Linear Algebra
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Lecture - 59
Determinants and Volumes

Last few lectures we have been seeing basics of determinants determine functions their properties and computational rules. Today I want to give the most important applications of the determinants in the computation of volumes and these application is used widely in many branches of science and engineering. So, I will recall some basics which I will not prove, but I will assume that for. So, today I am going to do determinants and volumes.

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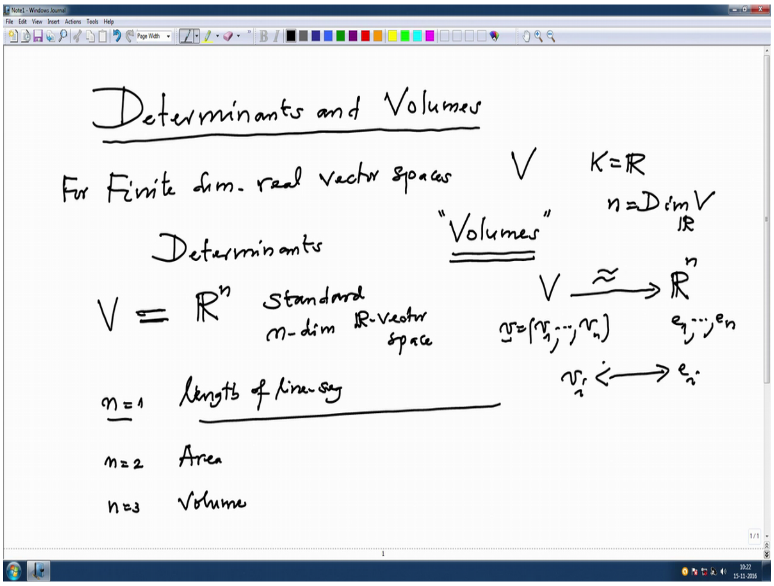

So, finite dimensional real vector spaces for one for finite dimensional real vector spaces and determinants and volumes.

Now, we have to recall a basics of volumes there is a very close connection, and this is what we are going to discuss in this lecture. So, first of all when once a finite dimensional real vector space it is V and a base field is K base field K is real number field of real number and n is the dimension V I am going to assume that V is actually R power n because otherwise the results will be return re in any case we know that the dimension is n. So, V to R n there is a isomorphism if I have a basis here v v 1 to vn and there is a standard basis here e 1 to en and unless we have an isomorphism which maps vi to ei. So, I will transfer everything from this case to the general case. So, we will concentrate on the standard vect this is called a standard real vector space standard n dimensional R vector space, and as you know the concept of volume is actually we have been studying from the school days it is concept.

So, for n equal to 1 that is a case of real line that is that will measurement of a line segment and n equal to 2 it is. So, this is lines length of line segment and in case of n equal to 2 it is a area, and n equal to 3 actually for n equal to 3 the word volume is used and we study all these computations in case in course of calculus usually. So, first I will recall what we assume about volumes or what do we what properties of volume I would like to use I will not construct what it is and define exactly what are the working property that we will use for the volumes.

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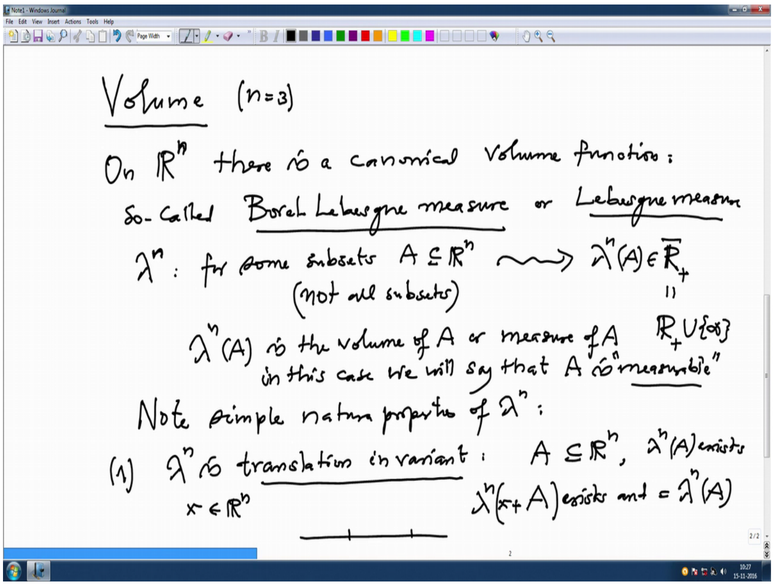

