**Linear Algebra**
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**Lecture – 58**
**The determinant of a linear operator**

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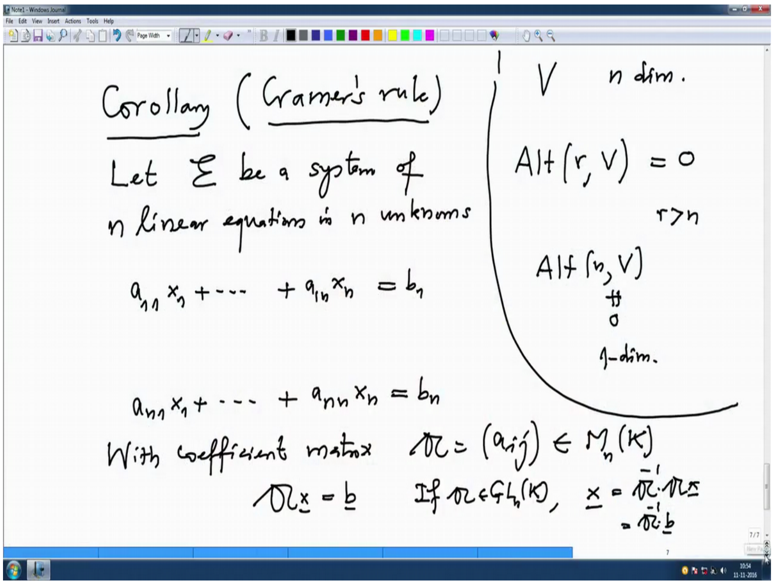


So let us write down the formulas the j th component of x that is. So, xj is nothing, but the you replace j th column by the column b.

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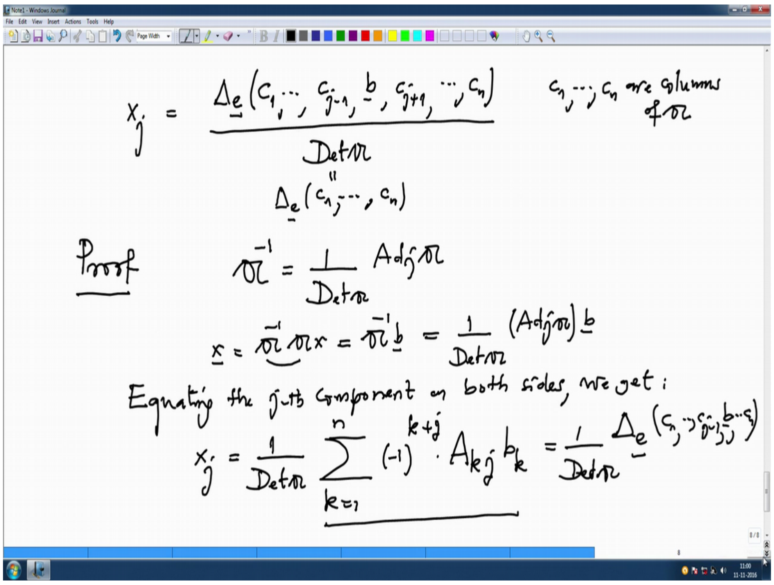

So; that means, determinant off. So, I will write in terms of the standard determinant function, determinant of do not change now I cannot use I will use c 1 c 2 cn for the columns of a. So, c 1 c j minus 1 and then b; b the vector on the right side c j plus 1cn. So, remember here c 1 to cn are columns of the mat coefficient matrix a, this is determinant of that and divided by determinant of a, I could have also written this as delta e of c 1 to cn a delta is a standard determine function.

So, how do you verify this? These very easy to verify. So, proof how we use to the fact that the inverse of the matrix we know a is invertible that is our assumption. So, a inverse is adjoint of a one over determinant of a times adjoint of a these, this follows from the adjoint determinant formula. Now, therefore, we know as I wrote earlier the vector x the unknown vector x is nothing, but a inverse of ax because this is a identity, and this is a x is b.

