Linear Algebra
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Lecture - 56
Properties of determinants and adjoint of a matrix

Now in this second part of the lecture we will prove very fundamental result which is used very often for a computation in determinants and besides that it is also interesting in it is own right, that it is what is called a product formula.

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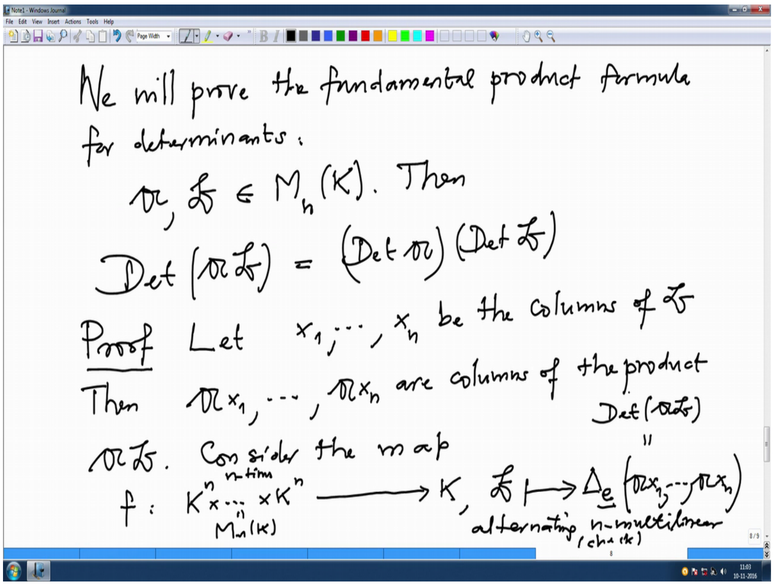

So, we now come to we will prove the fundamental product formula for determinants. And also we will use these product formula for the computation of inverse also. So, this is use very often. So, what do you want to prove? I want prove that if I have 2, 2 square matrices a and b of the same order then determinant of the product equal to product of the determinants ok.

So, proof, I will use the fact that these determinant function is alternating and n multi linear. So, let us denote let x 1 to x n be the columns of b. Then a x 1 a x n these are columns of the product a b. That is how the matrix multiplication is defined. And now consider the map, let me call it f this is from k power n cross, cross, cross k power n remember, this is n times, n times this is nothing, but M n k columns. So, what is the map and from here to the scalars? What is the map? Take any matrix b and map it to delta e, delta e is the standard determinant functions on k power n. And map these 2 evaluated these columns a x 1 a x n. This makes sense, this is a map from k k n k n n times to k or M n k 2 k this is a map define.

First thing I want to check that this map is alternating n, n multilinear. First let me check multilinearity, multilinearity in each variable, but you see if I change this matrix b. So, let me write formula so on. So, because this in each variable this is linear therefore, it is multilinear. To check it is alternating I have to check that if 2 components are equal, then they are equal 2 components are equal if b. So, this is clearly alternating in multilinearity. So, I will just simply say check yourself, is nothing much to check.

So therefore, if it is alternating then what is the. So, this is by definition this is nothing, but determinant of a times b, because these are precisely the columns of a, a times b and then this is a standard determine functions.

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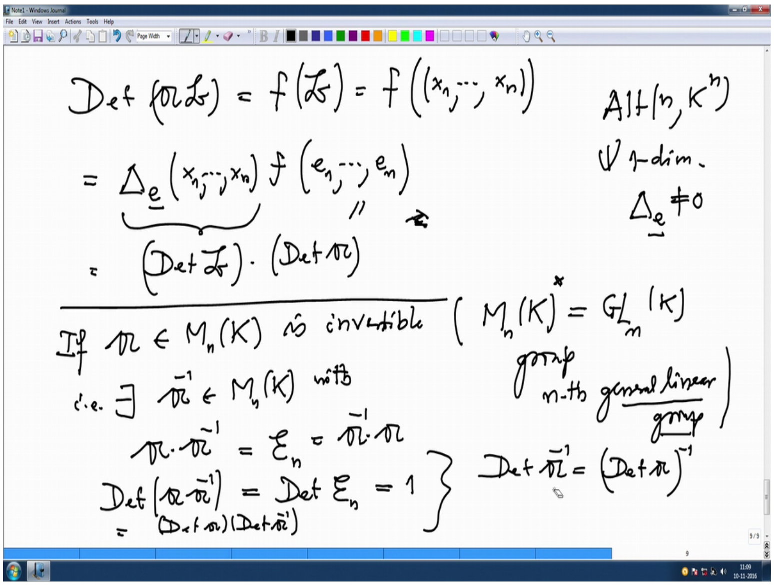