So, just. So, I am recalling about volume, and as you see these word was used this term was used only for n equal to 3 in general what terms is use is what is called a major . So, on Rn there is a canonical volume function what is that mean; that means, and this is usually called Borel so called Borel Lebesgue measure are even more generally Lebesgue measure. So, what do you do and use usually notation for that is lambda n is n n is for this dimension n this lambda is for lebesgue. So, that is a volume function.

So, it is lambda n associates for some subsets a of Rn it associates a number lambda n of a this is a number positive real number it could be infinity also. So, this could be I will write the notation for it is R bar plus. R bar plus is by definition it is R plus non negative real numbers and along with that this is in symbol infinite and that one is called is only the volume and this is this is associated for some subset only not all, and whenever it is associated one calls that lambda n A is the volume of A or also called measure of A. I will not go into you the detail that why it cannot be associated to all subsets and this then in this case in this case we will also say in this case we will say that a is measurable.

I will not go into this definition formally, but I will collect what should be the properties of lambda n and if you have 2 sets we has volume then what happened to their operations on the sets, and the building blocks where some subsets of Rn where the volume will exist and what will that be.

So, I will only note simple natural properties of lambda n and an axiomatically used it I will not try to prove this these properties, the first important property is lambda n is translation invariant. What does that mean? That mean if I have a set A I have a subside a of Rn their volume exist.

So, lambda n exists and suppose I have a vector x in Rn and I translate a by x; that means, I consider a new set x plus A this is the translated, then this is also has a volume and the volume does not change then lambda n exists and is equal to lambda n of A. This is very very reasonable because imagine if you have want to measure a line segment from here to here and this was I shift it by 2 meters then the length will not change.

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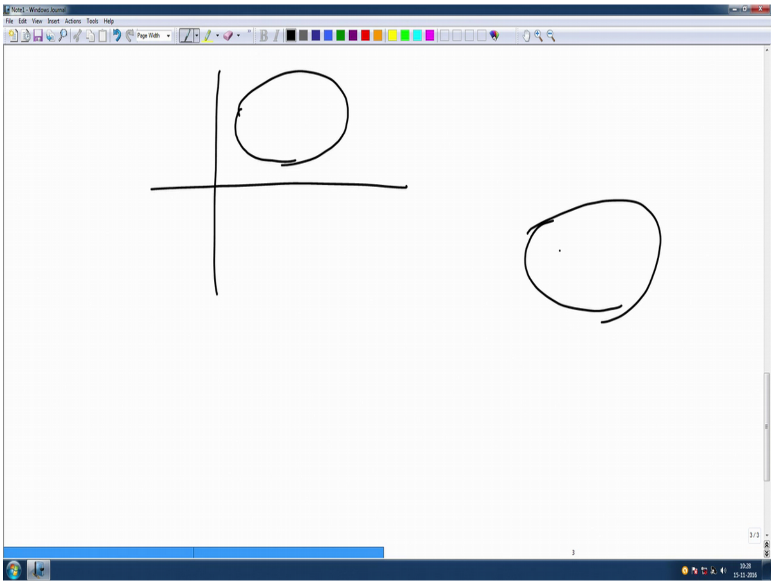

Similarly if I have a some portion of the plane suppose I want to compute this area and for sure I know this area exists, and if I would have translated the same subset here then the area is not going to change. So, that is why it intuitively it is clear that is the first important property.

