Linear Algebra
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Lecture – 54
Determinants (continued)

Now we apply the above determinant theorem and in general the theory of alternating forms to this specific vector space, the standard vector space K power n or also we will write K power I.

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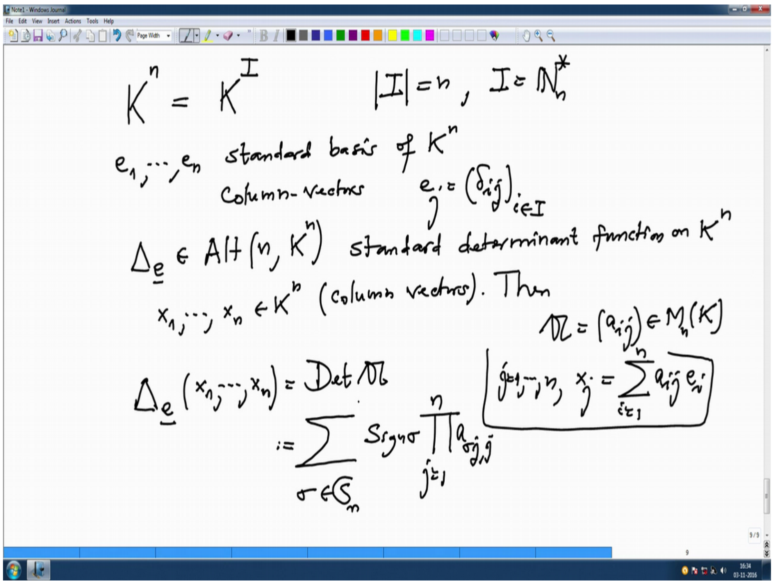

Where I is I has cardinality n or when I is specifically N star n, this is a standard vector space and we have a standard basis e 1 to en standard basis of K power n and remember that we are writing these basis elements as a column vectors.

So, ej is the column delta ij where i is varying in I it is a column; where j th position in that column is one and everywhere else it is 0 and corresponding to these standard basis we will have an alternating form which we have denoted by delta e. This is this is an element in alt n K power n this is called a standard determinant function on K power n. Everything is standard because it is with respect to the given coordinate system and what do we have seen in if I take x 1 to xn columns, column vectors in K power n column vectors arbitrary n column vectors then we have seen delta e evaluated on this tuple x 1 to xn is nothing, but determinant of the coefficient matrix a what is a? A is the coefficient matrix aij this is in M n K and this is aijs are determined by the equations xj equal to summation aij ei i is from one to n and j is sum o1 to n.

So, these equations define this matrix a and delta e evaluated on this column x 1 to xn tuple is determinant a and also we have seen this is nothing, but this is the definition of the determinant actually. So, this is nothing, but summation, summation in the running over sigma permutations of n letters sign of sigma, then the product; product is running from j equal to 1 to n a sigma j j this is the determinant.

Ah this is what how we defined the determinant of a matrix square matrix, you take the matrix look at it is columns x 1 to xn right columns in terms of the standard basis take the coefficient matrix and look at this formula on the right side that is called a determinant and we have also seen that with this definition of determinant of a matrix.

