Linear Algebra
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Lecture - 57
Adjoint-determinant theorem

So let us continue this lecture with what we have planned last time namely Adjoint determinant formula. So, I will prove it today.

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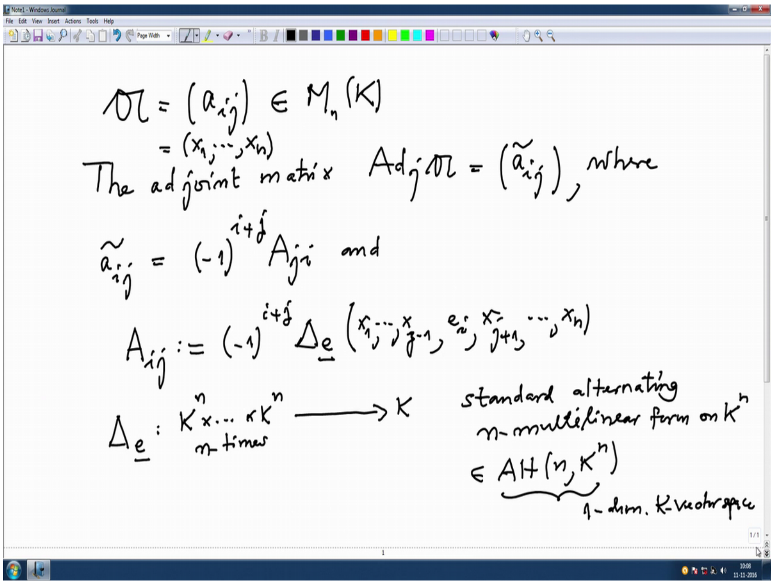

So, let us recall what we did last time last time for a a matrix square matrix a a i j in M n K we have defined the adjoint matrix, the adjoint matrix that we have denoted Adj a and this is defined a i j tilde where a i j tilde is defined minus 1 power i plus j capital A j i and what is capital A i j.

And capital A i j they are defined I will define them in terms of the standard delta function. So, we have delta e this is standard determinant function on K power n n times standard alternating n multi-linear form on K power n. So, this is an element in alt n K power n and this is one dimensional that is the most important that we have been using in this and this capital A i j is minus 1 power i plus j delta e evaluated on x 1 to x phi min a x j minus 1 e i x j plus 1 x n.

Where this x 1 to xn are the columns of a, so, I will write it here this is also it is x 1 x n. So, they are the columns column vectors of remember here. So, this is very easy you take a standard determine function and at the j th position you put the standard d i th a and evaluate the standard determinant function there and these values are some scalar and last time I showed you why this minus 1 power i plus j occurs. So, this was the adjoint matrix and now I want to prove. So, we will prove the adjoint determinant formula, but before that I will take this opportunity also to prove the expansion theorem.

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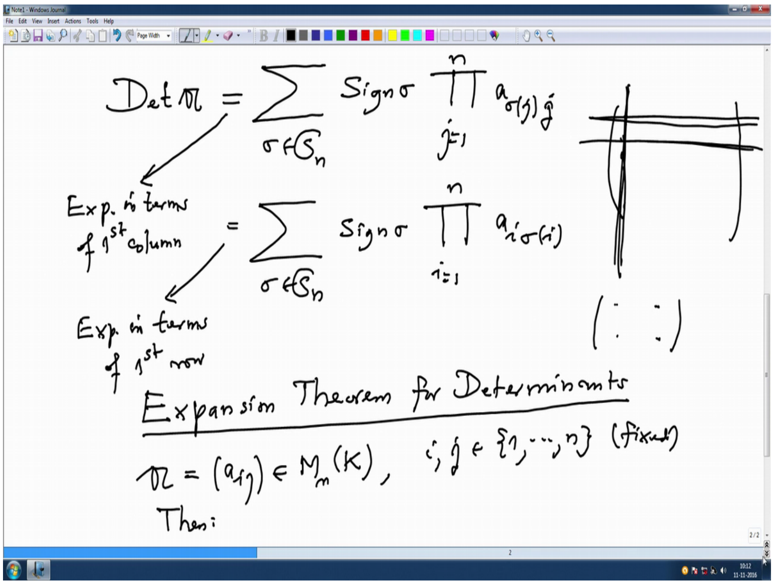

So, remember from a determinant of a by definition it was summation; summation running over sigma in S n sign sigma and the product is on j equal to 1 to n a sigma j j.