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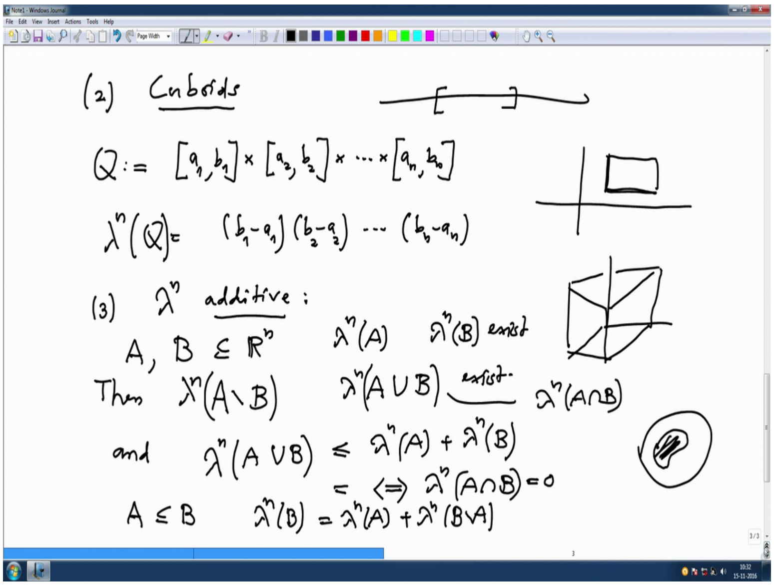

Second one is second one cubords, what are cubords? Cubords are so, for n equal to one the cubords are just a line the close interval for n equal 2 they are just a rectangles, and more generally for arbitrary n cubords I will denote by symbol Q this is by definition, you take the interval a 1, b 1 close interval cross a 2 b 2 cross cross an bn. So, n equal to 3 typically cubrod will fit in this and not necessarily of the all equal length maybe a different length, but cubord and this is cubord. So, what should be the volume of cubord; obviously, that. So, we will lambda n of Q is nothing, but you take b 1 minus a 1 times b 2 minus a 2 etcetera etcetera bn minus an these exactly matches with what calculation we were doing it in the school.

So, area of a rectangle for example, it is this length and this length and so on, this is what we will use we will use this for calculation purpose third one lambda n is additive, what is that mean that mean if I have a subsets A and B of Rn both have volumes. So, lambda nA and lambda nB exist that is given to us; that means, these sets are measurable. Then the difference A minus B and A union B these also lambda n exist then this exist and obvious formula that is lambda n of A union B is less equal to lambda n A plus lambda n B and equality olds equality holds if and only if lambda n of the intersection is 0 also I should have said lambda n of the intersection also exist.

So, and in particular if A is a subset of B, if A is a subset of B then lambda n B will be equal to lambda n A plus lambda n of B minus n this is also more or less intuitively clear because if you have a subset here a smaller subset, then the volume of the bigger one is the volume of this plus volume of the remaining part. So, these all these things are very very intuitive, the most difficult part is the construction of this volume function lambda n that is usually done in a first course on major theory. This is very very important also for when one wants to study probability theory in a very very meter way.

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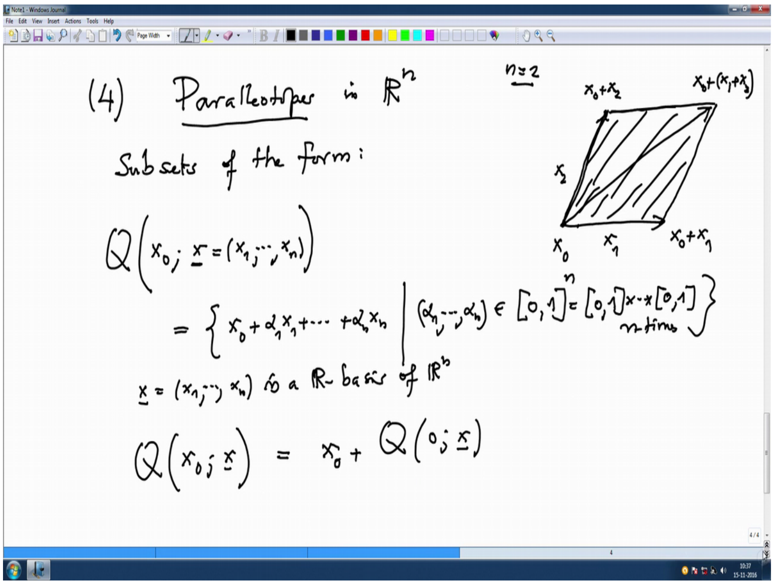