So, this is a inverse b. But a inverse is one over determinant times adjoint of a and these I have to multiply by the vector column vector b, and now I want to compare the j th component here, this is a column vector this result is also column vector and compare equating the j th component on both sides what do we get we get on this side x j and what do we get on the other side of course, this is a constant. So, it is 1 over determined as it is, and the jth entry in this product, but the jth entry in the product is the jth rho multiplied by this column vector b corresponding entry multiply and add up.

So, therefore, this is sum running from k equal to 1 to n the jth entry here is jth rho is index by a tilde j first, but a tilde j is minus 1 power k plus j times and then you have to. So, that is capital Akj times bk. This is precisely the jth no j rho multiplied by the column vector b that is this and what is Akj? Akj is precisely this. So, this is precisely what we are done this side is precisely one over determined it is same thing, but this side these one is precisely. So, this one over determinant as it is delta e you evaluated at cn to cj minus 1 and in b here cn, that is because you see what was capital akg that is e I here, but this b vector column vector. So, when you write it as in terms of the standard basis till we bks and then when we expanded it (Refer Time: 05:50) it like this. So, that proves thing.

So, remember this the most important I again stress that most important fact we use in all these computational rules is the alternating multilinear maps and their properties and the main theorem of determinant theory which says that given a basis there exist a unique alternating multilinear maps. So, that delta e evaluated at the basis tuple is one, and this gives a basis for the alternating forms.

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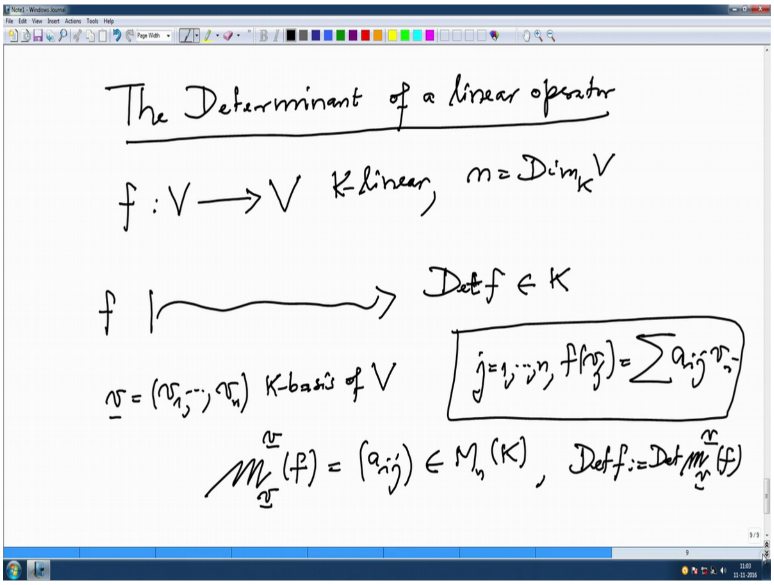


Now, let me briefly also now do it for the. So, I want to do it the determinants of a linear operator. So, given a linear operator f on a finite dimensional vector space V to V k linear and n is the dimension of V and I want to associate for this f a scalar, which one want to call a determinant of f this is a scalar and one can do this in two possible ways; one is take any basis v, v 1to vn k basis of v and take the matrix of f with respect to these basis. So, matrix of f with respect to the basis v that is these are notation aij, and remember these aij is a defined by the equations for j equal to one to n f of j equal to summation aij v i. These equations the coefficients of these n equations will give this matrix this is the matrix M K, and then define data f equal to determinant of this matrix f.

