# Linear Algebra
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## Lecture – 07
## Sum of Subspaces

You know our discussion about the subspaces. Recall that last time we define that, what is a sum of arbitrary family of subspaces.

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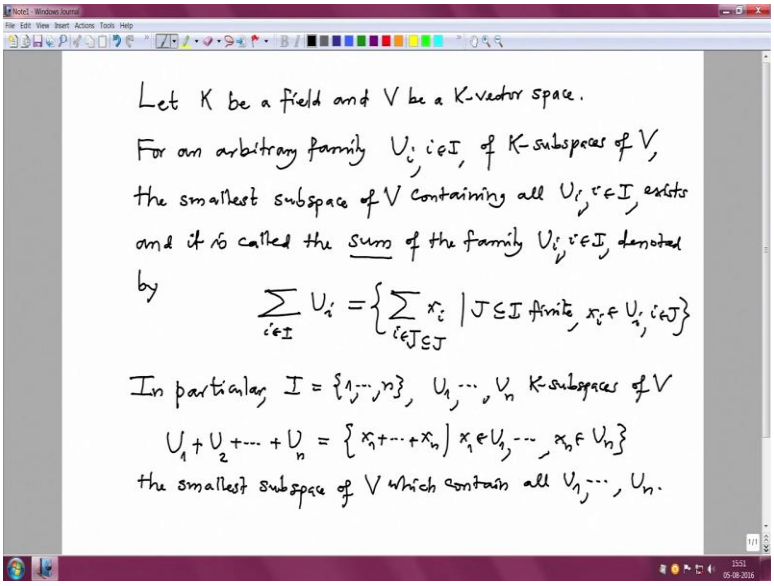


So, let us recall it. As usual K be a field and V be a K vector space. For a family, for an arbitrary family U i index by the set I of K subspaces of v, the smallest subspace of V containing all U i s exists, and it is called the sum of the family U i. Usually denoted by sum I in I U i. This is also one can describe element wise, this is nothing, but all finite linear finite sums, where j is a finite subset of I, and each x i belongs to U i or I in j.

So, we have seen last time that the right hand side is a subspace that just follows from the subspace criterion, because if you have two finite sums, then the sum of that is also finite sum. Also scalar multiple of a finite sum is a finite sum so; that means, this right hand side is a subspace, and it is clearly a, it clearly contains all the U i s, and it is clearly the smallest one. So, therefore, by definition these two subspaces are equal.

So, in particular let us understand for finitely many subspaces, in particular when I is finite, let us say 1 to n and we have a family, we have n subspaces U 1 to U n K subspaces of v, then the sum U 1 plus U 2 plus plus U n is the subspace of the sums x 1 plus plus plus plus x n, where x 1 is in U 1 x n is in U n. This is the smallest subspace of V which contains all U 1 to U n.

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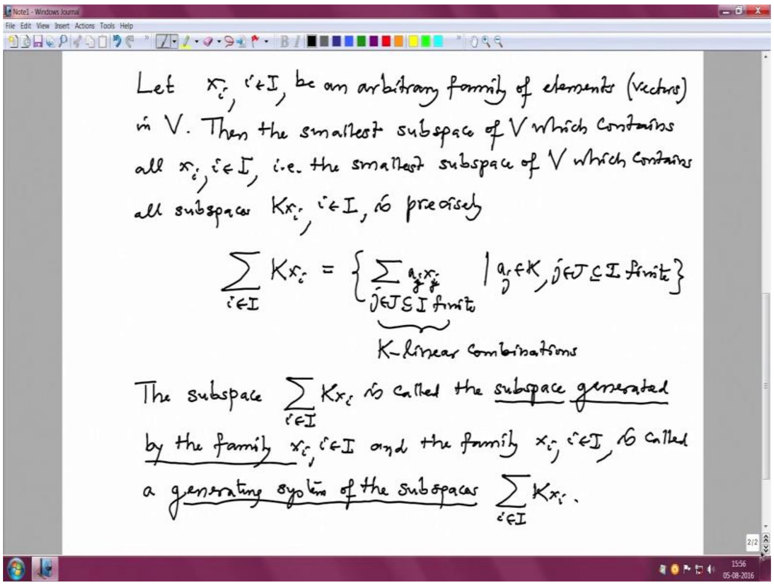


So, let us continue; I want to give couple of remarks. So, how do we construct from a finite family of vector. So, let x i i in I be a finite arbitrary actually I do not want to take (Refer time: 05:50) be an arbitrary family of elements. Remember we are also calling elements of vectors; this is as vectors in v.