And without this many applications in engineering and mathematics will not be precise anyway. The fourth one these three properties and the fourth one now we are interested in so called paralleotopes p a r a l l e o topes in Rn. So, first of all what is the paralleotop in Rn. So, you need these are subsets of the form subsets of the form.

So, I will first draw a picture the picture is so, there is a point here I am drawing a picture in n equal to 2 this is a point x naught remember I we I do not want to still give a important to 0 as a origin. So, x naught is the point here and x 1 is a vectors this is x 1 is a vector in this. So, x 1 is a vector. So, and this point is therefore, x naught plus x 1 similarly we have another vector x 2 which starts from here. So, this is x 2. So, these tip is x 0 plus x 2 and then you complete a paralyze. See if you want to add these 2 vectors there the parallelogram who say that complete this parallelogram and this is the x x 1 plus x 2 these vector now this parallelogram. So, this point here is x naught plus x 1 plus x 2 see that is the definition of this some vector and now the question is. So, this subset here, this is up in case of n equal to do we keep calling parallelogram and the question is what is the volume of the parallelogram.

So, these are high dimensional I will write now in the notation form. So, in a high dimensional case it is Q x naught semicolon x, these x is a basis x 1 to xn this and this is by definition what these are all sums like this x naught plus alpha 1 x 1 plus plus plus plus alpha n xn where these alpha 1 2 alpha n, n tuple these vary in 0 one power n power n means product of 0 1. So, many times n times this is a parallelo this a paralleotope in Rn where x 1 to xn x x 1 to xn is a R basis of R n you see here this is x 1 and this is x 2 if x 1 and x 2 are in the same line; that means, they are linearly dependent and it will not be basis in that that case it will be a generally generate case degenerate case, and the volume will be 0. So, that we are not interested in computing 0 only there is nothing to compute this is a paralleotope and then we are interested in using the determinant function to compute the volume of these paralleotopes, that is the connection between the major theory and the determine function

So, one thing is which is; obviously, this Q x naught x this paralleotope is same as 0 plus not 0 x naught x naught plus Q 0 x. So, this means wherever I have a paralleo parallelogram I will translate it to wherever the origin is, that will not change though the area will not change in general volume will not change therefore, in our calculation we will use this is the property earlier stated that it is translation invariant. So, no, I will not go much into this. So, I will not go more than these about the volume.

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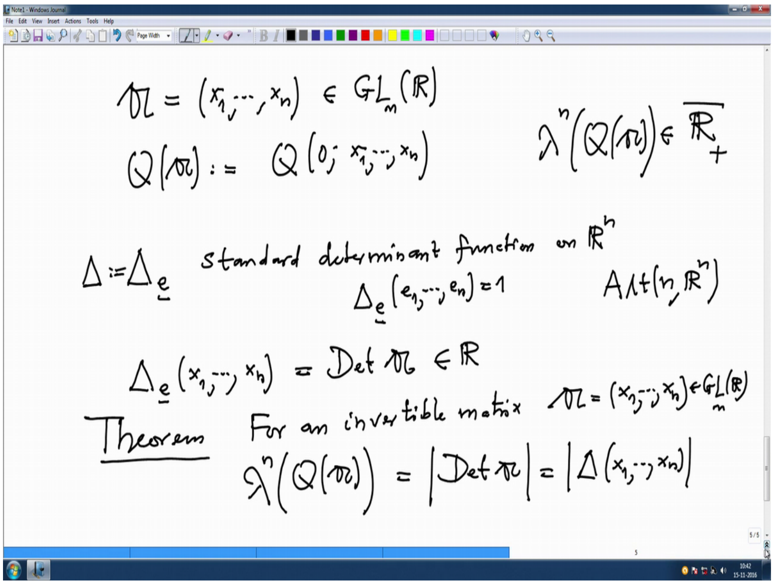

So, now in general the notation I will put it for a matrix a I will denote the columns of the matrix by x 1 to x n, and assume this matrix is in GLn and then I will put Q of a to be Q of 0 x, x 1 to xn.