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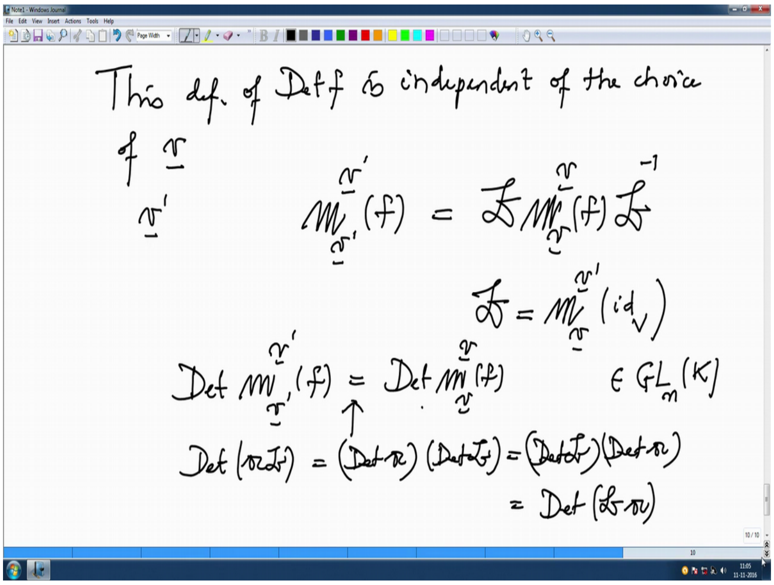

But, now, one has to justify that this is these definition does not depend on this basis v. So, this definition of del f is independent of the basis of the choice of the basis v, but for this we remember we analyze that if I change a basis what happened to the matrix, and we know the matrix if I have another basis v prime in the matrix of f with respect to the basis v prime this is a matrix b m v v f b inverse where b is the transition matrix b is the transition matrix from v prime to v of id, and b is invertible and therefore, these we object earlier now when I apply determinant here determinant here then I use the product formula and therefore, and these determined and these determined are inverses of each other and so on. So, that will prove that determinant mv v prime f equal to determinant m v v f; for this equality we have use determinant of ab equal to determinant of a times determinant of b and this is also an equal to determinant of b times determinant of a because it is a field. So, I can multiply either is. So, this is determinant of b times a.

So, determinant v make determinant b times a. So, therefore, this when I then bb knows we get cancelled and then you get this. So, product formula was very important if you do it this way this is one possibility other possibilities do not do this way do it abstractly do it just by or philosophy use alternating linear maps alternating multi linear maps. So, let me do it that way, and then product formula also I will show you it can be proved much easier even much easier than the before. So, therefore, what I am trying to do you now I am trying to define determined of a linear map directly, without going through basis choice of basis and then write the matrix of f and so on.

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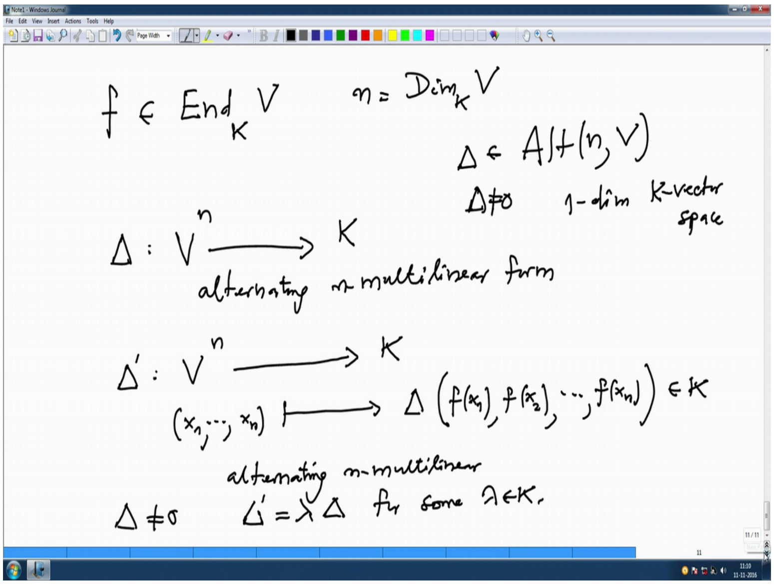

So, now only thing we have given is f, f is an endomorphism it is a linear operator on a finite dimensional vector space v and let us call n to be the dimension and I am going to use the main theorem of determinant that is alt n V is one dimensional K vector space therefore, any nonzero form any if delta is here and delta is nonzero then these delta will be basis; any nonzero vector in a one dimensional vector space is a basis of that vector space and we have also enough recipes or enough nonzero alternating n multi linear forms on v. In fact, each basis will give such that was the main theorem of determinant functions alright.

