Linear Algebra
Prof. Dilip P Patil
Department of Mathematics
Indian Institute of Science, Bangalore

Lecture – 06
Examples of subspaces (continued)

So, let us continue with some more examples; second example.

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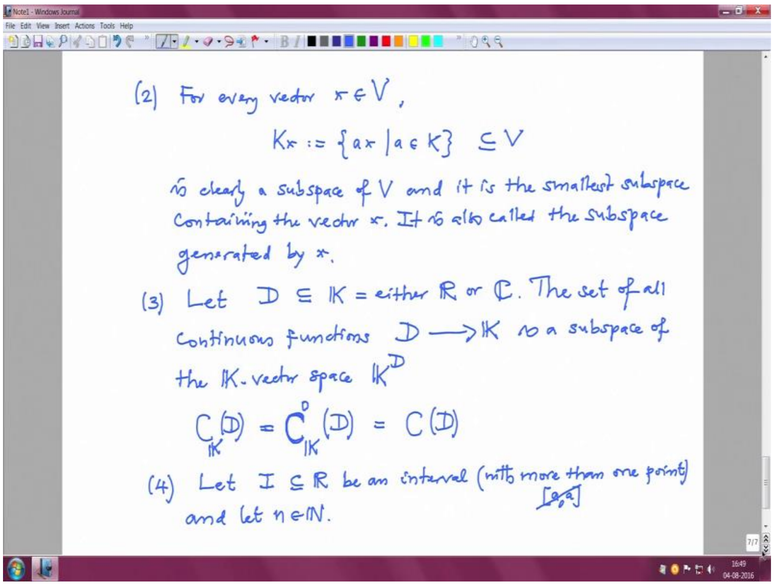

For every vector x e vector space V is scalar multiples of this vector K times x. This is all those a x as a here is in the field k. This subset is clearly a subspace of v. This one can see easily by using the subspace criteria we have given, and this and it is the smallest subspace containing vector V vector x. So, it is also called as vector subspace generated by, it is also called the subspace generated by x.

Now, some more examples I want to know for, we will use them further in our lectures. So, let D be a subset of this special symbol for K, where ever I write such a K K with double line this is either R in the field of real numbers or in the field of complex numbers. You can take either of them and the statements I will write they will be valid for either case. So, the set of all continuous functions, on D D to K, is a subspace of the K vector space K power d. Last time I defining what is K power D they are all functions from D to K and we have seen that from vector space over k, and among them we are taking continuous functions.

So, this is immediate because in the first course on analysis, we check that sum of two continuous functions is again continuous, continuous function multiplied by scalar is also continuous function. So, therefore, by subspace criteria, the space of all continuous functions on d, it values in K. it is a subspace. These subspace I am going to denote by C, C for continuous, K here; that is K value and domain is D, D or 0 here.

So, sometimes I will draw, I will just write C D, continuous functions on d, or sometimes I will also I write it C suffix K just to remember where the values are, values of this functions. So, this is a standard notation, and first course on only C C is divided to study of this subspace.

Continuing further, forth example: let is now take the interval, let I content in R be an interval, which has more than one points, with more than one point. So, I want to avoid the situation like intervals like this a, I want to avoid this, it is a, it should contain more than one point this contains just single terminal, and let n be any rational number.

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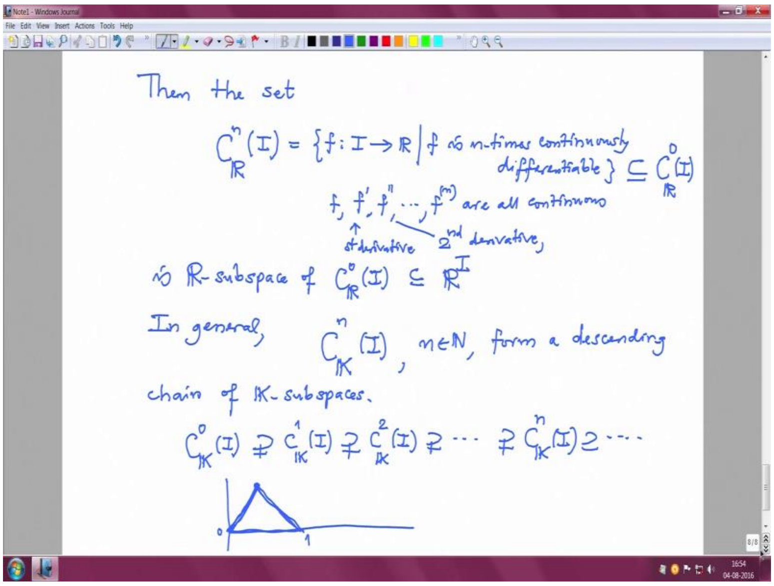

Then the set C n R I; this is the set of all f I to a I to R real valued functions on I and f is n times continuously differential; that means, f f prime exists and it continuous f double prime exists and it is continuous it is (Refer time: 07:20) f n are all continuous measures, and these are the derivatives. This is the first derivative; this is second and so on.