So, these determinant of a times b. So, determinant of a times b, this is f of the matrix b that is the definition of f of b. And this is f of x 1 to x n a columns of b, but we have seen that any alternating n multilinear map is a scalar multiple of the standard determinant function. Because this Alt n k power n this is one dimensional and this standard determinant function is a nonzero element in this therefore, this nonzero element will be a basis of this vector space, which is one dimensional and f is one of them therefore, f has to be a scalar multiple of delta a and b. Scalar multiple that we also know how to do it this is delta e of x 1 to x n times f of e 1 2 e n.

This, these scalar is to precisely the determinant, determinant of b. And this 1 is by definition determinant of a. Remember this is by definition e 1 to e n have the columns. So, this is b is e 1 to e n that is b identity. So, these determinants of a. So, that proves. So, you see this proof is not very difficult, but it depends, is easy because we know these vector space is one dimensional. And the main theorem of determinant theory says that there is a standard determinant function and therefore, every other alternating multilinear map is a scalar multiple of that and we know how to calculate, by using the properties of alternating multilinear maps. So, that proves the product formula.

Now suppose a this square matrix a is invertible, if a invertible matrix. Remember our notation that set of invertible matrix is a set of inverses in this ring M n k is a ring, it is non cumulative the inverses in this ring, that is inverse is here that is our this is our notation and this is precisely we are denoting GL n k. And this is clearly a group, because it is invertible elements in a multiplicative monoid, and it is a group this group is also called general n nth general linear group. This is very, very important group even in other branches of mathematics analysis and league algebras league group etcetera so on. So, if it is invertible then I know there exist in inverse. So, that is there exists a inverse in M n k with the property that a times a inverse equal to identity, also equal to a inverse times i. Such a inverse is unique and we need to write down the both the conditions in the definition, but we have checked in our using the theory of final dimensional vertex precise.

One in one equality it is enough to conclude the other also because linear maps which are injective are surjective and bijective, surjective, injective they are all equivalent. And these I keep calling a pigeonhole principle linear algebra. So, you have this. So, when you apply determinant on both sides we get determinant of the product equal to determinant of the identity matrix and determinant of the identity matrix is; obviously, 1. Because, if you want to formally say it you these are upper triangular matrix with 1, 1 on the diagonal, and we have just proved the 1, 1 on the diagonal the product of 1, 1 which is 1. So, on the other end these product is, these product is the product of the determinants.

Therefore all together these use equation in the field. Now, therefore, we get a nice formula determinant of a inverse equal to determinant a inverse so; that means, if your matrix is invertible whether you apply inverse first and then the determinant, or determinant portion the inverse these 2 operations are compatible ok.

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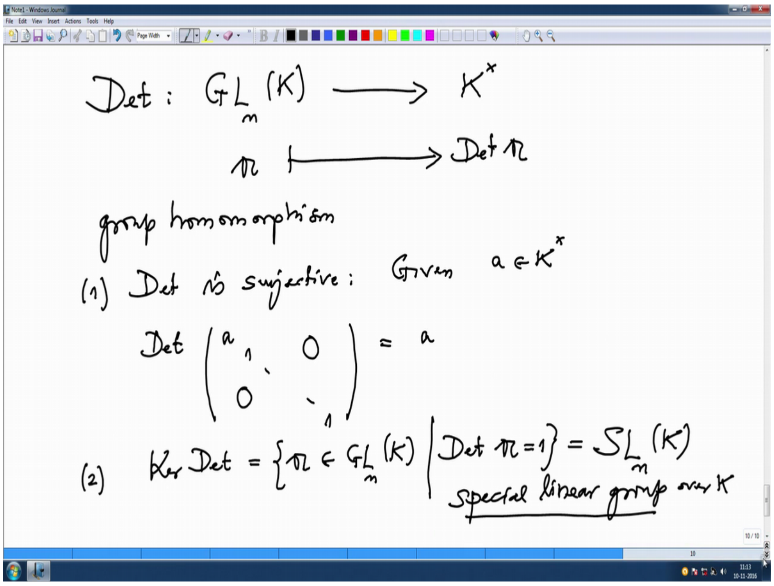

So, moreover also think of, so a determinant gives you map therefore, from GL n K to k cross. Any a goes to determinant, we have noted a invertible matrix cannot have 0 determinant because the product is 1, and if one of them a determine is 0 product cannot be 1, you are in a field it is important and the product formula tells you this is actually a group homomorphism. And whenever you have a group homomorphism then you can talked about the kernel image isomorphism theorems and so on.