So, now take any delta. So, delta is V power n to keep is a form and so, it is alternating n multi linear -form. So, for this you choose you take an alternating n multi linear form on V and now I will use this given f and these delta to get a new alternative form. So, I am defining now a new alternating form which I will call it delta prime which is again from V n to K and what do I need to define I need to define it for many tuple x 1 to x n, where does it go? First you apply f to this vectors they are vectors in v. So, I will apply f frst. So, that is I consider f of x 1 f of x 2 and f of x n this is now the new n tuple and I apply these given delta to that. So, I will give the a scalar.

So, I have a map from V n to K I want to say it is alternating; that means, what if two components are equal then it is zero, but if two components are equally two components are equal here and therefore, delta because delta is alternating it is zero. So, it is alternating and I want to check that it is n multilinear; that means, it is multi linear in each variable; that means, if I change one of the component here remaining components are fixed linearly then this will linear, but then f is a linear map. So, therefore, in that component linearity will be clear. So, therefore, it is n multi linear form. So, therefore, if delta where nonzero if I would have chosen this delta to be non zero, then this new multi linear form should be a scalar multiple of this. So, therefore, if delta is non zero because this is the basis of this alternating and this is another element. So, this delta prime has to be equal to lambda times delta for some scalar lambda in k and not only these lambda will be unique.

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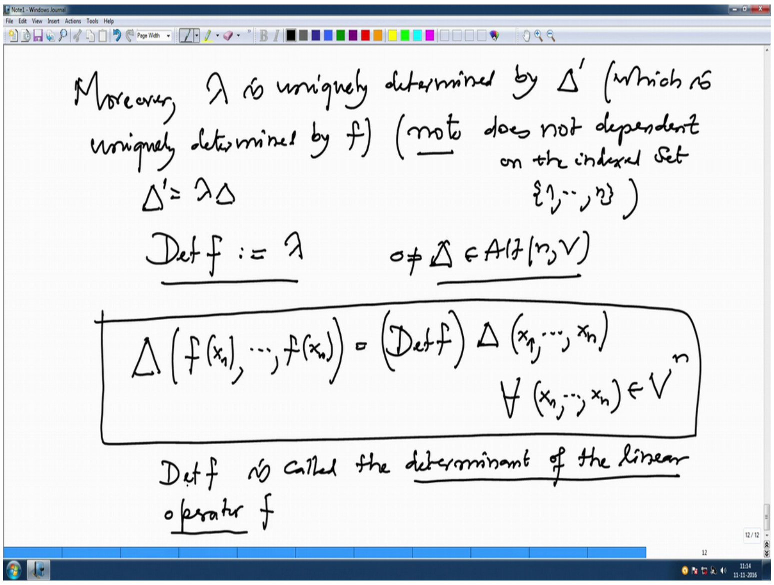


So, moreover I will write on the next page moreover lambda is uniquely determined by delta prime, but delta prime which is uniquely determined by f. So, this lambda is uniquely determine by f and not only that it does not depend on a. So, note because it does not depend on the indexing set I use one to n, but I one could use arbitrary indexing set does not depend on the index set, which are using this case it is 1 to n 1 could if necessary one could you will different indexing set, but the cardinally it is important.

So, this lambda I will define then determinant of f is by definition that unique lambda see remember here the only thing I will use that is if I have one dimensional vector space then any nonzero vector is a basis of that; that means, every other vector is a unique scalar multiple of this nonzero given nonzero vector this is the only fact I am use.