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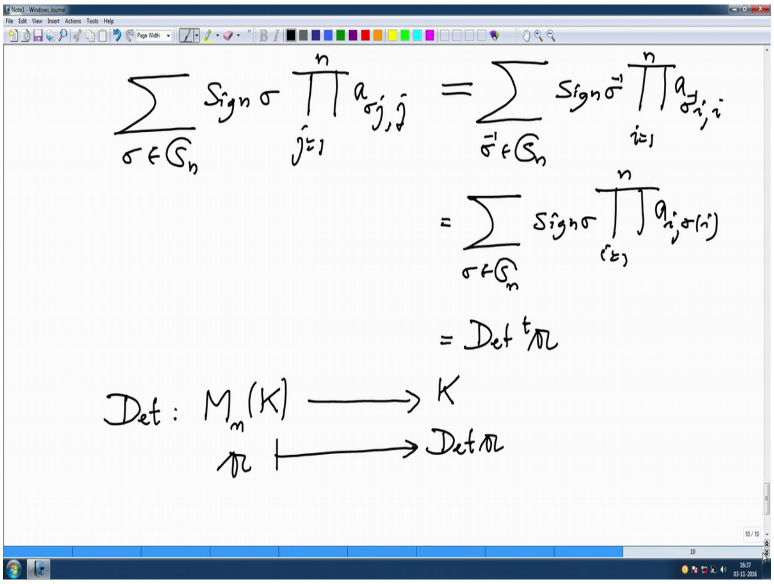

We also know that this formula this sum sigma in sn sign sigma product j is from 1 to n, a sigma j comma j this when I instead of sigma I write sigma inverse in the sum this sum changes to now I will without log I will write the same sigma in a later sigma in sn sign of the inverse and sign is original permutation is same. So, this is same as this. So,

If you want write like this sigma inverse here sigma inverse here then the product; product is now running from I equal to 1 to n a sigma inverse i comma i, but then when I switch to this what will happen is this sigma in sn or this sign is sigma in sn here sn sign of sigma product I is from o1 to n a i sigma i. So, this is by definition the determinant of the transpose because the indices of change this is the so, that means, we have define a map det think of det as a map from M n K to K k (Refer Time: 05:556) any matrix say map it should determinant and we will study the properties of this map.

So, for example, I want to show that this map preserve the multiplication also these map also characterize the invertible matrices. So, we have some couple of basic things to be proved the following. So, first of all I want to note what happens when you change a basis of the determinant functions.

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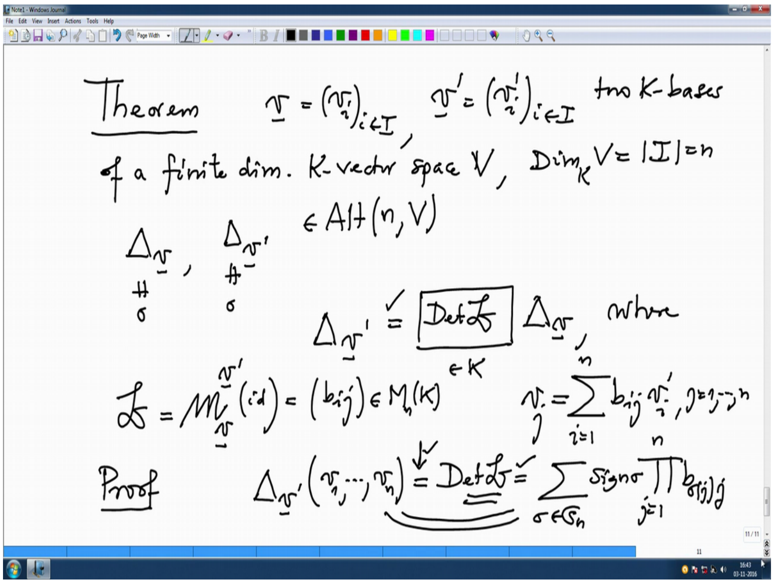

So, let us write it is a theorem suppose I have two basis tuples v is one basis tuple vi, i in I and another is v prime which is numbered by the same indexing said vi because we know any two basis of a vector space have the same number of elements two K basis of a finite dimensional vector space v K vector space v the dimension of V, is cardinality I which will let us put it n, then we know we have a two determinant functions one corresponding to v delta v and another one corresponding to v prime both are nonzero because this evaluated at v 1 to vn is one this evaluated at v 1 prime v 2 prime etcetera is. So, the both these are nonzero elements in the vector space alt nv.

