introduction to determinants

So in this lecture I will discuss as I said in the last lecture I will discuss determinant functions.

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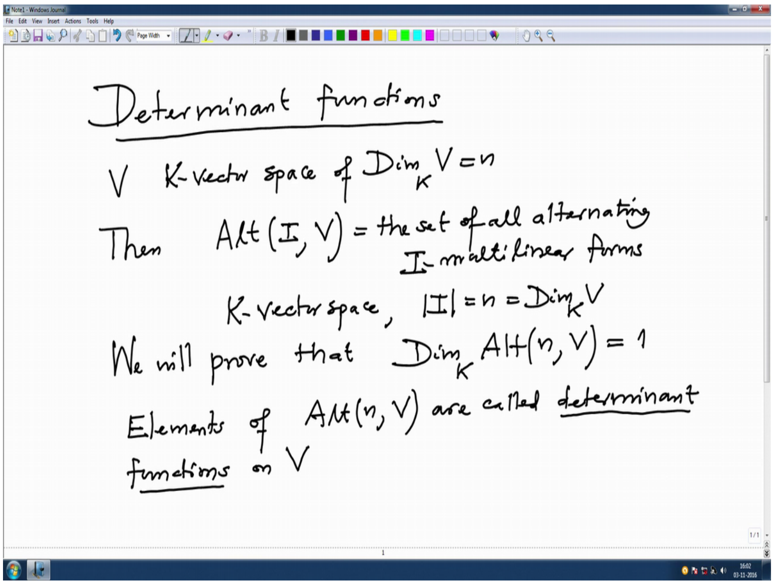

So, as we saw last time that when V is a K vector space of dimension n, then the alternating multilinear maps alt iv alternating linear alternating multilinear forms. So, I do not write that K in the notation these are is the set of all alternating I multilinear forms, this is interesting this is a K vector space, and I am going to discuss only the case when cardinality of I is also n which is the dimension because we have seen in the last lecture that when cardinality of I is more than the dimension then 0 is the only alternating multilinear map on V.

So, therefore, they are zero vector spaces. So, we do not need to study them this is the first interesting case when cardinality I equal to the dimension, and in this case I will prove that this is one dimensional vector space we will compute we will prove that dimension of this vector space dimension K, now I will write instead of I will keep writing nV this is one. So, this is what I am preparing today for in any case elements of these vector space alt nv are called determinant functions on V on v.

So, they are multilinear alternating multilinear n forms I will now keep using the n forms on V and they have the properties of the determinants as we will see. So, the first easy lemma that I have already done it earlier, but I want to do it again more.

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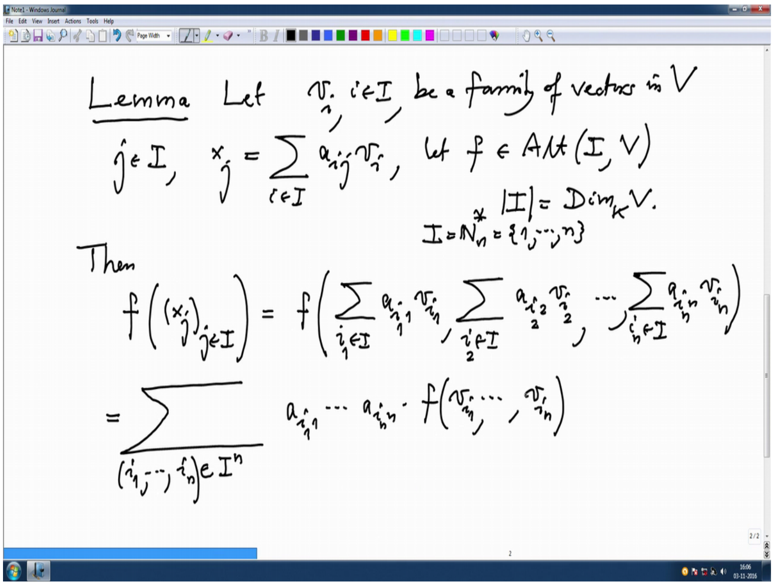

So, let us let vi i in I be a family of vectors in V and given for any j in j a j in I let us write xj to be the combination of this vis aij vi and let us take let f be an alternating multilinear form.