So, this way we have defined determinant of a linear operator directly without going to any basis. So, therefore, the definition of the determinant, when you plug it in there therefore, we get like this. So, delta of, this will delta of, so we get a formula delta not delta determinant of or delta of. So, delta of f of x 1 to xn this is equal to determinant of f times delta of x 1 to xn this delta was given to us in alt nv nonzero. So, we are given this and this was the new delta prime now this was. So, therefore, when you say when you say delta prime equal to lambda times delta; that means, on any tuple they are equal. So, when you evaluate this equation on the tuple x 1 to xn you precisely get this and this is true for every x 1 to xn in V power n this is how the determinant is defined.

So, del this determine this is called determinant of f is called the determinant of the linear map the linear operator f, and now I will show you quickly that why I these deter why these definition is quicker than earlier definition. See it is always better to do it coordinate free this is a coordinate free definition the early definition was using coordinate coordinate means choosing a basis ok.

So, let us let us quickly show you let me quickly show you that the two things I want to show you, the first of all it should also match with the earlier definition.

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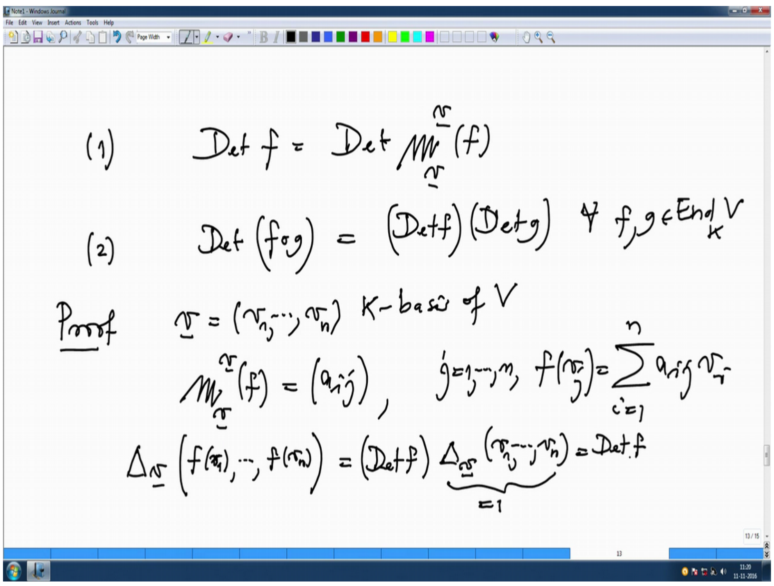


So, I want to also show you quickly one I want to show you determinant of f is nothing, but determinant of the matrix of f with respect to any basis this is also we will have to show. Secondly, I will show you the product formula and secondly, I will show you determinant of f composed g is same as determinant f times determinant g for every fg linear operators on V. So, proof. So, if to proof one I have given a basis V, v 1 to vn k basis of [vocalized-noise], then as I said the matrix of f with respect to these basis is the matrix aij where aijs are defined by this equation j equal to 1 to n summation f of v j equal to summation over I one to n, aij, vi then and what do you want to prove; we want to prove this determinant equal to this remember these determinant is defined by I will you show this determine is defined by oh what happened it got stuck because that is it got stuck.

It is stuck (Refer Time: 23:03) yes, but do not to save this right.

Oh yeah we need to save, but they will have say.

Sir, it is on the space that is why.

Is it because of this.

Yeah, yes yes.

Yes.

Yes (Refer Time: 23:40) (Refer Time: 23:40) now it is working.

Now, it is working we can put it here.

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Shall I shall I continue.

Done.