So, therefore, one must be a scalar multiple of the other and an I want to find what scalar multiple. So, that scalar multiple I want to find. So, that is I want to write delta v prime as some scalar multiple, this is some somebody in K I will calculate that times delta v prime; that means, this is a scalar multiple of this not v prime v.

I want to find what is that scalar so obviously, how will the scalar will come out from? Scalar will come out from so, called a transition matrix. So, let me write the answer here. So, this answer is determinant of b, where b is the transition matrix from v prime to v of id if you remember this this is nothing, but bij, and bij this is a matrix M n K and this bij that defined by the equation vj equals summation bij vi prime, i is form 1 to n and j is 1 to n.

So, when you write a basis vectors in the basis v in terms of the basis vectors in v v prime then you get coefficients and that is this coefficient matrix it is called a transition matrix from v prime to v because upper index is always written from where it is, and these are in terms of this you see because of this.

So let us prove this formula this is very easy proof; proof is just only one line. We just every evaluate both sides see these are deter both are alternating maps and they are uniquely determinant by the basis values and we know that delta v prime left hand side delta v prime evaluated at vi prime i in i is 1 to n this is one that is how this is the determinant function corresponding to this basis, but this is equal to now just plug it no I want to compute sorry I want to compute let us I want to compute left side delta v prime on the basis v 1 to v n.

This is this side is what if I want to evaluate delta v 1 to vn this is one and I have to prove that this is. So, enough to note this actually we have to prove this equality once I prove this equality, this equation is clear because you evaluating both sides on v 1 to vn will give you this and it is uniquely determined by this tuple. So, and this is what this is. So, how do you do this? This is just substitute each vj here and expand that generalized distributive law and then you will get this value that is summation anymore sigma in sn sign sigma product; product is from j equal to 1 to n, b sigma j j, but this was precisely the definition of the determinant. So, this is clear and this this equality we know from the general distributive law therefore, this is clear and therefore, this equality is clear everything is very easy.

And now if I want to shift this constant to the other side I know this is b is a transition matrix of the basis from the basis v prime to v, but then it has to be it is invertible matrix because it is a transition matrix.

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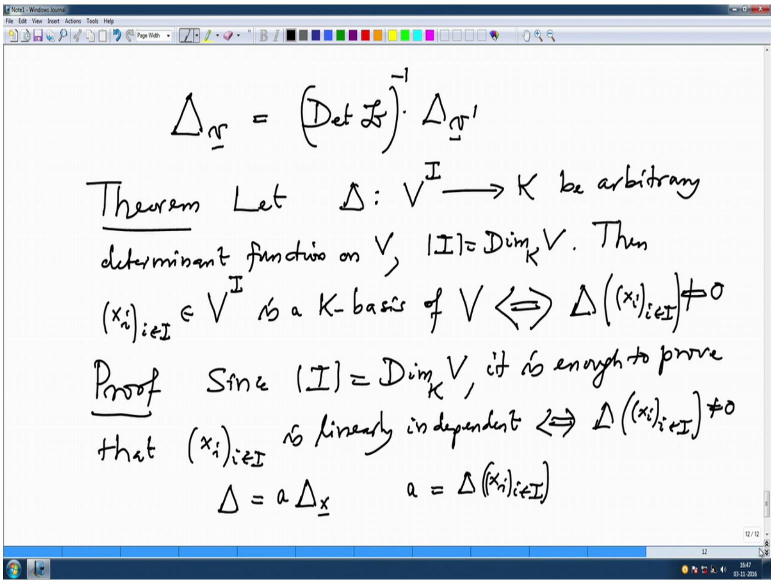

We have proved that it is in in gl gln K therefore, also we could write the same equation as same equation we could have written as delta v prime no delta v, delta v equals det b inverse times delta v prime.

