Linear Algebra

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Lecture - 55
Computational rules for determinants

Today we will see the Computational Rules for Determinants.

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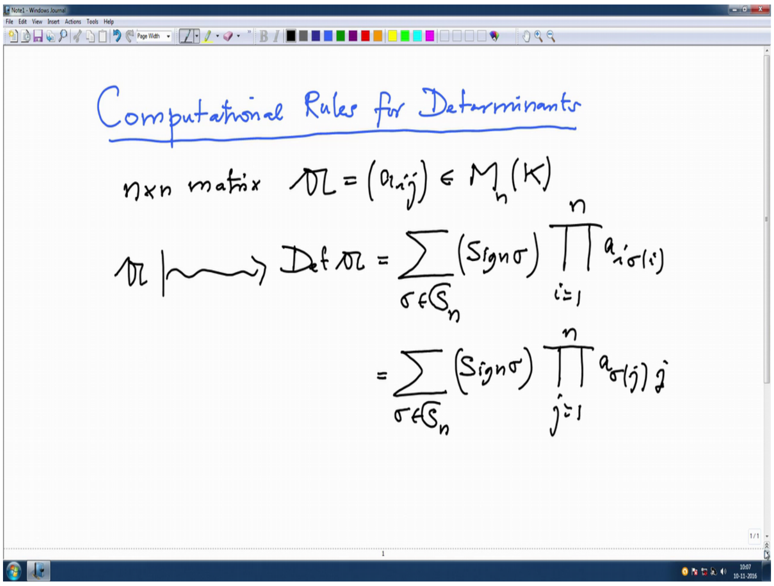

So, let me just briefly recall that we have defined determinant functions and now we want to use this to compute determinants of matrices. So let us recall that; what was the determinant of a matrix. So, for a determinant of a matrix we need a square matrix n cross n matrix, which is a integer a ij and integer in some field k. So, this is an element of M n K and to these a we have associated a scalar which we have called a determinant and this determinant is by definition it is a summation sigma in S n permutations sign of sigma product i is from 1 to n a i sigma i.

Also we have seen that this is also equal to this equality follows by replacing sigma by the inverse of sigma. And this is sign sigma product j equal to 1 to n a sigma j j. And we have also seen when we constructed determinant functions we have also seen how do these right hand side came out of the alternating multi linear forms on an n dimensional vector space. And that motivated us to consider this scalar and that is called a determinant. And we have seen some properties of the determinant functions. And today we will leave some computational rules which will be useful for computation.

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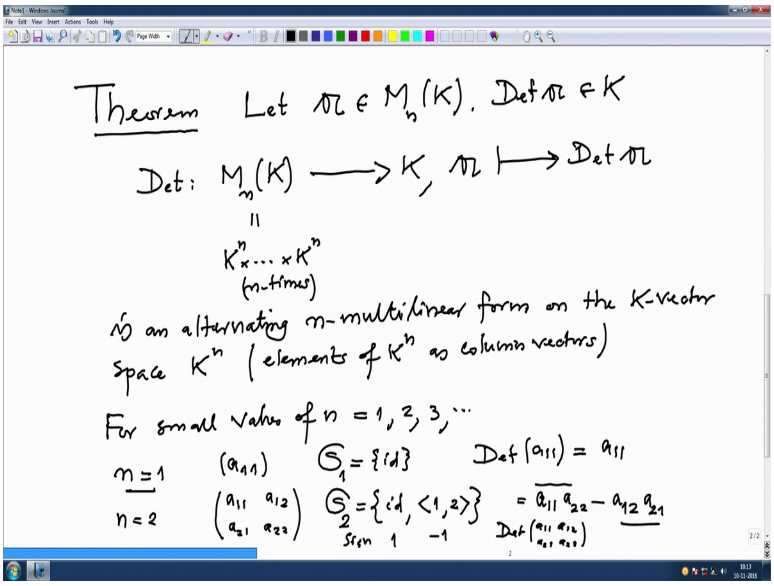

So, first of all I want to think so let me write in the form of a theorem. So, that says the following. So, this theorem: think of determinants as a map. So, let a, be a square matrix. So, think of then we have seen the determinant the determinant of a is a scalar; that what we have just recalled a definition. And think of this Det as a map from M n K to K K goes to the determinant of a. And think of these M n K as K power n cross K power n cross K power n n times. And this K n the vector space of n dimension.

