Linear Algebra
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Lecture - 60
Determinants and Volumes (continued)

So, having seen theorem about volume in connection between volume and determinants now, one could also extend this to more general setup where we will be apply this to many more general things.

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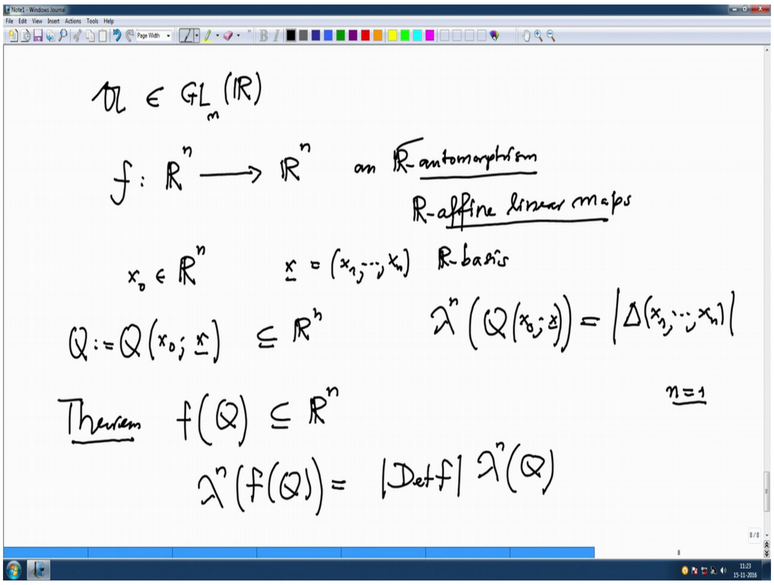

For example: I want to instead of a matrix GL m; in a GL m are non singular matrix in GL m R. I would like to consider a map f from R n to R n which is an automorphism so an R automorphism. Or even more general than this. See automorphism then you have to fix a origin, but did may not the map may not fix any anything. So, I even want to consider what is called R affine linear maps. That means, a translation of a linear map; now first translation and linear map.

So, for simplicity I will assume automorphism because your translation and linear maps that we know our big major be use very well. So, when we will make proofs for general case you will reduce to the automophism case anyway. So now, x naught as usual x naught is a fixed point in R n and x is a basis of R n x n to x n R basis. And from this we have these parallel out of Q x naught x this is a parallel out of in R n and we have seen the volume of this parallel probe is nothing but, so lambda n of Q x naught x, this is nothing but mod of the delta of x n to x n. And we I just want to stress that the proof of this formula we have used elementary matrices to we do the problem to essentially n equal to 2.

Now, what do we do? If I apply f to this parallel trope that mean let us call this is Q. So, f is a automorphsim from R n to R n, so if I apply f 2 if I look at the image of Q under f that is f Q. This is subset of R n. And now I want to prove that we already know Q either volume lambda n lambda n Q exists, so this also should have volume and how to compute that volume. So, then this is a theorem lambda n of f Q is nothing but mod of the determinant of f times lambda n Q. I will see the other side it is it lambda n Q we compute and determinant of variable also you compute independently because then you know the matrix of f and choose the basis in so on and so on. See this is what we have done.

See for example, suppose just to motivate: suppose I take n equal to 1 and linear map I just take a multiplication by a big number of small number; that means, I am shrinking the length, shrinking or enlarging- then what happen to the volume? This formula tells you what to do. In general if you have n R 2 if you have a subset and you know the volume of that of obvious suppose you know that and if I shrink the map automorphism of R 2 is shrinking or enlarging and this is a 2 obvious cases or maybe I will rotate and shrink or many other possibilities that will be come under this automorphism. And to compute the volume of the new picture that we get carried by an automorphism that will be easy to calculate like this.

So, again I will not do this, this is a similar proof like the earlier one. Now also one more theorem I want to state which is a very interesting and that will give some more concepts.

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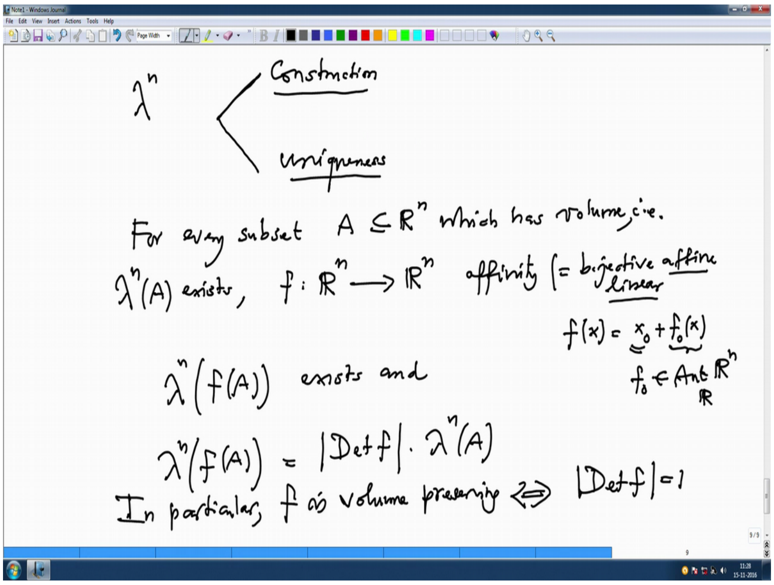

So, remember I mentioned that this volume function lambda n. This is there are two parts of when you one to study this first is a construction part. So, construction is really done by the assumption that we would like this function to have. For example, translation invariant, keep going should have volume mod, and additive property, countable additive property and so on. These are must and then we construct volume function which assigns some subsets volume and not all, but then the uniqueness such a volume for initially unique or not; that is another question.

So, this question I am not rotation this, this questions are typically the first course on major theory. But this assuming we have the answer for this, then we can say more. So, that is what I want to state the theorem. So, theorem: I will not state as a theorem, but I will just state as assertion. So, for every subset A in R n which has volume that is lambda n A exists. And suppose a for a automorphism not necessary f is the affine linear from R n to R n I will write here affine affinity. Affinity means, a bijective affine linear. And what is the affine linear just one minute; that means, it is f x looks like a translation first x naught plus f naught x, where this f naught x is now are linear. So, fix and it translates now all points are translated with this x naught this f naught is an automorphsim not R R n.

So, this is also good. Anyway, when we are this and see they are many more in this now because origin is not fixed you can always work with more coordinate free approach.

Then, the question is whether lambda n of f of A the image of A under f whether these exists and if it exists how do one compute it. So, these always exists, and this equality lambda n of f of A is equal to mod of the determinant of f times lambda n of A. In particular f is volume preserving if and only if leave mod of the determinant is 1. Volume preserving means, by applying f you do not change the volume of a subset. This is very important property because when you make a coordinate change and you define a this map and then you do not want volume to be changing.

So, these are very important for application, so more than that I will not say. And the next topic actually here should be not only the affine, but the projective, but I will not deal in this course with the projective once. So, this I will probably do in the next course which I will call geometric linear algebra.

So, with this I would stop to other course, and we will see you in the next course.

Thank you very much.