So let us see in this case first of all this map is surjective. I say 1 this map is surjective. So, what we I need given any scalar I want to find a determinant whose find a matrix invertible matrix if determinant is that. So, given nonzero scalar you got the matrix any six, look at the matrix in put a on the top end you on the main diagonal and put remaining into is to be 1, 1, 1s in this is the special matrix. The determinant of these matrix is; obviously, the product of the determinant a product of the diagonal matrix which is a.

So therefore, a even the image of this map. So, that is the determinant map is surjective. Secondly, what is the kernel of the determinant map? Kernel of the determinant map is a very important group which is which is the set of all matrices invertible matrices such that it goes to identity. Remember identity in this k cross group is 1. So, that mean all those invertible matrices with determinant is 1. These are this is a special group therefore, it has a special notation and special name, means called a special linear group SL n k this is special linear group, special linear group always over k.

So, remember this when k is a finite field. Then the GL n K is the finite group, and in this case it will be interesting to compute it is order and other invariants of the group, which I will not do it in this course. Similarly for SL n and so on. So, though I so we know the kernel we know it is surjective therefore, we can also apply isomorphism theorem which will say that GL n mod kernel because it is SL n k is the kernel of a group homomorphism, it is normal subgroup therefore, this quotient make sense.

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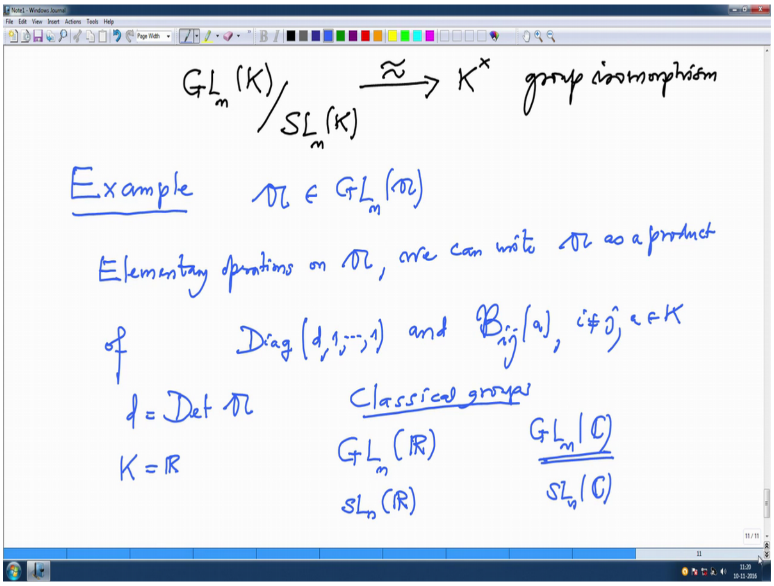

This is a quotient group and this group is isomorphic to it is image, but image is everybody. So, this is k cross. So, this is a group isomorphism.

Ok now I want to recall let me write as an example. This is what we have done it in when we studied elementary matrices. So, I will just summarize in this example with this new language what we have done in that. So, remember if you have given invertible matrix a over a field, you have by using elementary operations on a. Elementary operation means row column and all those operations. We have written this a So, we can write a as a product of a diagonal matrix diag d 1, 1, 1, 1, 1 and elementary matrices

So, let me write a we can write a as a product of the diagonal matrix. And elementary matrices B i j a where i not j and a is in k. If we did it very carefully in that elementary matrix is product, normally you get the diagonal matrix, but then you have shifted the diagonal entries successively to the upper position then we brought this to the top position. Now because the determinants of this elementary matrix says is 1 and when a the product of this and these this d these entry d is nothing but the determinant of a and if this original matrix where I have determinant one then I actually this is identity matrix.

So, you will not require this. So, that show that that shows 2 very important observation which I want to say in terms of group theory. So, this means the group GL n a is generated by these elementary matrices, and this type of diagonal matrix generated by them because every matrix every g element in GL n we are write in a product of this. So, therefore, the matrix GL n is generated by this type of diagonal matrices and a elementary matrices of these are elementary matrices of special type they are not elementary matrices their a 1 of type 2 i thing in the notation. So, this is very important when for example, when you prove.