And you can think the tuples as think of these elements of K n as columns. So, with that this is an alternating n multi linear form on the k vector space K power n. And now you should remember here elements of K power n we are thinking as column vectors. So, this is what we have proved earlier. Now, I want to consider the properties and properties of this pretending map.

For example, how do you characterize invertible matrices? How do you find the inverse of invertible matrices? Also whether this these induces whether this map determinant map, whether it is product preserving and properties like this. So, before I continue I just want to mention that for small values of n namely n equal to 1, 2, 3, what we define is the usual definition of the determinant that one learns in the school. And this is I want to check this little bit. For example, for n equal to 1 it is 1 cross 1 matrix. That means, the matrix is just a 11. And if you see the formula that is if the summation is running over S 1 permutations of later one, but these only one permutation of later one that is id-identity.

And therefore, the summation will only one term, and therefore determinant of a 1 and the sign of the identity permutation is 1 therefore this is nothing but a 11. Similarly for n equal to 2, you have the matrix we usually write in the school as a 11 a 12, a 21 a 22. And in this case now the group involved permutation group is s two which has exactly two permutations: one of them is identity and the other is the transposition 1 goes to 2. These are the only two permutations of the set 1, 2.

And the sign of identity is 1 this is the signs and sign of the transposition is minus 1, and then the sum is running over sum as two terms. The corresponding to the identity term is so a 11 times a 22, because the second index is the image of the first one. And the second term in the summation comes to the minus n because sign over transformation is minus 1, and then this is a 1 and 1 goes to 2; therefore, a 12 and a 22 goes to 1 therefore a 21. This is the determinant therefore. In this case determinant of a 11 a 12, a 21 a 22 this is this one.

And this is usual one, a product of the diagonal that is this term; product of the diagonals and product of the anti diagonal with the minus sign that is this term. So, this was what usually was mentioned at least when I was in a school in 11th standard.

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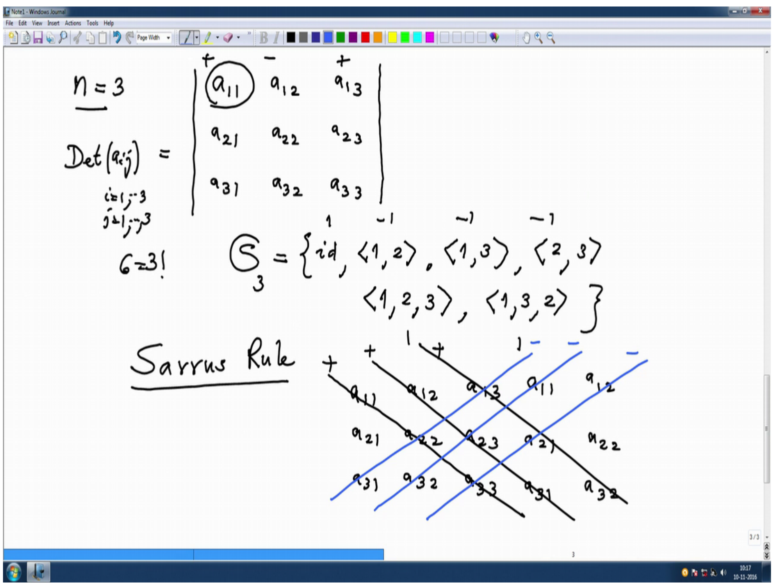

Similarly, for n equal to 3; I will not check this, but n equal to 3 what one should check is and if you write like this a 11 a 12 a 13, a 21 a 22 a 23, a 31 a 32 and a 33 these determinants. Sometimes I will one writes instead of Det one writes these two vertical bars, this is by definition the determinant; determinant of this matrix a ij it is this ij is 1, 2, 3. And we want to find this determinant.