So, this is also sub set of C 0 R I and it is a subspace, set is a subspace is a R subspace of C 0 R I which is also subspace of R power I. this is ambient vector space and this is subspace and this is further subspace on that. And actually I could have done the same thing for instead of real numbers complex numbers also. So, in general we have a big family, in general, just to how do one check this is, one needs to note that if you have two differentiable n n times continuous suitable functions on an interval, then that sum is also continuously differentiable, that we have seen again in the first course on analysis, and the derivative is the sum of the derivatives.

Similarly, for the scalar multiple; so this is obvious from the subspace criteria. So, in general we have a big family C n K I n e n is form descending, chain of K sub spaces ; that means, C 0 is here, these are continuous functions on I K values continuous functions on I, then comes C 1 once differentiable and difference, once continuously differentiable and two times continuously differentiable, because you know if it is differentiable then it is continuous.

So, we have a descending chain like this. If you studied more analysis, then you note that at every stage this inclusion is proper; that means, we have to construct functions which are n times differentiable, but not, n times continuously differentiable, but not n times continuously differentiable, and constructions of such functions is usually done in the analysis course. I just want to see here the first inclusion is easily, because if you look at this function the graph of the function graph is, this is known you can take on the interval; 0 1, you take this ten function, this function is continuous, but it is not differentiable.

At this point it is not differentiable, because differentiability means tangent to be well defined, and here the tangent is not well defined. So, this is the example for this now from here to here, you will have to work more and more harder to find functions and so on.

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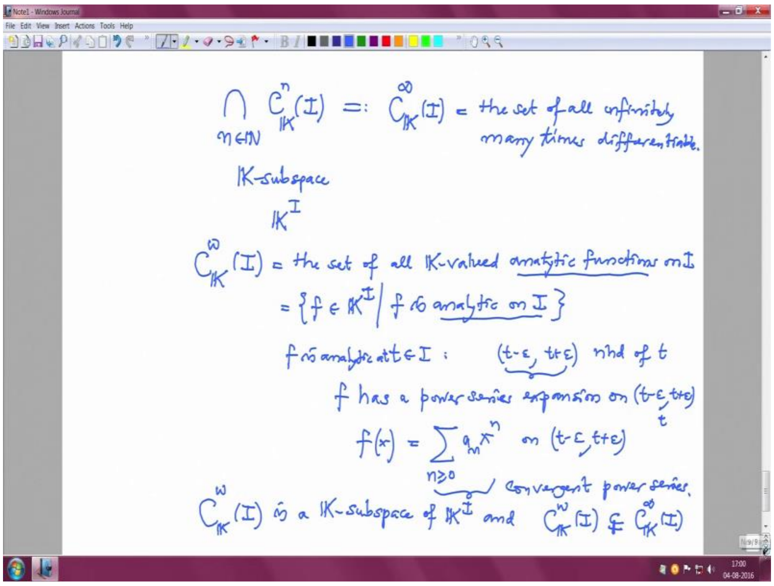

Continuing still further, we can say something more, if you take their intersection C n K i. So, this is the intersection of all the sub spaces this is also subspace, because we have seen just above the intersection of arbitrary family of sub spaces again subspace, this is the K sub space. Again if you like we are working in this big ambient space function space K power i. This by definition is denoted by C infinity k, this is the set of all infinitely many times, times differentiable. So, this is also subspace, and such subspaces are very important to study in analysis. Even 1 can also construct examples which are n times differentiable for given n, but it is not infinitely many times differentiable.

One more subspace I want to define and then we will go on to some other examples; C omega K i. This is the set of all K valued analytic functions on i. What is the analytic function on i; that means, function f. So, this is although the f in K power I, f is analytic on I, analytic on I means it is analytic at every point in t in I, f is analytic at t means at in a small interval around t t minus epsilon to t minus t plus epsilon e this is small neighborhood of t. In this interval f should look like a power series function, f has a power series expansion, on this small interval t minus epsilon or t plus epsilon where t is a pointer. so; that means, f of t looks like or f of f of x, f of x look like summation a m x power n on t minus epsilon t plus epsilon, where this side is a power series conversion power series.

I will not go much into other details. So, this is. So, what to notice this is a subspace C omega K I is a K subspace of K power I and C omega K I is in fact, a subspace of C infinity K i. So, this means every analytic function is infinitely many times differentiable, and you might have learnt in analysis course that the derivatives given by term by term differentiation of the power (Refer time: 17:10) expansion. Also here this inclusion is proper; that means, this is more tougher to construct example of a analytic function, example of a C infinity function, which is not analytic such functions is very arrive in probability. So, enough of these function spaces, this example was completely R elated to the subspaces which arise in analysis.