And also this is equal to when you replace sigma by sigma inverse in this summation then it is sigma in S n sign sigma product i is 1 to n ai sigma i. Now this one this one is a determinant expansion of the determinant in terms of the column the first column this is expansion; expansion in terms of first column what does one mean by that if you write the columns like this; this is the first column and what do you do you take the first entry remove this row remove this column and take the minor then go to the next entry with the alternating sign then this entry remove this row remove this column remove this row remove this column. And whatever the mine n minus 1 minor comes that determinant. So, it is one can define it inductive lee also like this and this is these are the terms of those the sign come because its alternative know.

Similarly the second equality is by using the first row. So, first row that is you odd usually it was took it took standard way to define like two by two we use to be like this in the school take this entry at time these take this entry with a minus n time these and so on. So, this is the expansion in terms of the first column and this is expansion in terms of first row and somehow it was preferred rows, but if you see our exposition the columns are better exposed for some reason.

So, now I want to use the adjoint entries to also give expansion of the determinant in terms of any i th row or any j th column not necessary first once. So, that is the next theorem I have written. So, that is, so the theorem I want to prove is this is I will call expansion theorems theorem for determinants. So, you have a matrix a, a i j and i and j are in between one and n they are fixed and it fixed. So, in the definition i was 1 and j was also 1.

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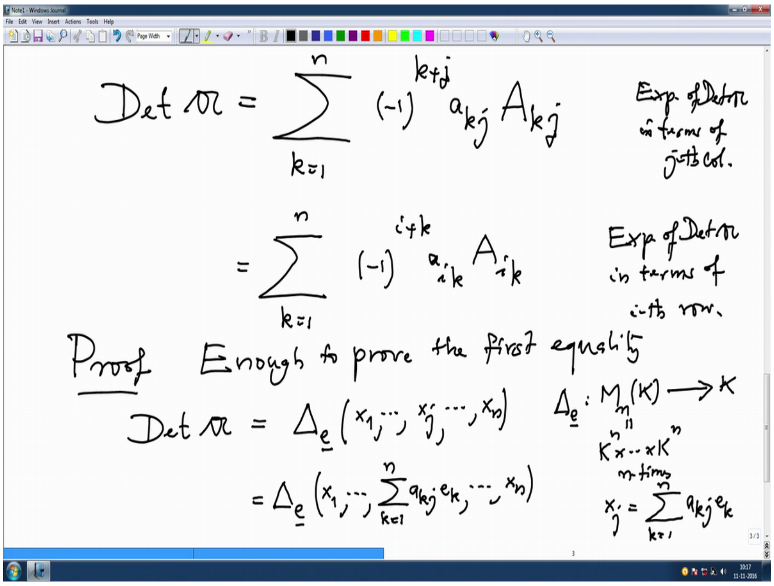

Then determinant of a equal to summation; summation is running from K equal to 1 to n minus 1 power K plus j entry of the K j th entry of the matrix times this a k j this is expansion in terms of the j th column.

So, I just remember this expansion of depth in terms of j th column. Similarly now expansion in terms of the i th row that is summation K equal to 1 to n minus 1 power i plus K a i k capital A i k. This is expansion of det in terms of i th row both are very important because for some reason if your matrix has i th row or j th column as many 0 entries then it is better to expand in terms of that row or that column because then the computation will become faster instead of applying Gauss and bringing of them to this position that position directly you can.

So; that means, save time for calculation that is there and later on if I have more time or maybe in the next course I will also prove expansion theorems when not in terms of one row or one column, but in terms of bunch of rows and in terms of bunch of columns. So, that will be also interesting, but I do not know whether I will finish in this. So, let us prove this it is enough to prove the. So, proof enough to prove the first equality because the second equality will be same when you replace a by a transpose and we already know a; and determined a and determined transpose of the same value. So, it is always our philosophy that uses the theory results to make the computation faster. So, this is ok, so usually enough to prove the first equality.

Now what is the determinant I will use alternating multi-linear form. So, determinant a is by definition delta e evaluated at the columns x 1 2 and j th column is somewhere here x j x n a delta e is to the standard and it is you think of it is a form it is a map form m n K to K this m n K you can think of k n cross k n cross k n n times this is alternating multi-linear form this is standard alternating, we define these by using the standard basis of k n and the standard basis also we think like columns. In this so, what is given? So, in this I am going to put. So, what is x j; x j is the j th column of a. So, that is delta e I am going to put it x j.