So, the usual recipe given was the a 1 fix this guy, then you take the 2 by 2 determinant this is with the plus sign then with the minus sign this guy and then remove this row and this column in this row and then that one with the minus sign and again with the plus sign say. So, it will have for these will have two: terms one with the positive sign, one with the negative sign. For these also have two terms: one with positive sign, one with a negative sign. For these also you have two terms.

So, in all together there will be six terms. Six terms comes because the permutation group involved here is S 3. And S 3 especially six element, three factorial is the order of S 3 and you can write down the elements of S 3: one of them is identity then we write down the transpositions that is 1 goes to 2, 2 goes to 1, and 3 is fixed. Another transposition 1 goes to 3, 3 goes to 1, 2 is fixed; then another transmission 2 comma 3. These are the transmission.

Now come the three cycle that is there two of them: 1, 2, 3- 1 goes to 2, 2 goes to 3, and 3 goes back to 1. And then 1, 3, 2. And when check that these are we know for sure that these are three factorial element that is six. And these are also six element so that they are different, but that is clearly different and this will have signs; this has sign 1, this has minus 1, this has minus 1, this has minus 1, this has 1 and 1.

So, three signs with come with the positive, three is comes the negative. And those are the term, but these also instead of doing this there is a there is a rule which is attributed to this Sarrus: this if one uses this rule there is less likely to make any errors in numerical calculations. So, what do you do is- you write down these columns a 11 a 21 a 31, a 12 a 22 a 32, a 13 a 23 a 33; these are the original matrix. And write down the two columns again next to them.

So, what I written in I have written the columns first and second column again. And what do I do? I look at this product and there with a positive sign, and then I look at these products with a negative sign and that is the determinant. So, see one advantage is here you are only doing the same, here you are doing these and these n n minus n so these may be a possibility to do error that can be minimized if one use a this Sarrus rule.

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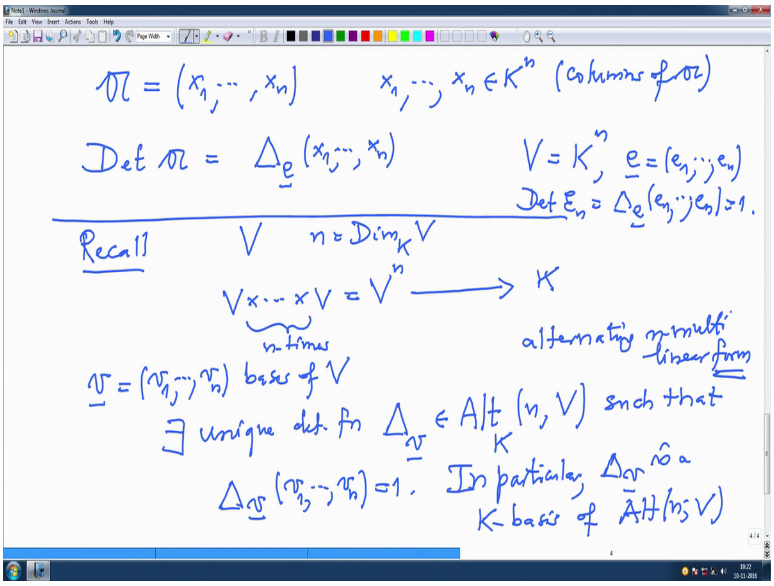

Now another thing what I will quickly recall is. So, if you have a matrix a I will write as columns so x 1, x 2, x n; where x 1, x 2, x n are elements in K power n they are columns of a. And then you note the determinant of a the way we have defined is nothing but delta standard bcc of this columns of x n. Let me just recall quickly here.

Remember here, when we defined determinant functions that on a vector space V of dimension n the determinant function is a map form V cross V cross V n times. These I will keep writing V power n to K which is alternating and n multi linear form. Form word is used when the values are scalars. And you have to check that these were the main theorem of determinants theory that we have whenever we have a basis v v 1 to v n basis of V. There exists a unique determinant function that will depend on this V so that you called it delta v; delta v is an element of alternating in multi linear form. So, that is an element in Alt K n V so n is the dimension such that delta v evaluated on the tuple v 1 to v n is 1. So, these are the main theorem.