So, I want to specifically note this in a form of theorem. So, let me write this as a theorem, which is very often used. In fact, many people take these has a definition which is not very correct. So, let delta v arbitrary alternating form on v be arbitrary alternating or arbitrary determinant function on v at simply mean this cardinality of I is equal to dimension v and delta is alternating multi linear and then a tuple x xi is arbitrary arbitrary tuple in V power I is a K basis of V, if and only if delta evaluated at the tuple xi is nonzero.

This is in fact, this is one way to prove that any two basis of a finite dimensional vector space have the same cardinality that is namely the cardinality of you know the smallest cardinality where there is a non zero alternating form already we have. So, proof see there are correct in number. So, I only have to prove either they are linearly independent or a basis or a generating system because the cardinality since cardinality of I equal to dimension it is enough to prove that the tuple xi is linearly independent if and only if dealt of this xi i in I is nonzero, but this is what we proved it earlier.

Because the this way or definitely we have proved it, we are proved a contra positive actually we have proved that if the tuple is linearly dependent and the delta of the tuple is zero and otherwise you see this delta; delta function if it is a basis this, this if it is a basis then delta must be some scalar multiple of delta x if it is a basis, where a is actually delta of xi. So, if a is nonzero that is this is nonzero, then it is a multiple it is a nonzero multiple of this determinant function which is nonzero. So, this is the proof.

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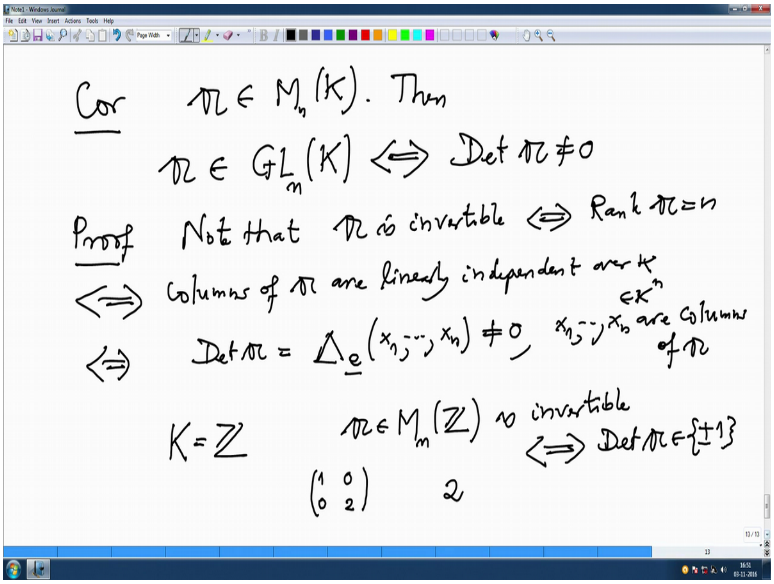

So, when in the corollary in the matrix format suppose you have a matrix a square matrix then a belongs to GLn K if and only if determinant of a is nonzero. So, proof note that. So, the definition in GLn K means a is invertible a is invertible if and only if rank of a is n that is if and only if columns of a are linearly independent over K, that is if and only if det a which is by definition delta standard delta function evaluated at the columns x 1 to xn is nonzero. So, where x 1 to xn are columns of a these are in K power n.

So, that I just want to make one very important comment here many people take this as a definition of GLn K that is should not be; this is a theorem which characterizes invertible matrices by means of this scalar being zero or nonzero. So, this is very important because for example, when you pass on from in general case from a field to arbitrary commutative ring, then it is very important for example, when you take K equal to ring of integers and you consider a matrices over ring of integers and which matrices are invertible are not the matrices with determinant is a nonzero integer, but matrices with determinant is actually a unit in z. So, only possibility is 1 and minus 1. So, matrix a in Mnz is invertible if and only if determinant of a is either plus 1 or minus 1.

So, even for example, if we take two by two matrix say 1 and 2 and 0 0 here diagonal matrix, the determinant is two and this matrix does not have inversing z with integer of entries in z, it has one over two will come in the deter in a in a denominator of the entries where the matrix does not have integer entries it has only a rational entries. So, in that sense this is very very important to write in this way because for generalization this will come in handy.