So, what is x j; x j is therefore, summation; summation is running over K equal to 1 to n a k j e k you have use the standard basis elements to write this column in terms of the standard basis. So, the coefficients are obviously, the entry in the j th column now first entry is a 1 j th second entry is a to j and so on. So, I am just going to put that in this summation K is equal to 1 to n a k j e k and all others.

Are same columns x 1 2 other than j th entry I am not change now what do we do with this we use the fact that the delta e is alternating and multi-linear.

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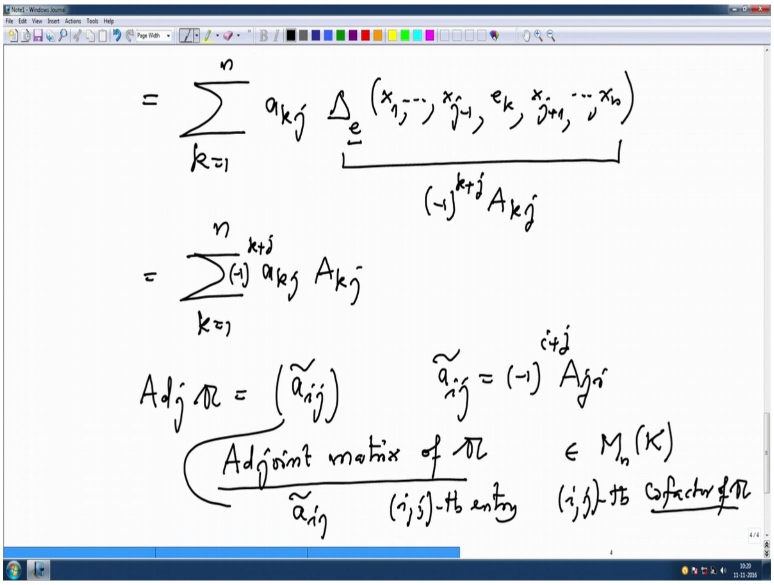

So, I will take this sum out. So, therefore, this is equal to summation is anymore K equal to 1 to n delta not delta. So, the coefficients first, so, a k j delta e x 1 to x j minus 1 then here is e k x j plus 1 to x n this is just a multi-linearity and what is this? This is operationally what do I define capital A k. So, this is minus 1 power K plus j capital A k j. So, that is it, this is summation K equal to 1 to n A k j these when you take it out minus 1 power K plus j capital A k j, this is what we wanted to prove.

So, this shows if your definitions are in order the; proves are also easier. So, now, now we come to the adjoint determinant formula. So, remember we have defined adjoint of a d j; this is the transpose of so that is a i j tilde and a i j tilde is minus 1 power i plus j and not capital A i j, but capital A j i it is a transpose of and now what I want to prove is this is called adjoint matrix of a this is also matrix in m n these also the same size square matrix and these i j th entry here this tilde a i j this is i j th entry of this i j th entry is called a cofactor i j th cofactor of a these are the standard terms one usually use I will again not go to a numerical example.

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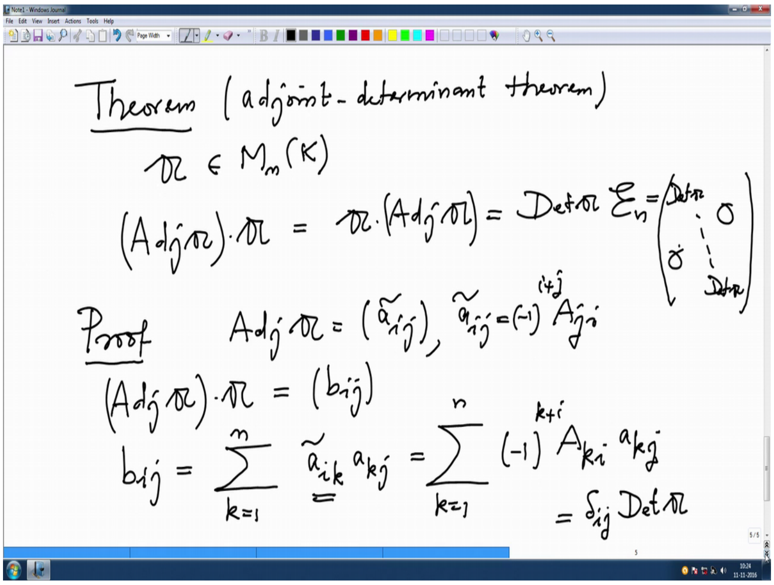

So, that the main theorem is this following theorem these also I will keep referring adjoint determinant theorem. So, A is any matrix square matrix with entries in arbitrary field K then if I take the adjoint of this matrix and multiply the matrix a.