Now, because we are assuming the matrix is in GLn, these vect these column vectors are linearly independent because the rank is n therefore, column vectors are linearly independent therefore, they will form a basis and therefore, this make sense and this is one side this is a paralleotope and the lambda n of Q Qa this is the number this is a number now we are interested in computing this number effectively this is a volume of the paralleotope.

On the other hand what do we have? On the other hand we have the standard determinant function delta e this is a standard determinant function on Rn and what does it do it with of course, it is unique with the property that delta e of e 1 to en this is 1 and this is a basis for the all alternating n multilinear forms on Rn and when I evaluate these on arbitrary tuple delta e of x 1 to xn this is nothing, but the determinant of the matrix a this is what we have proved.

So, the column, these determinant we think of it is a function of columns, and it is alternating (Refer Time: 22:21) and multi linear and so on this is what we have proved and now I want to prove the following theorem. So, the theorem is the most important theorem. This theorem says for an invertible matrix a which is with column vectors x 1 to xn this is in GLn R volume of the parallelepiped associated to a; that means, associated to be vectors column vectors x 1 to xn these volume is nothing, but mod of the determinant. We know this determinant is a real number it could be positive it could be negative, but we know by definition volume is always positive or negative. So, we have to take the mod. So, this is. So, this is mod of and hence forth I will not write this delta e is fix there will abbreviate to delta because e is fixed.

So, this is delta of x 1 to xn and take the mod of this, and see I will prove this, but before I prove it just imagine.

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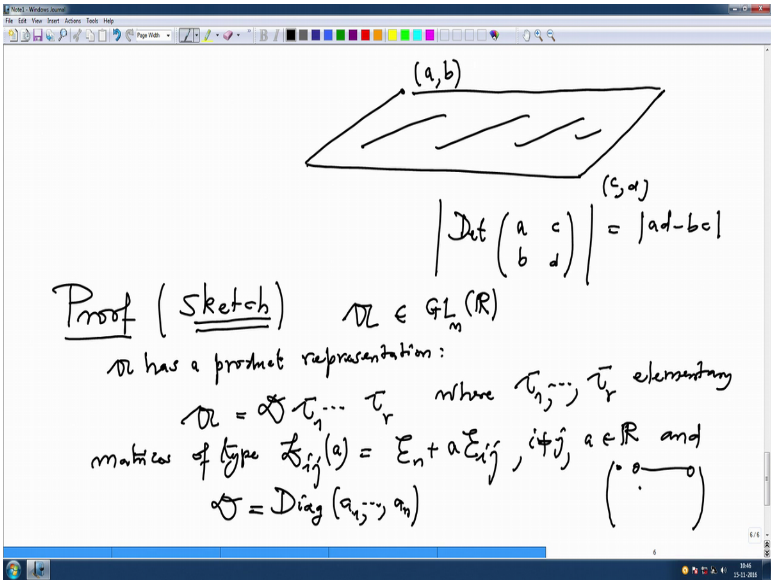

Suppose I give you 2 vectors in Rn one has coordinates ab and the other has coordinate cd and there will look at the parallelogram is a parallelogram and what is the volume? Volume say it says that I have to compute the determinant, these I have to write as a column ab these also I have to write as a column cd and take the determinant of this matrix and take the mod same. So, this is obviously, mod of a d minus b c.