So, I am we take cardinality I equal to dimension V then if I evaluate f on this tuple xj I want to know what do we get. So, this is equal to f of let us write this I is for a simplicity I will write the set I equal to n star n, which is one to n and this x 1 is a combination j equal to 1. So, is the combination and in this sum i is varying. So, I will call it I one in a i. So, it is a i 1 1, v i 1 similarly x 2 which is i 2 in i a i 2 2 vi 2 and so on, i n in I, a in n vin and now we use the fact that f is multilinear. So, what will happen is all these ais will come out.

So, this sum will run over all the tuples i 1 to i n in I power n ai 1 1 a i n n then f of v i 1, v in this is very very simple so far these we have seen so far, but in this now who will survive because if two in this sum is running over all the tuples in I power n, but some of the tuples may have some two different coordinates may be equal and in that case then f of this will be 0.

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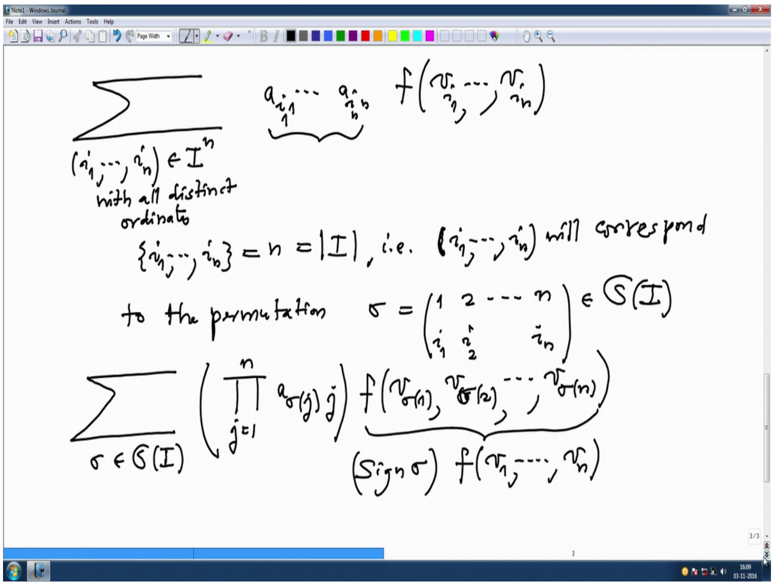

So, that will not contribute anything. So, this sum is equal to sum i 1 to in I power n not only that with all distinct coordinates then this is ai 1 1, ai n n times f of vi 1 V in.

So, a repeated coordinate tuples we have removed, but then remember this we can think of a tuple where all coordinates are distinct then the cardinality of the set of this coordinates this cardinality will be n which is cardinality i; that means, you can think these are the permutation of 1 to n. So, that is this tuple i 1 to in will correspond to a permutation, to the permutation sigma where sigma is defined like this image of 1 is i 1 image of 2 is i 2 image of n is in this is a permutation on i.

So, this sum is nothing, but a summation now I write instead of the tuple with notation I will write sigma in si and then this is a product product is running over j from one to n a sigma j j this is this this one times f of v sigma 1, v sigma 2, v sigma n, but we have seen that whenever we have a multilinear map f and when you permit the coordinate and if it is alternating then this sign will come out. So, this is same as sign of sigma times f of v 1 to vn.

So, that gives us that tells us that this this coefficient which is for the for the if you call this matrix a to be aij.

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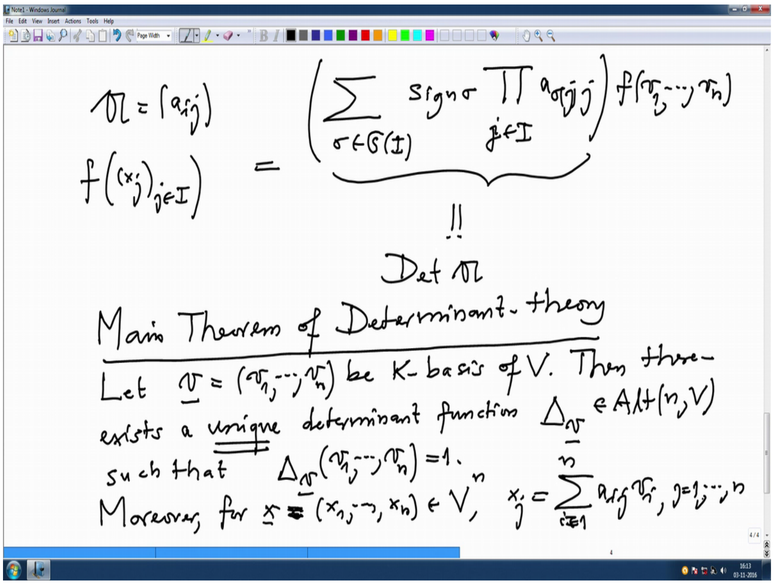

Then the summation running over a permutations of I sign sigma time this product j in I a sigma j, j you should write bracket here. This is important this is the coefficient of that term f of this this times f of v 1 to vn this was the value of that tuple f of xj. So, this one could define this one as a determinant of the matrix a.