Or do the other way that is multiply a on the left this, this is nothing, but determinant of a times identity matrix. So, remember this matrix is a diagonal matrix with entries on the diagonal determinant a determinant a and 0 is the diagonal matrix. So, this let us prove this first and then we will draw the consequences proof, alright.

So, remember the definition of adjoint is i i j th entry is tilde a i j tilde and a i j tilde is minus 1 power i plus j capital A j i. So, when you multiply again I will prove only one of the equality and the other equality will follow by replacing a by a transpose. So, I will prove only the first this equality this equal to this that equality I will prove and this equality will follow by replacing a by a transverse or by using the row or if I use a column expansion for this then i 1 uses for this equality the row expansion similarly. So, what is the product a d j a times a this product is again a n cross n matrix. So, let us call it b i j and now is how are the b i js computed then b i j will be equal to summation over K equal to 1 to n.

So, that bi b i j th entry will be the the K th row here multiplied by the no i th row here multiply by the j th column here corresponding entry and added it. So, I have to look for the i th row here. So, i th row here will be index for shielding cbi i K tilde times j th column there. So, that is a k j this is the b i j the i j th entry of this product, but this is same as I just plug-in what is tilde that is same as summation K is 1 to n minus 1 power this I am writing K plus i this here capital A; capital A k i times a k j, but this one now we look carefully this one the only term which will survive here for i equal to j because if i different from j this one; this one is a repeated column here in the cofactor. So, this is nothing is only survived for i equal to j. So, there and in that case because of the expansion theorem will be the determinant.

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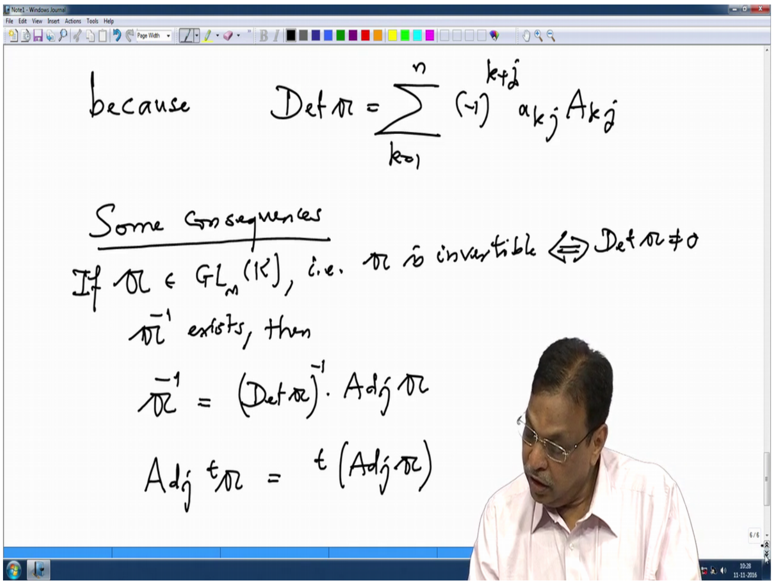

So, this is nothing, but delta i j where delta i j the canonical delta times the determinant of e because so, let me write the reason because we are proved in expansion theorem about the determinant of a is summation K equal to 1 to n minus 1 power K plus j A k j capital A k j. So, that is when i equal to j it is precisely this equality which are wrote which is we proved in expansion theorem that will precisely give for i equal to j it is determinant and for i not equal to j it is 0. So, that proves what we want. So, that proves what we want these product we have proved it is equal to this matrix determinant than the diagonal.

Now, is some consequences, so, first of all suppose you have a invertible matrix if a g l m k that is a is invertible remember that we have proved earlier that this is equivalent to checking the determinant to nonzero these also let me just remind you this also we proved by using the standard determinant function. That means, a inverse exists this invertible means a inverse exists and we want to compute the inverse for it then we can actually write down a formula for inverse then a inverse is matrix is nothing, but determinant inverse.