So, it is finitely generated in particular not finitely generated these, these matrices are not finitely many a number unless the field is finite. So, if field is finite k is finite then there actually finitely. And you know the generators also very specifically and SL n k is also generated now you do not need this. So, SL n k is generated by these elementary matrices. So, for example, when you want to prove when k is in a in a classical case k equal to r for example, and if you want to prove results about GL n R or SL n R and similarly for GL n C and SL n C. Suppose, So in this particular case these groups are not just groups. They are also they also have topology which will in a read in a read by the usual topology of R n or C n, and these are even the group operations are continuous that mean these are league groups. And if you want to proof for example, GL n C is connected you use this statement which will be very need to check that GL n C is connected, because you connect only elementary matrices. And once you connect the generator then whole all elements will be connected path connected and so on. So, this is very, very important when you study what is called classical groups. With that I will I will not make more dimension this ok.

So, the next is, next is now we can write more, more, more results about these group theoretic results, but I will not do that for a next one also 1 could describe the center of this group and so on. And their quotient groups and so on, but I will not do it in this course orders and so on and it is connection with the other group other finite groups, which I will not do it now.

But now I will concentrate on the finding the inverse. Finding the inverses, finding now finding the inverses of a given invertible matrix.

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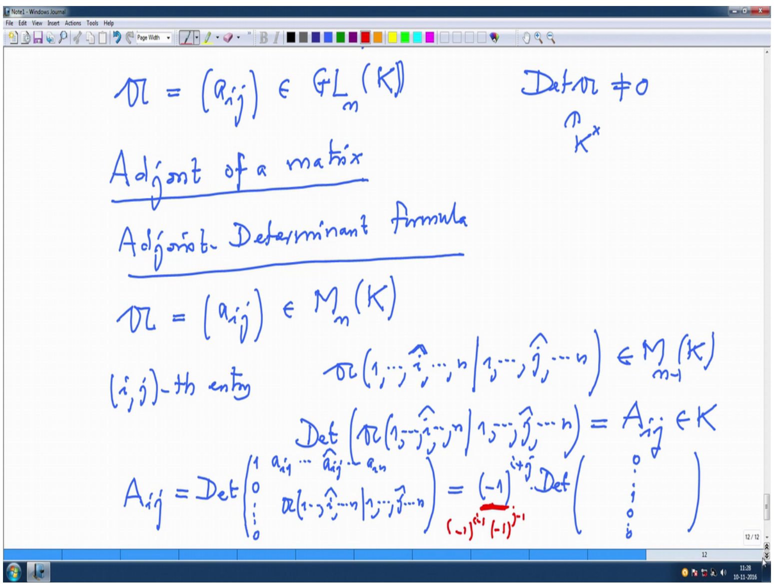

So, these are the computational problems. How do we find the inverse of a given matrix a? So, a is a matrix given, and we will also know for sure that it is invertible. And how do we check that? For this checking is easy we just compute the determinant of a and check determinant of a is non zero; that means, it is an element in k cross. And we know that if determinant of nonzero then the matrix is invertible this we already noted earlier.

So, having noted this first thing to have to note this and then you want to write the inverse. Explicitly in terms of the entries of a and for this what we need is what is called adjoint, determinant adjoint of a matrix and the relation between adjoint and the original matrix. So, these which I will call it adjoint determinant formula, so these 2 topics first I will in spend some time to define adjoint. And then prove the adjoint determinants formula. Actually it will also be not bad idea to do it like what we did it for the function first or for the linear maths first. And then prove the formulas for that, but I will do this time with the coordinates. So, I am now preparing to define adjoint of a matrix.

So, start with a matrix a the adjoint of a matrix is always defined. So, start with a matrix a is a i j this is n cross it is a square mat for adjoint it defined for a square matrix. So, start with a matrix and what do I do is I fix a i jth entry. So, fix I jth entry; that means, I fix ith column in ith row and jth column. So, and from the matrix a i remove ith row and jth column so; that means, what and the notation is I will now use 1 i ith row is here and n these are all rows and I will put a cap on i this means i removed ith row.