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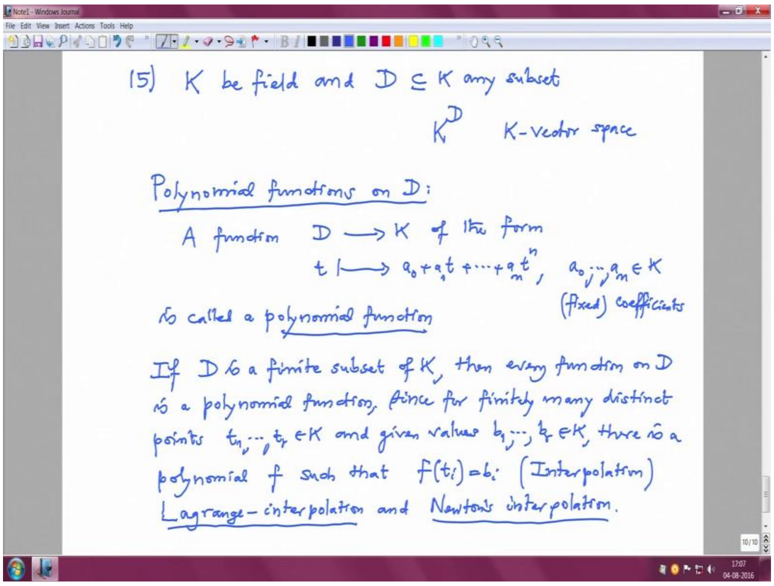

Not let us coming back to the next example, I have forgotten the numbers which are the number in fifth example. This is sort of example from algebra. So, let K be field, it could be finite field also. In fact, to later some time that this is more interesting when K is a finite field, and D any subset of K any subset, then we have this big ambient space K D these are all functions, all K valued functions are K and we have seen last lecture that this is a K vector space. Functions are added like the values are added, also scalar multiplication of a function is means defined by using multiplication in the field k. Now among all the functions I consider what is called a polynomial function.

So, polynomial, let us recall what is a polynomial function, on d; that means, we have a function of the form, function of the form, a function from D to K of the form any t going to a naught plus a 1 t plus plus plus plus a n t power n, where this a 0 2 a n are elements in k, and they are fixed, a 0 to a n is fixed, and these are called coefficient, coefficients. So, such a function is called a polynomial function. If D is finite, if D is a finite set, finite subset of k, then every function on d, is a every function on d, is a polynomial function. Why that, because, note here this is something like interpolation. So, for since I will write here the reason, since for finitely meaning distinct points t 1 to t R in K and given values b 1 to b R in k.

You know there is a polynomial f; such that f of t I is b i. So, this is if you remember it is a interpolation, there are two ways to construct such polynomials; one is called Lagrange interpolation, and the other is Newton's interpolation. More about this I will write in either in supplement or in assignment. I just want to recall that both of these also involve derivatives.

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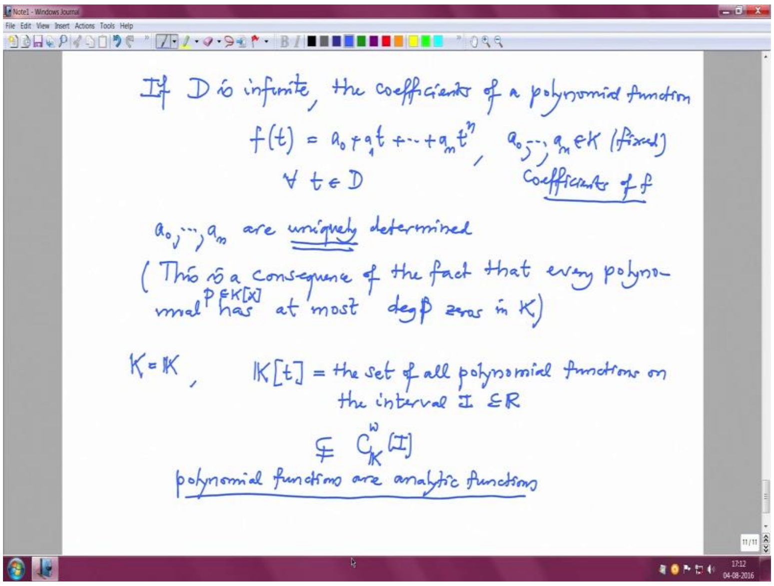

Continuing further if D is finite, if D is infinite sorry D is finite is over. If D is infinite subset, then the coefficients of a polynomial function. Remember I have written polynomial function as f of t equal to a 0 plus a 1 t plus plus plus a n t where this coefficients a 0 to a n or elements in k, they are fixed and we are calling them coefficients of f and this is valid for all t in d; such a function you call it a polynomial function. The coefficients are uniquely determined; if D is infinite a 0 to a n are uniquely determined. This follows from the fact that if you have a polynomial, then it is determined by it is n values, because if you have a polynomial which vanishes at lower than the degree 0, then the polynomial is 0. This, this is a consequence of the fact that each every polynomial has at most degree f degree. Let me call the every polynomial p with coefficients in k, and at most degree p 0es in the field k.