And then as a consequence we have proved that these alternating n linear forms a vector space of dimension 1. And therefore, this delta v because it is nonzero element because on the tuple we want to be n it is nonzero this will be basis of particular delta v is the basis of k basis of Alt n V. And so now our vector space in this case because we have fixed a coordinates or vector space V is K power n, and the basis I am taking a standard basis e so that is e 1 to e n.

So, corresponding to the standard basis we have a standard determinant function that is delta e. And when you evaluate this delta e on e 1 to e n obviously you get the determinant of a identity matrix and that should be 1, because we see it. So, this delta determinant of identity matrix is nothing but delta e on e 1 to e n which is 1 and in general determinant of arbitrary matrix is delta e of the columns.

So, therefore this is because this is an alternating and multi linear map I have already the built in properties for the determinant; namely if I have two columns are equal in the determinant is 0 because it is alternating map. If I change one column by adding one column to the scalar multiple of the other column then the determinant will not change, because it is alternating when you expand it those property.

And whatever I do it for columns similarly for the rows. So, elementary operations on columns; the first two operations will not change the determinants when you exchange the rows or columns the determinant will change by a sign, when you multiply a particular column by a scalar the determine will change by that scalar. So, all these properties are built in this definition. So, I will not repeat them; I will not explicitly write and repeat them here.

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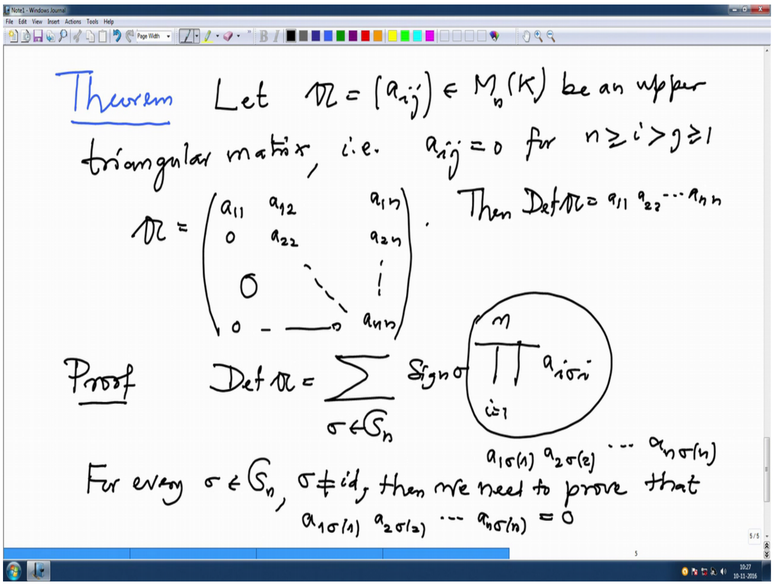

So, the next one: now let us compute determinant of some special matrixes for example: let me write this as a theorem. Some of these one may feel they are obvious, but obviously one need to prove them formally and it is very important. So, let a be m cross n matrix a ij, be an upper triangular matrix. Upper triangular means that is a ij's are 0 for n bigger equal to i strictly bigger than j because equal to 1.

That means, in the row number is bigger than the column number the entries are 0. That means, the matrix a will look like this on the diagonal it is a 11 a nn, below diagonal will be 0, and here it is as usual entries a 12 a 1n and so on. Here it will be 0 a 22 a 2n and so no. Here there all 0s. This is an upper triangular matrix this we below main diagonal all entries are 0. So, this row number is 2 column number is 1.

So, row number should be bigger than the column number strictly bigger than we need support triangular when it is the other way when the column number is bigger than the row numbers then it will be lower triangular. And whatever we prove it for upper triangular the similar result will be true for low triangular as well, because we can simply work with the transpose of the matrix. And we noted that the transpose of the matrix and them original matrix the determinants are the same, because of the two equalities. And what is the determinant of this? So, that is this one and determinant of a Det a is nothing but the product of the diagonal entries. This is very easy to prove, so proof.