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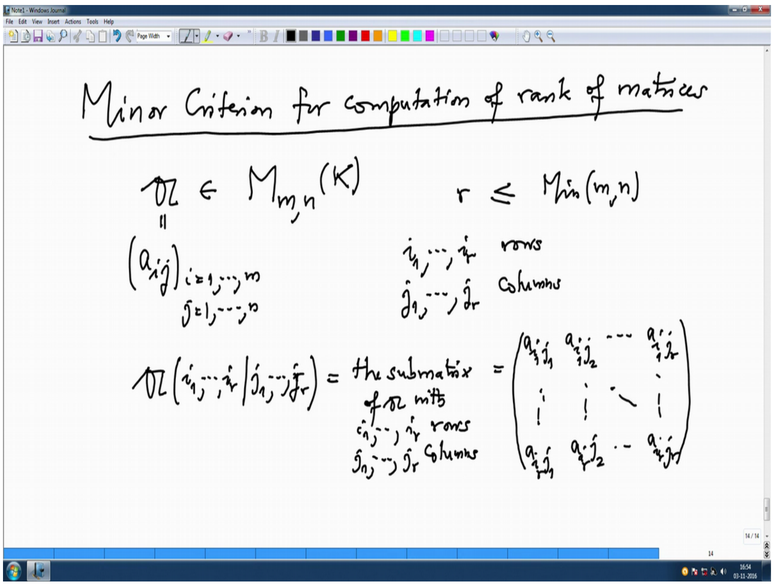

So, now the last observation here; even the rank we can compute by using determinants. So, that is known as minor criterion for computation of rank of a rank of matrices. So, first let me recall what a minor is. So, when you have a matrix you know this matrix. Now you can take it general matrix a is a matrix with m rows and n columns if let us write this is aij. So, i is running from 1 to m, and j is running from 1 to n. If I fix any r which is, so rank cannot be more than the minimum and for any r which is less equal to the minimum I want to choose r rows and r columns, and form a sub matrix of the given matrix a so, that I will write it as. So, suppose I choose the rows i 1 to i r rows and j 1 to jr columns.

So, I only take those entries from these rows and these columns. So, in that matrix I want to denote by i 1 to ir j 1to jr j 1 to jr this is the sub matrix the sub matrix of a of a, which consists of i 1 to ir rows and j one to jr columns and j 1 to jr columns . So, that is also you can write it. So, the first row is i 1 th row. So, ai 1 first column is j 1. So, ai 1 j 1; ai 1 row is same column is j 2 and so on. So, ai 1 jr and similarly i 2, i 3 ir; ir j 1, a ir j 2 and a ir jr actually if you notice lot of energy goes in deciding what is the matrix. So, it is better to deal with the linear maps.

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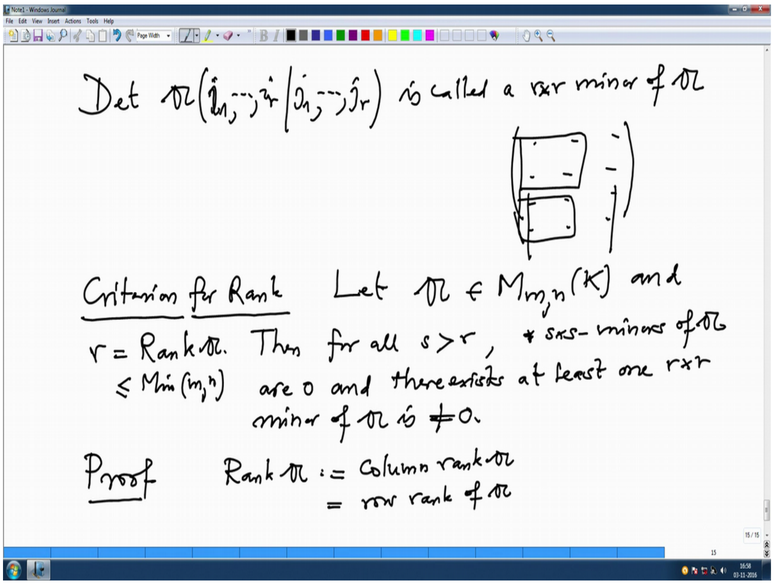