It is so important because of this formula. So, before I go on to the formal property of these I want to state the main theorem of determinant theory so main theorem of determinant theory. So, in the above observation if you would take v 1 to vn as the basis of v; that means, with any alternating multilinear form on v this is form is uniquely determined by it is values on the tuple basis tuple because any other tuple you write in terms of the basis v 1 to vn as sj equal to aijvi summation and then we were to know the value of this is determinant a times this value.

So, it is uniquely determined by basis value of an alternating form on a basis vector. So, let us start with the basis. So, let V equal to v 1 to vn be a K basis of V, then there exist a unique determinant function, I will denote it by this will depend on the basis v. So, delta v delta v is the determinant function means it is an element in alt nv such that delta v evaluated at this basis vector v 1 to vn is 1 unique is very important. So, moreover actually we can write down what is delta v; moreover for any other x tuple at x is x 1 to xn in V power n write each xj is summation aijvi this we can do it because I is from one to n because v 1 to vn is the basis and if I take any such arbitrary tuple.

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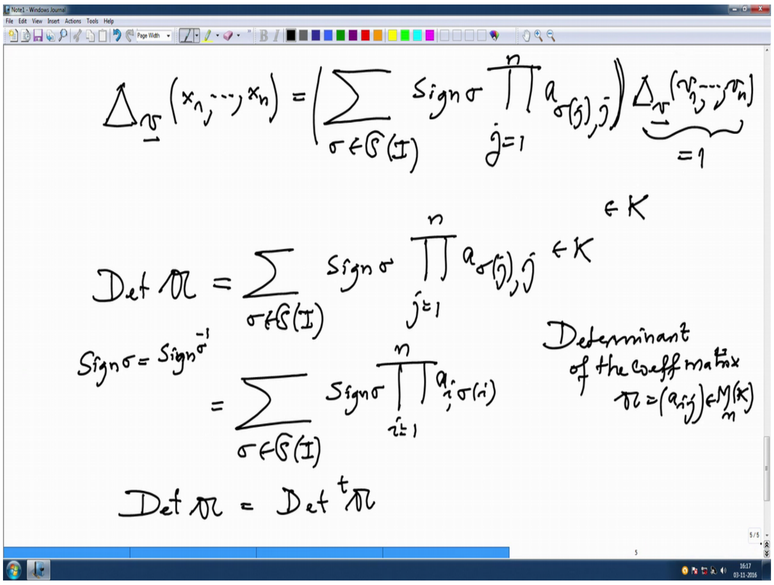

Then delta v evaluated at x 1 to xn is equal to summation is (Refer Time: 14:51) sigma sign sigma product is running over j 1 to n a sigma of j comma j this delta v evaluated at v 1 to vn, but this is we are remaining one. So, it is this quantity and this constant this is a constant it is a linear form. So, the values are constants values are scalar. So, this linear form has a important than this one I will call this is the determinant of the coefficient matrix. The determinant of a will be then called this quantity sigma in si sign sigma product j is from 1 to n a sigma j j which is a scalar it is called a determinant of the coefficient matrix a equal to aij this is a matrix in Mnk and by the way.

So, this it is also immediate that when I in this sum this sum is running over all permutations. So, when I replace sigma by sigma inverse in this, then the sign of sigma this is because sign of sigma and sign of sigma inverse are same that is immediately from the cycle decomposes the canonical cycle decomposition of sigma the number of cycles are same disjoined circles are same, the only thing they are written in a reverse direction and sign will does not change and then these product changes i is from 1 to n ai sigma i.

So, this is in fact, the determinant of the transpose matrix. So, actually immediately we will get from here that determinant of a equal to determinant of the transpose matrix. So, that is immediate from the definition. So, we only have to prove the existence of delta v and uniqueness of delta v. So, uniqueness is clear because on any tuple this these value these value is one. So, it is uniquely determined by this formula in fact, because this is one ok.