So, by definition determinant of a is the sum, sum is running over the permutations sign of permutation then the product i is from 1 to n a i sigma i. Instead of writing this I could also write spell it out that is a 1 sigma 1, a 2 sigma 2, and so on a n sigma n. So, these some this summation is running over n factorial permutations.

And the commands are sign and then this (Refer Time: 21:55). So, we just have to see what happens. And this term, this a 11 a 22 a nn this is corresponding to the permutation sigma is identity, because when sigma is identity sign is 1 and then this sigma 1 is 1 sigma 2 is 2 and so on, so it is the required.

So, all we need to check is for every permutation sigma in S n if sigma is not identity then we need to check that the sum one corresponding to be sigma in this summation is 0. So, then we need to prove that a 1 sigma 1 a 2 sigma 2 etcetera a n sigma n this term this product is actually 0. And this product is where there elements of the field and the product is 0 so it is enough; if we prove that one of them is 0.

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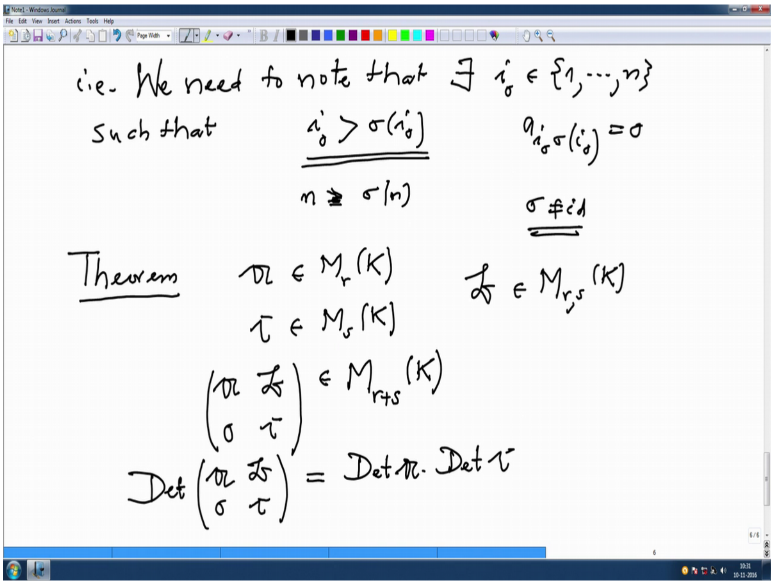

That means, we need to find that is we need to note that there exists an index i naught from 1 to n such that i naught is bigger than sigma i naught. Sigma i naught is a column index and this is the row index. So, if we can prove this there exists an index for which this is 0. So, that will mean that a i naught sigma i i naught this is 0, because it is a upper triangular, but this is very easy because you start from the top end so if look at the index n and compare n and sigma n; sigma is not identity if this is equal this n sigma n cannot exceed n. So, this is always.

So, if it is equality here go for the lower index and keep comparing them. And all of them cannot be equal because sigma is not an identity. Therefore, we can we can definitely come to a situation where index row index is bigger than the column index. As I mentioned earlier, similarly for the row triangular matrices; so finding I determinant of a upper triangular matrix let me use a triangular means either upper or lower triangular. And I will not mention the numerical examples because one can check them easily once you know the theory correctly.

So, the next theorem I want to mention about the block matrices. That means, I have a matrix i have matrix is a is r cross r matrix; c is s cross s matrix square matrix is these two are square matrices; and b is r cross s matrix M r s K. And using these three matrices I form a another square matrix which will be like this a here, c here, 0 matrix here- this is

0 matrix, and this is a b matrix. Note that the number of columns is r here, number of columns is also s here.

So, it has r plus s columns. And let us check the rows these has r rows and this has x rows. So, it has r plus s rows and r plus s columns will be the square matrix of order r plus s. And in this case you want to prove a formula determinant of this matrix a 0 b c with determined is nothing but determinant of a times determinant of c. This is very useful for computational purposes.

So, what do we do? As I mentioned in the beginning that we do elementary operations of rows or columns and I will stick to one of them or when I do both I will mention it. In any case when I do row operations on a for example from a I will come back to a prime transform a to a prime and what the usual procedure what we adopted in a gauss elimination.