So this is called r minor of a determinant of this determinant of this square matrix now i 1 ir j 1 jr this determinant is called a r by r cross r minor of the matrix a. There are many r cross r minus of a see for example, even if you take three by three matrix or four by three matrix this is four rows and three columns. If you want two by two minus it can be this or it can be this or it can be this column with this column and so on there are so many possibilities. So, there are many r cross r r minus of the matrix now they are criterion says for rank. So, let a be a matrix with m rows and n columns, and r is a rank of a then of; obviously, we know these r is smaller equal to minimum of m comma n, then for all s bigger than r, s cross s minors of a are 0, all s cross s minor where s is bigger than r they are all zero and there exists at least one r cross r minor of a is nonzero then the rank is r.

So, proof this is I will not give the details, but I will just remind you that remember for doing the rank we have proved that rank of a the definition was the column rank that is the number of maximum number of columns, which are linearly independent and we have proved that by going to the transpose we have proved that this is same as the row rank. So, use this act to check this this is very easy to check.

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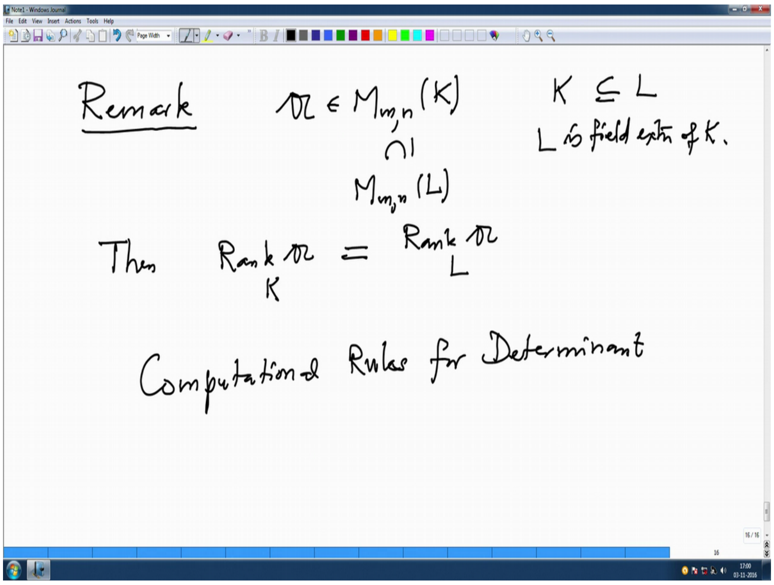

Now I will also want to make one remark which I do not remember if I made it earlier, but it is very important. Suppose we have a matrix a m cross n matrix M n K matrix and K is a field and suppose I enlarge the field l l is a. So, L over K l is a field extension of K then; obviously, I can also consider these as a matrix this is contained in Mmn L this is a matrix with entries in l also a typically rational matrix you can think as a real matrix or rational matrix you can think as a complex matrix.

Then rank of a as a consider as a K matrix and rank of L consider as a matrix in matrix or L these two ranks are same, this is very very important these are already we have applied these in a Gaussian elimination also when I said that if you have a system of linear equations with rational coefficients if it has a complex solution does it have a rational solution answer is yes it has a rational solution also, so this ok.

So, then next time I will consider how do we make competition with the determinant. So, next time we will do computational rules, rules for determinant; where I will also prove I will we will also consider adjoint of a matrix and determinant adjoint formula and so on, so next time.

Thank you.