So, remember a determinant of f I just wanted to show these determinants of f is this is connected by this formula. So, any nonzero alternating form when I plug in the tuple f of x 1 to xn then you get determinant of f primes delta of x 1 to x n. So, that is what I am going to use now we are here. So, I have to compute what is instead of taking arbitrary nonzero alternating and multi linear form, I will take the one which corresponds to this basis; that means, I am going to take delta v that we know it exist and it is a unique with the property that when I evaluate this on the basis we want to even you get one. So, therefore, determinant of f is equal to delta v evaluated or f of x 1 to xn v 1 to vn not determined of f is equal to this that we compute this by using this, this is determinant of f times delta v evaluated at v 1 to vn, but this is one therefore, this is determinant of f. So, determinant of f is a value of delta v on the tuple f of v 1to vn and now I will substitute f of v 1 equal to from this equation f of V to from this equation and so on.

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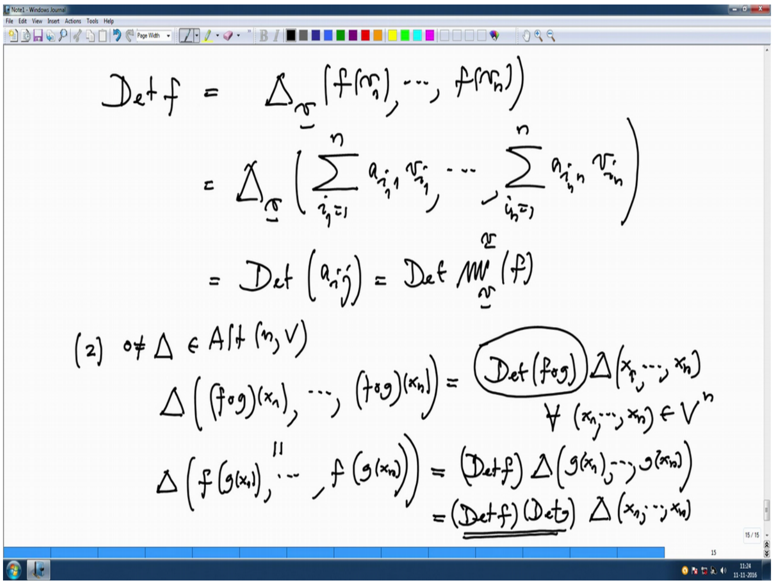


So, there therefore, determinant of f which is equal to delta v of f of v 1 to f of v n, this is same as delta v the first component is v 1. So, that is summation no I 1 from 1 to n a i 1, 1 vi 1 and so on summation I n one to n these we have done earlier also such a calculation i n n, v i n and these when you use the alternating multi linearity of delta v you will get determinant of the matrix aij, that is how we have defined the determinant of a arbitrary matrix. So, that proves these to and this is determinant of m v v f. So, this show that this new definition matches with the earlier definition with respect to the basis f.

Now, to prove the product formula n what do I do prove. So, to proof two you have to prove that if I given any nonzero delta, delta in alt nv nonzero, then I how is the determinant of the product relative. So, determine delta of when I apply for any tuple x 1 to x n if I apply f composes g one x 1 etcetera f composes g on xn this what comes out is the determinant of f composed g n delta of x 1 to xn this is true for all tuples x 1 to xn n in V power n. This is a definition we new definition we proposed that given any nonzero delta, delta f apply the linear map first and then what scalar comes out to this.

Now, I want to compute this in a different way I first want to do this first. So, this is by definition delta of f of g of x 1, f of g of x n this is same as I will take out f first when I want to take out f first; that means, I have to take out the determinant. So, this is the deter determinant of f times delta evaluated at g of x 1 g of xn and now I will take out g. So, when I want to take out g the determinant of g will come out.

So, this is equal to det f times det g times delta evaluated at the tuple x 1 to xn and this is true for all x 1 to xn, but now if you here it is these constant and the other side it is this constant and we have noted the constants are unique one dimensional vector space constants are unique therefore, this equal to this. So, that proves the product formula that proves the product formula. So, this way is one can avoid the using basis. So, I would stop here we will continue next time.