This fact may be I will prove some time in a later section, but right now we have used it in this example, because D is infinite and such a polynomial of there are two, if the coefficient, there is other set of coefficients, these also satisfy this equation in the difference is a polynomial function, which is identically 0 or so, but it has D as more than given n points. Therefore, it should be 0. So, identically 0; that means, all the coefficients are equal.

This about study of the polynomials I will do it in a different section, but this was just to give an example. So, if I denote set of all polynomial functions temporarily by K t, where I am again I want to recall this was for so far for a general field, but now I take K equal to either field of real numbers or field of complex numbers and then these K t is denote the set of all polynomial functions on the interval I contained in R. Then this K t is actually subspace of C omega. Just remember just now we have just seen this C omega K I is the space of all analytic functions on the interval I, and we have seen it is a subspace, and analytic means every point have a small interval around it, where this function has a power series expansion, but these are actually polynomial functions.

So, in particular they are power series expansion so; that means polynomial functions are analytic functions, so polynomial. This means, this inclusion simply means polynomial functions are analytic functions, and here also there is a proper inclusion because simply, because you would have seen exponential functions are analytic functions, but the exponential functions are not polynomial functions. So, that is a reason this inclusion is very proper.

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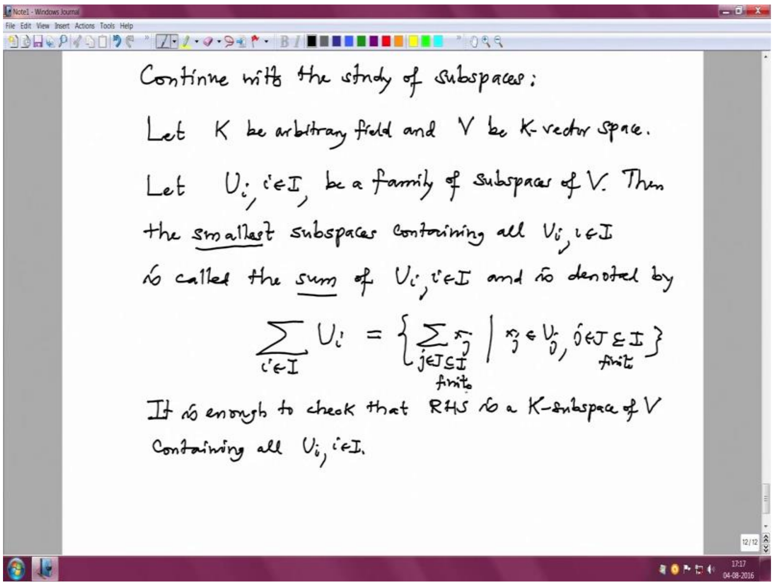

Now, with this I would like to continue the study of subspaces. So, given; continue with the study of subspaces, if you have a let always I will once I will recall K be arbitrary field, and V be K vector space, let U i i in I be a family, of subspaces of v. there the smallest subspace containing all these V is; such a such a smallest subspace exists, because V is 1 subspace of V which contain all V I s. So, definitely there is 1 subspace and when you want to take smallest you want a section of all of them, which contain all the v's, then this is we have seen the intersection of the subspaces, this subspace, and this subspace exists. Therefore, it is a smallest one, this subspace is denoted by is called, first it is called the sum of U i s and is denoted by, it is called a sum, denoted by this U i i in i.

Actually, we can write down the elements of this subspace. This is precisely the set of all sums x js, where j in j, where j is a finite subset of I and x j they belong to u j j belongs to j and j is running over all finite subsets of I. In other words you look at the finite sums of the finitely many, sums of the many elements from V js. So, how does one check equality, it is very simple. see if we could check that this is a subspace this is a right hand side if you could check that it is a subspace, and if you could also check that it contains all the u Is, then by definition this has to be equal to the left hand size, because left hand size is by definition it is a smallest subspace.

So, it is enough to check that right hand side is a K subspace of V containing all U i s, but this is obvious, because you see if you want to check that all U i s are contained this.

So, fix an I and we can (Refer time: 34:20) I they are; obviously, the finite sums, vector itself is a single tense. So, single tense therefore, all U i s are contents here, now to check it is a subspace we have to check that two elements of such, two elements in the right side, the sum is also there, but if you have two sums it is a very finite sum. So, it is there scalar multiple is also (Refer time: 34:47) there. So, it is a subspace. With this, I would like to end this lecture and continue next time.

Thank you very much.