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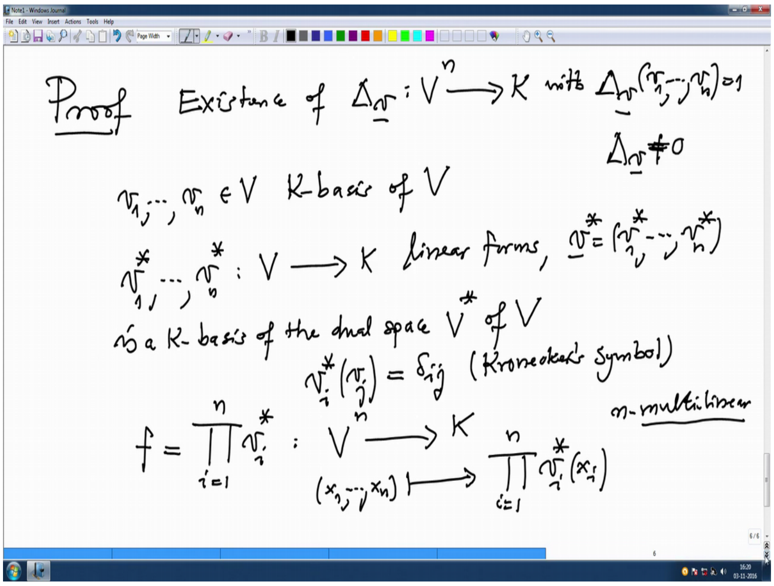

So, proof of the theorem though existence existence of of delta v it should be an alternating I from V power n to K with delta v 1 v 1 to vn is one. So, in particular delta v is near zero it is a nonzero not near zero, it is nonzero alternating multilinear form. So, this also we are already constructed. So, the only thing we have given is the basis v 1 to vn is a basis K basis of V and that look at the dual basis that is v 1 star vn star they are linear forms on V linear forms and they form V star which is v 1 star this is the basis of the dual space is the K basis of the dual space of V V star of V and remember they are defining their defining equations are v star i evaluated at vj is precisely delta ij this is kronecker delta kronecker symbol.

That means v star evaluated at vi is 1 and v star evaluated at any other basis element is 0, and last time I also told you a recipe if you have given. So, many linear maps then we can construct a multilinear map that is a product map. So, the product map f equal to product i in i is from 1 to n vi star think of this is a map from V power n to K defined by any x 1 to xn this is map to product i is from 1 to n, vi star evaluated at xi this make sense and this is clearly n multilinear because vis are linear and the product.

So, it is because of the distributive and so on and once I have a linear I have a multilinear map then we know how to make it alternative last lecture we have this recipe Af, Af is by definition summation sigma in si sign sigma.

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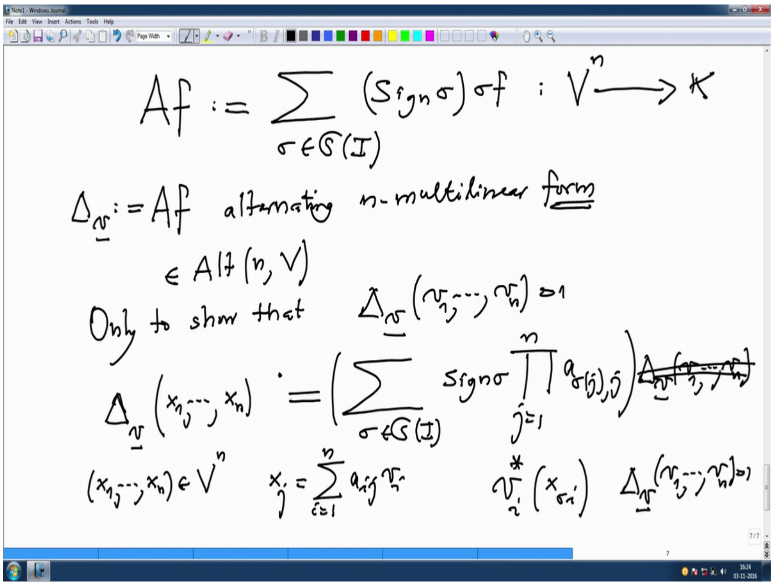

And then sigma f this is an I from V power n to K all these sigma fs are multilinear. So, this sum is multilinear. So, Af is definitely multilinear and we proved Af is actually alternating alternating n multilinear form, form is because the values are in K and this is what I want to check I want to take delta v by definition this because it is alternating a multilinear it definitely belongs to alt nV. And all that we want to show that now is a delta v enough therefore, only to show that delta v evaluated at v 1 to vn is 1 .