Then the smallest subspace of V which contain all x i s, means that is the smallest subspace of v, which contains all subspaces K x i i in I, because if it contains all x I, then it contains all subspaces cyclic subspaces generated by the x i s. So, this is is precisely this notation K x i I in I .This is precisely finite sums j in j finite a I x i a j x j; such that a js are some elements in k, x is among the x js and j is a j is a j in j j is a (Refer time: 08:26) or finite subsets.

Such a sum such a finite sum is also called K linear combinations. So, this subspace, this subspace K x i i in I is called the subspace generated by the family x i. So, any element in the subspace generated by this family is precisely a finite linear combination among the x i s, and the family x i i in I is called a generating system of the subspace, yes.

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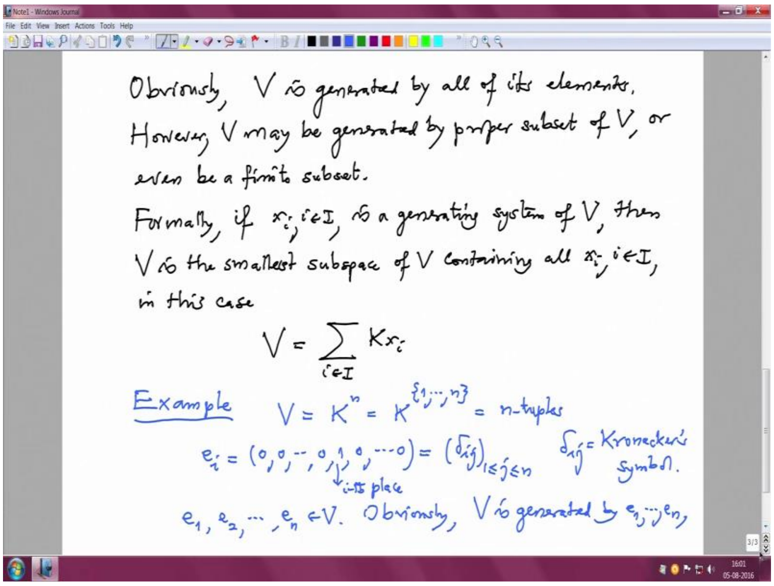


Obviously, V is generated by all of its elements, but it could also be generated by a proper subset of v; however, V maybe generated by proper subset of v, or even a finite subset. So, formally if x i s is a generating system of v, then V is the smallest subspace of v, containing all x i s. In this case V is K x i i in I this is useful for checking whether a system of elements is a generating system of V or not. So, just to give an example, let us take our modern example of a vector space; V equal to K power n. This is also I just want to remind you can think of them as (Refer time: 13:12) from 1 to n to k, these are n tuples.

So, in this we have this e 1 e I e is 0 0 0 i-th place 1, and everywhere else it is 0, this is i-th place. this is also denoted compactly by delta I j 1 place equal to j place equal to n, where delta j is the standard Kronecker symbol. Then we have this n vectors e 1 e 2 e n; obviously, V is generated by e 1 to e n. So, we need only to check that, every element of V is a linear combination of this e 1 to e n. So, that will mean that V is the smallest subspace which contains all these case.