So, irrespective of the origin, this is very very very important for calculation and an similarly for higher dimension. Higher dimension the more calculation with the determinants will be very helpful which we have spent large couple of lectures only on the computational rules for the determinants ok.

Now, what is the main ingredient of this proof? So, proof of the theorem I will sketch I will not write all the details, but the details will be obvious what I leave it for checking. The sketch of the what will I what will be the machinery that I am going to use in this proof the main machinery the following essentially I am going to reduce to the dimension 2 essentially and how will I do that. So, look we have a matrix a which is invertible matrix and if you remember what did we prove about this, this a as a product demco has a product decomposition a has we have proved this has a product representation, what is that a equal to diagonal matrix times c 1 to cr where c 1 to crr elementary matrices elementary matrices of type bija this means this is the matrix I naught equal to j. So, this is en plus a at the ij plus a eij and i naught equal to j and a is a scalar and d is a diagonal matrix and d is diag a 1 to a n. I have chosen the co column operations you see you could have chosen the row operation and then this will come on the other side, but it is a choice I made it here this is what we have proved it because.

So, let me recall quickly what did we do we have a non singular matrix we use column operations to bring column element nonzero here and then kill use column operation to make these all zeros, similarly the same and then once you have done these you go backward now. So, this is very very easy and now what I will analyze is what happens between the cubords or the parallelepipeds by these matrixes ok.

So, now these s and these s first of all these s they are arbitrary real numbers, I want to assume they are a smaller equal to one that is because when I raise it, I know the I have to use the properties of the elementary matrices.

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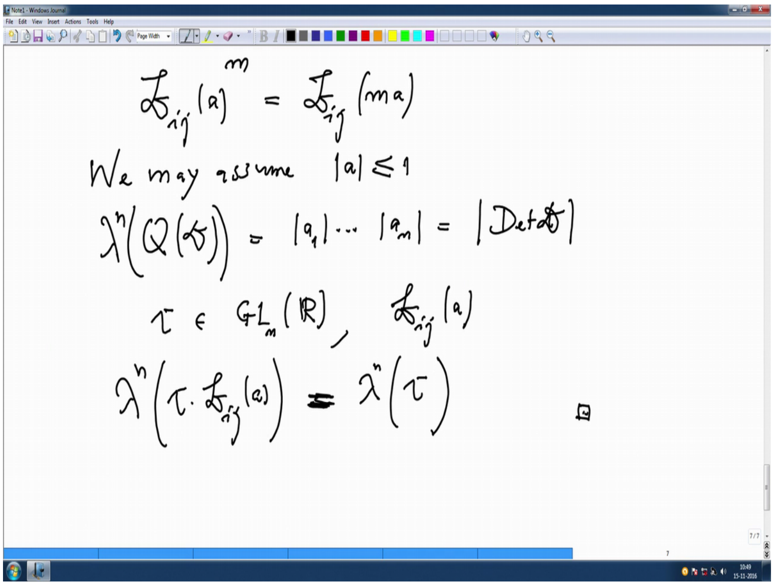

You when I have a elementary matrix of this form this type, and suppose I raise it to power m then this is nothing, but a bij m a this properties I know.

So, therefore, I will choose and I will assume we may assume I will not write more we may assume these real number a the mod of that is smaller equal to 1. They I that is where I have to use the archimedean property of real numbers and so on and what is the obvious thing is if you have a diagonal matrix, what is the what is QD then QD will have volume; obviously, this is mod a1 mod a n this is like a cubord right. So, this is on the other hand this is also nothing, but mod of the determinant of a for diagonal matrix this are associational retrial.

Therefore it is enough to prove the theorem for only one matrix. So, I have only have to prove that if I have c is a matrix in e GLn, and I have an elementary matrix bij a then volume of c times bija equal to volume of c no I have to prove that only for this let. So, I have to prove that this is Q this c I have to prove only this equality and this you know now only i and j are in word only i and j are in word. So, I will only concentrate on that and I will reduce the problem 2 n equal to 2 n where it is very very clear. So, (Refer Time: 31:10) in the next half we will see some more applications of this formula.