But well let you evaluate delta v let us evaluate actually you know how to evaluate delta v let us evaluate not only at this tuple, but any arbitrary tuple. So, delta v evaluated at x 1 to xn where x 1 to xn is arbitrary tuple V power n, and because each xj we can write in terms of the basis aijvi i is from 1 to n this by the above because it is alternating and so on, these we know the formula this is summation running over sigma permutations sign sigma product j equal to 1 to n a sigma j j times times what? Actually this is this is generally I directly wrote it this is and then this thing you validate delta v evaluated at v 1 to vn.

So, and actually you could have written one more step in between here see if I had to I want to write one more step here I write this this this came out of the fact that they are vi vi star evaluated at x sigma I because that is what the definition of this is, first you permit the tuple and then apply this and so this is comes out this. So, and then this will go or disappear. So, we get this formula in particular when I put x x 1 equal to v 1 xn equal to v 1, we will get delta v evaluated at v 1 to vn equal to one because in that case these aij s are diagonal one one one one one and so, this sum this this sum will be actually one because they will only be the ones on the diagonal and everywhere else it is 0. So, it is only one only one term will survive in this sum mainly sigma is equal to identity all other terms will vanish ok.

So, that proves this is the required determinant function that we were looking for that on delta v on v 1 to vn is one and so that is what I will prove now to prove that it is one dimensional.

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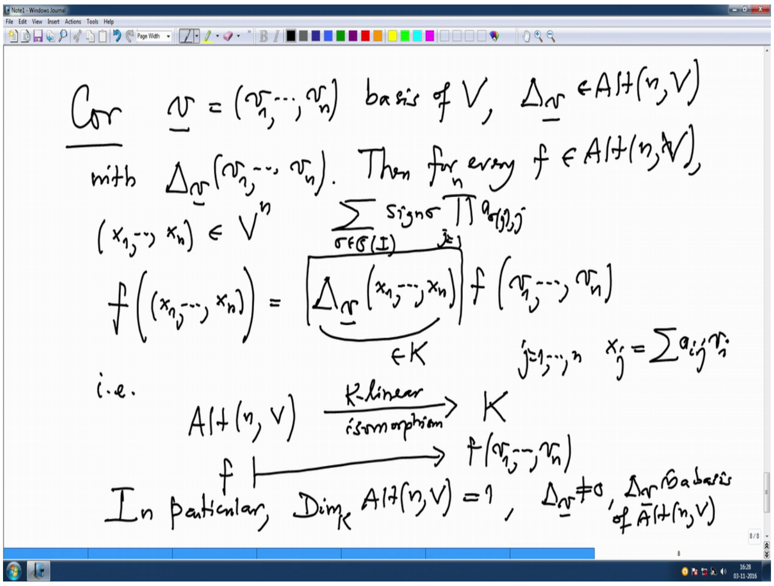

So, let us write one corollary to this let take v to v to vi basis v 1 to vn basis of v, and delta v be the corresponding alternating form with unique delta v evaluated on v 1 to vn is 1, then for every f in alt nV and every tuple x 1 to xn in V power n we can write down the value of f on these tuple and that is nothing, but delta v evaluated on x 1 to xn f of v 1 to vn that simply means. So, that is this is a constant. So, every multilinear form f is uniquely determined by it is value on a basis.

So, so; that means, we can give a map let us give a map from alt nV 2 K any alternating form maps to this v 1 to vn this is an isomorphism the K linear isomorphism. In particular it is one dimensional in particular dimension of the alternating maps alternating forms is one and once the dimension is one we know delta v this form is nonzero. So, this must be a basis on delta v is a basis of alt n V, and this formula I just put the same way you write x 1 to xn in terms of combination of if each xj you write as aij vi j is from 1 to n, and apply the earlier process and this is what we have defined this is nothing, but which will this is nothing, but the sum this quantity we have known this is the sum over sigma in sign sigma product j is from one to n, a sigma j or j this is just that generalized distributivity formula and that that shows everything.

So, we will continue after the break.