Similarly for 1 j I will removed jth column. So, I got n minus 1 rows and n minus 1 column. So, I got a square matrix of order. So, this is a matrix in m n minus 1 k. Some sometimes so the determinant make sense, the determinant of this matrix. So, determinant of this matrix a 1 this n 1 j this is one scalar. The scalar I am denoting by capital A i j it depends on I it depends on j and it is some scalar. If I, if I want to write this a scalar as a determinant of from 1. So, so this is also can be thought as keep the same thing I want to make it. So, I will, I will write a formula and then check. So, this is a i j is also determinant of course, keep this keep this matrix and I want to enlarge it by a enlarge it to n cross n matrix by adding row and a column.

So, I add the first column I added as the first column, first column is 1, 0, 0 and here I add to the ith I put in the ith, ith row on top here, but only the entries from these columns so; that means, I am adding this row as a i 1 etcetera; obviously, I will omit the entry a i j and then you keep a i n. Note that this is ith row except I have not taken the jth column instead of that I put this column. And here is the matrix a, this matrix.

So now, node these equal it is clear because I will expand in terms of this first row. When I expand in terms of the first row it is one time this determinants which is, which is this and if I want to bring this now in the middle in at the jth row. This first column if I want to bring it as the jth column and these entry 1. I want to bring it at the jth ith row then what do I would do I would do change interchange rows and columns and if you notice I have to change I have to change i minus 1 interchanges of rows and j minus 1 interchanges of the column.

So, this will all together be minus 1 oh, minus 1 power i plus j and then the matrix obtained from a by remaining ith the jth, jth column to be just 1 and i jth k and 0 everywhere else. So, this is precisely the determinant of I will write only the jth column that is 0, 0, 0, 1, 0, 0 this is a jth column, and remaining entries are like matrix a. So, this is called co i jth cofactor. So, remain note the sign the sign is very important to note. And how did we get this? We got this because what happened on this sign, this sign is we got it by interchanging ith row successively i minus 1 through i minus 1 interchanges of this row and each time we will get minus 1 time.

So, minus 1 power i minus j will come from row changes and then again change it to jth column so; that means, i minus 1 changes j minus 1 changes of the column. So, this will be this sign. So, altogether it will be see minus 1, 1, 1 minus 2 that will even.

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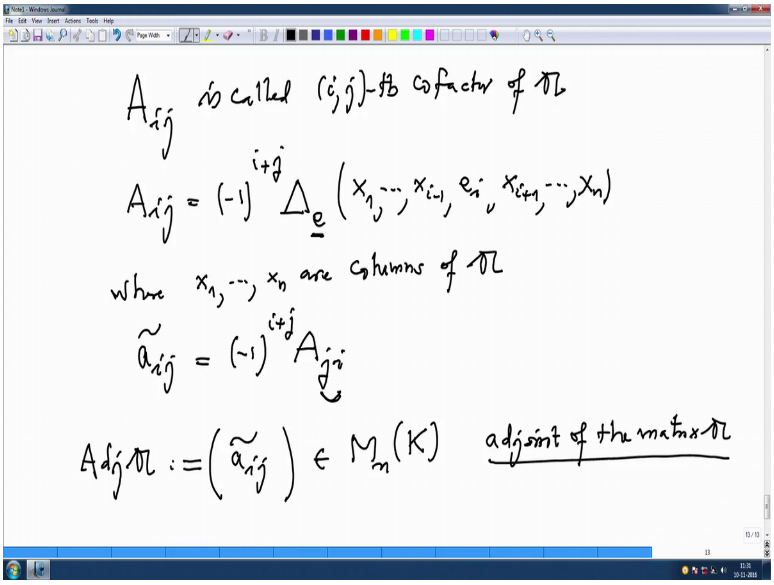

So, it will disappear and then it is same as this. So, this is called this capital A i j is called i jth cofactor of the matrix a. And note that because of that this i j a i j which is just now I said minus 1 power i plus j and times what? So, I want to write precisely I want to write this, this determinant. So, what are the columns? Columns Are the original columns of a, only this column is change.

So, that is delta with this is the standard determining function then x 1 x i minus 1 and here it is e i next is the x i plus 1 to x n. Where, x 1 to x n are columns of a, alright so, and the matrix now I take transpose of this matrix. So, I will take a i j. So, let me define entries a small suffix a i j tilde. This is by definition minus 1 power i plus j and in capital a j i. Note the change in the indices. This matrix a i j, a i j tilde this matrix, this is a matrix m cross n matrix, and this matrix is called a adjoint of a. So, the notation is Adj a this is called adjoint of the matrix a. And now next time I will prove so called adjoint determinant formula, and essentially we will prove that these adjoint matrix will help us in computing the inverse of a matrix, when a is invertible ok.

Thank you.