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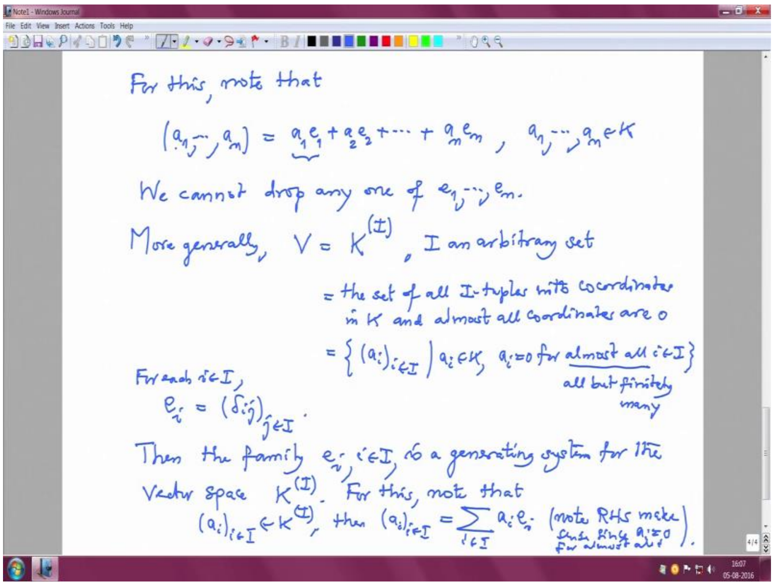


So, for this note that any tuple which is a 1 to a n is same as a 1 e 1 plus a 2 e 2 plus plus plus plus a n e n with this a 1 to a n or a elements in the field. This is obvious, because you see when the two, when you want to check that a 2 tuples are equal, you have to check that all coordinates are equal. So, for example, a 1; this is the first coordinate on the left hand side. On the right hand side the first coordinate is only contributed from this term, because all other e 2 set up to n have move first coordinates 0. So, when you multiply this and our vector space structure is component wise addition and component wise multiplication. So, this is obvious.

So, this checks that 1 to e n is a generating system for the vector space K power n. also note that fewer elements will not work. no we cannot, drop any one of e 1 to e n to get a generating system; that is obvious because if you for example, if you drop e 1, then you can never get e 1 as a combination of e 2 to e n, because all e 2 to e 1 have coordinates the first coordinate 0. More generally you can generalize this example more generally if you take V to be K power round bracket I, where I is an arbitrary set. Let me remind you K power round bracket I is precisely, this is the set of all I tuples with coordinates in k, and almost all coordinates are 0. So, formally you can write like this, this is a I i in I a I is there in K and a I equal to 0 for almost all I in i. Remember the phrase almost all means all, but finitely, maybe it does not mean infinitely meaning.

So, in this case you can continue denoting the same way e I for each I in I e I is delta I j this is very j in i; again the same Kronecker symbol. So, though it looks a infinite tuple, though it looks a tuple index by I, I could be infinite, but in this tuple really only finitely only 1 is non zero, namely I equal to j case j equal to I case. So, then the family of this vectors e I i in I is generating system for the vector space K power i. Again for this, like we have noted in a finite case, just note that if I take any tuple a I i in I in K I power K power on bracket I then this tupple a I is nothing, but summation I in I a I e i. Note that this sum on the right hand make sense, because only finitely many I s could be non zero. So, note r h s make sense; since a I equal to 0 for almost all i.

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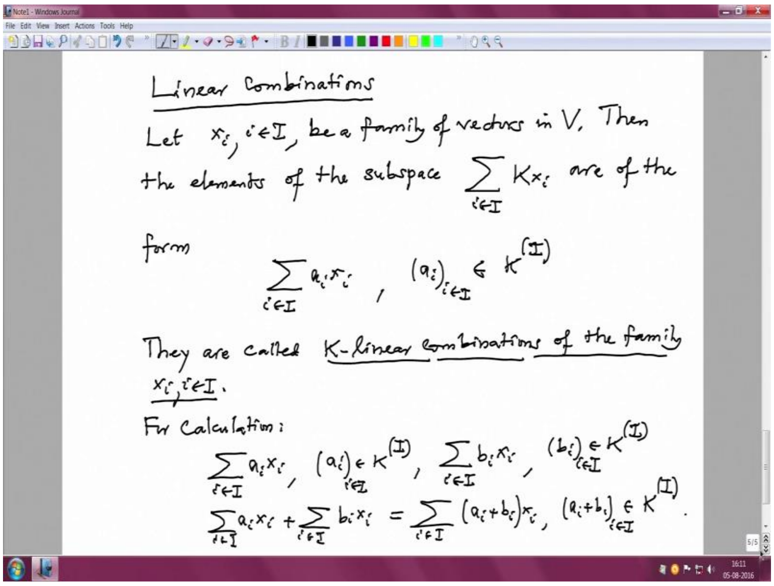


Now, for the future use I would like to formally write, so linear combinations. We have seen, if let x i i in I be a family of vectors in v, then the elements of the subspace sum K x i are of the form summation a I x i i in i. And now I can write this a I i in I belongs to K power round bracket i; that means, this sum is really finite sum, this are called, these elements, they are called K linear combinations of the family x i .