Determinant to the non zero scalar, so, therefore, this makes sense in the field this times the adjoint matrix because when you substitute this in that adjoint determined formula you get the identity matrix. So, therefore, this is the inverse and its uniqueness and so on all that so on this is this is how one usually computes the determinant the inverse now I want to write one corollary. But before I write one corollary also you; I want the note explicitly that adjoint of what happens to the adjoint of the transpose this is same as transpose of the adjoint just found the first equality if you replace a by a transpose you will get this also so on.

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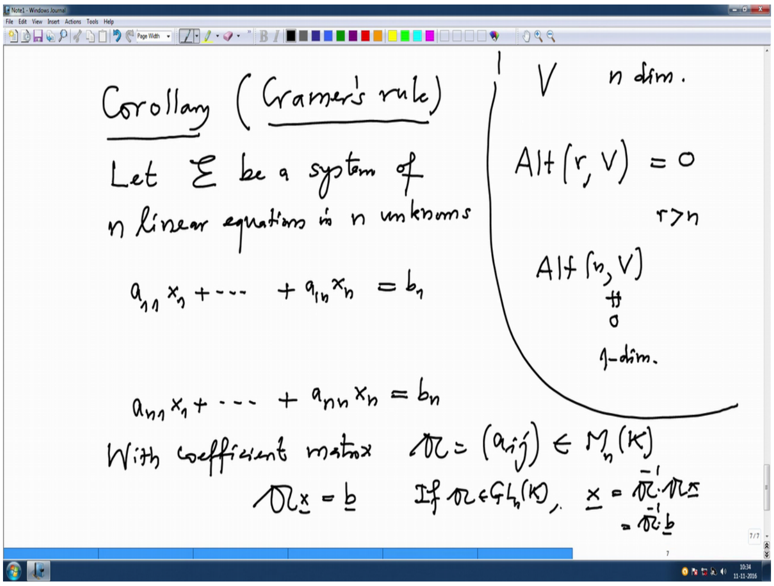

Now, another thing I want to deduce is. So, let me write the this as a corollary this is usually known as Cramer's rule, many many such computational rules in standard references one usually see as a statement, but not the proofs one reason could be because calculation becomes more in Basier if one does not use standard determinant functions. So, and we are deducing them is easily because we are always using standard determinant functions and also I want to take this opportunity to make one more comment that remember when V is n dimensional then I am going to shielding this alternating let us say r linear forms on V and we have checked that when r is bigger strictly bigger than n these are all 0. So, the first time it is nonzero is also that determines of form determines the dimension of the vector space. So, that is the first time alt n v nonzero and this is one dimensional and then when you want to study earlier cases that are also very nice theory, but one for that one needs one need to do so, called higher multi-linear algebra.

And those will be related to in general r cross r minus of the given matrix. So, things becomes also be safe for calculation, but if you keep using the alternating forms it becomes much literals, but unfortunately in this course I will not be able to do that and ah. So, that will also it is the next step will be to study instead of by linear forms. So, that will be the next step to study concretely more by linear forms and that will usually lead to inner product spaces and so on and so on. So, that will be the next course beginning. So, Cramer's rule now, so, we have the system of linear equations.

So, let e be a system of n linear equations in n unknowns and that I am going to the as usually when writes like this a 1 1 x 1 plus, plus, plus, plus a 1 n x n equal to b one and this is the first equation and n th equation is an 1 x 1 plus, plus, plus, plus, plus, a n n x n equal to b n and we want to solve this equation so, usually the coefficient. So, thus the coefficient matrix is with coefficient matrix coefficient matrix a which we are denoting a i j this is now n cross n matrix over of m n K and remember.

When we use gauss elimination we have noticed that if you one want to write down the solutions of this linear system of equations then first one or one finds one solution by a if adopt by trial error see there are also method, but let me say by suppose you know one solution; that means, if you know a system is consistent then all solution we can write down by using the home corresponding homogeneous system; that means, by computing the kernel and then we could and then we translate by one vector by one solution and then we get all the system and now we are we also know in this case.

So, when this system we have we have written it as ax equal to b, where b is a column vector x is also think of is a column vector and we know if x is invertible then there is a unique solution not x if a is invertible if a is in gln then; obviously, we know how to find a solution there is only one solution mainly x equal to q multiply this equation by a inverse. So, a inverse ax because a inverse ax is this and this is a inverse times b and from there you can compute this is column this is also column. So, you compute the corresponding component and you get the explicit formulas for x i th and this is what I want to write it down, but this I do it after the break.