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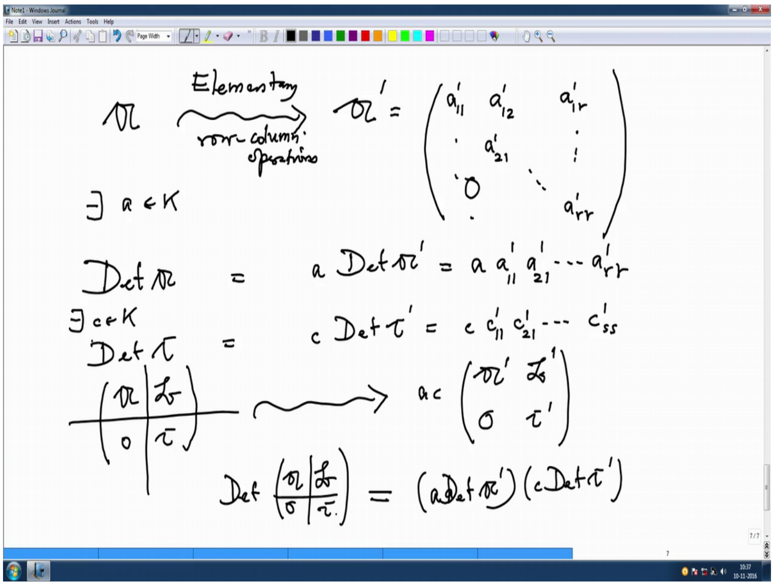

So, use the pivot element and bring it on the top. These entries will check if this entries will change. So, by row and column operations and with the minimal exchanging the rows or columns or also minimally multiplying by scalars we can transfer this matrix a to the matrix triangular matrix like this, which r cross r matrix. And if I would I have use elementary operations which I have to multiply by scalars then the determinant will change by a multiplication of that scalar. In particular if I want to change the columns or rows then the determinant will be minus.

So, in any case what will be the difference between the determinants: determinant of a and determinant of a prime they will differ by only by a scalar. So, this equal to a times for some a in the scalar there exists when I make these operations. So, these are elementary operations row or columns; operations to bring it to this form which is triangular upper triangular so determinant will change by this. And we have seen the determinant of a upper triangular matrix is nothing but the product of the diagonal entries.

So, this is a times this may entries a 1 a 11 prime a prime 21 or a prime rr this is all the determination. Similarly for c I will do the similar thing. So, for c similarly they will exist a scalar c such that the determinant of c will change it to c times determinant of the new matrix c prime which will be upper triangular, same organ. So, this will be c times c prime 11 c prime 21 c prime ss, same organ.

Now what do I do? I have this matrix this block matrix a b 0 and c. Now what did we do? We have made row and column operations on this to get the matrix a prime. Now I do the same row and column operations which I did for bringing a 2 a prime. So, this will not meddle this will only meddle with these blocks. And similarly for c I will do the same row operation here and same column operation that will not change the form of a which i brought it to a prime. This b might change but we do not care about b.

So, by the same row and column operations these block matrix will get transformed to the new matrix which will has to be multiplied by this a and c, and then this matrix will get of a get the form is a prime some b prime and c prime here and 0. So, this new matrix now because a prime is upper triangular c prime is upper triangular these new matrix is also upper triangular.

Therefore, the determinant of this matrix is nothing but the product of the diagonals here. And that product of the diagonal here is nothing but a time the product of the diagonals is nothing but the determinant of a prime and similarly this one. Therefore, these determinant is nothing but a times determinant of a prime times c times determinant of c prime.

Note here I have use the commutatively of the field. So, you one might think that things do not work when your base field is not commutative. There are interesting examples of fields, they are not called fields skew fields skew field means it is a ring may not be commutative, but every element, every nonzero element has a inverse. These are also interesting object to study especially in physics because quaternions, but a linear algebra will not work very well with that. So, the determinant of this is this. And then we just now prove that these determinants are equal to this determinant. And therefore this is nothing but determinant of this. So, this proves the theorem.

Now, we will take a short break and then we will come back.