So, for calculation we will often if I have 1 K linear combination a I x i and another one. So, this means first of all a I should belong to K power round bracket i. Do not forget this round bracket, that is very important, because we need only finitely many a I s could be non zero not more. And another one b I x i i in I with b I tuple, also the same K power I round bracket. Then they sum, this is by our rules distributive and rearrange the brackets and so on. This is nothing, but sum I in I a 1 plus b i. This is where we are using the vector space rule, remember the first lecture four properties those. So, and note that again that these a I plus b I tuple. They are added in K by using the addition in k, and they will also have the property that only finitely many components are non zero.

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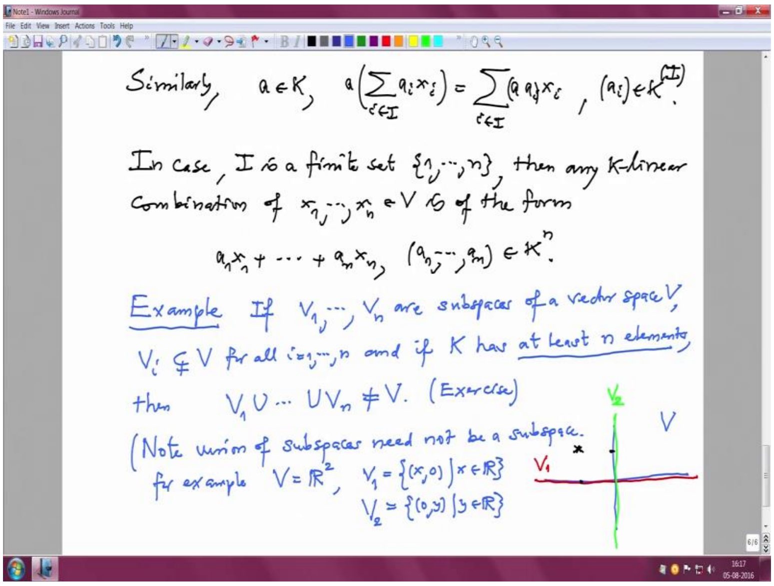


Similarly, for the scalar multiplication; a in K a times a i x i minus. This is summation a a i x i. I could write a bracket here, minus and a I belongs to K power round bracket i. So, in case I is a finite set, say 1 to n, then we have only finitely many elements in any K linear combination of x i to x n, which are these are n elements in the vector space is of the form. Now, we can afford to write it explicitly a 1 x 1 plus plus plus plus a n x n, where this a 1 to a n, they are n tuples.

So, I would like to end this section. I would like to discuss one example couple of minutes. So, one example, very important example and I will keep using it many times in various arguments. So, it says that if we have, if V 1 to V n are subspaces of a vector space v, and all of them are proper, each V i V i is a proper subspace, if and if K has the field K has at least n elements, then the U V n can never be v. note that here I should remind you note that U n of subspaces, need not be a subspace simple tuple ,for example, if you take V equal to r 2, and take two subspaces V 1 is x axis. So, that is tuples of the form x comma 0, x is varying in V r, and V 2 y axis that is 0 comma y, y varies in the real numbers. So, pictorially it looks like this V is r 2

So, it is a plane, this is v, and V 1 is this axis, this is V 1, and V 2 is y axis. So, if it is a subspace, if the U n is a subspace, then for example, one element here and one element here we could add it, but that will be this, that will be this element which is; obviously, not in the v. So, therefore, U n is not a subspace, and the checking this example, I will leave it as an exercise. Remember K has at least n elements is very important condition, without that this will not be true.

We will take a break and then continue.