Linear Algebra
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Lecture - 08
System of linear equations

Come back to the second half of this lecture. In this second half I will consider linear equations systems of linear equations.

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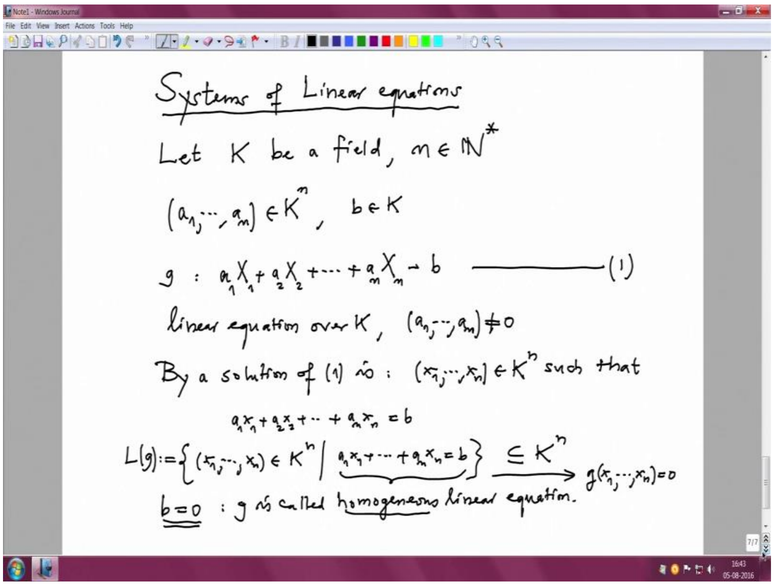

So, first 2 minutes I would like to set up there the notation that we will keep using it in the section and couple of definitions for the completeness.

So, let as usual K be a field and I am going to consider linear equations in N variables. So, N is some natural number non 0 and for each tuple a 1 to a n in K power n and another constant b scalar b in K the equation we look is linear equation, we look is a 1 x 1 plus a 2 x 2 plus, plus, plus, plus, plus a n x n equal to b, this is the equation in variables capital X 1 capital X 2 capital X n, the coefficients are in the field K and when we say linear equation over k, but to we will also have to assumed that this tuple a 1 to a n is not a 0 tuple because if it was 0 then the left hand side will be 0 and there is nothing to study.

So, this is a typical linear equation and we want to find the solution and what does mean by a solution by solution of the let us call this equation as one of one is what is precisely a tuple again x 1 to x n, now small x 1 to x n is tuple is in K power n such that when I plug in instead of capital X s small x s then a 1 x 1 plus a 2 x 2 etcetera, etcetera, a n x n is b, note that there could be many solutions. So, in general I would like to write for this right now I do not want to do this was solution basis therefore, a subset of x 1 x 2 xn all those tuples in K power n such that a 1 x 1 plus, plus, plus, plus a n x n is equal to b.

So, this could be a visual set, but I would like to study this subset of K power n for example, you need a subspace if it is the sub space how big it is; how does one find it and so on. So, this is the typical and instead of one equation I will take many equations much of equation in n variables and we will would like to study how big the solution plays out as and find out etcetera, etcetera.

This is the thing for example, if b equal to 0 for a moment suppose b equal to 0 then this let me temporary call it this equation as g and this is a L g, g is the solution space for the equation g strictly speaking when I write g is should bring this b to this side and equate to 0 so, but this term which is if b is 0 then this equation is called a homogeneous linear equation g is called homogeneous linear equation and allow me to change here b 1 the other side. So, that there will be a no confusion this is minus b 1.

So, we are looking for in this case also we could write it instead of writing this long we could also write g of small x 1 x 2 to x n is 0. So, this is 0 set of g a linear equation and it is.

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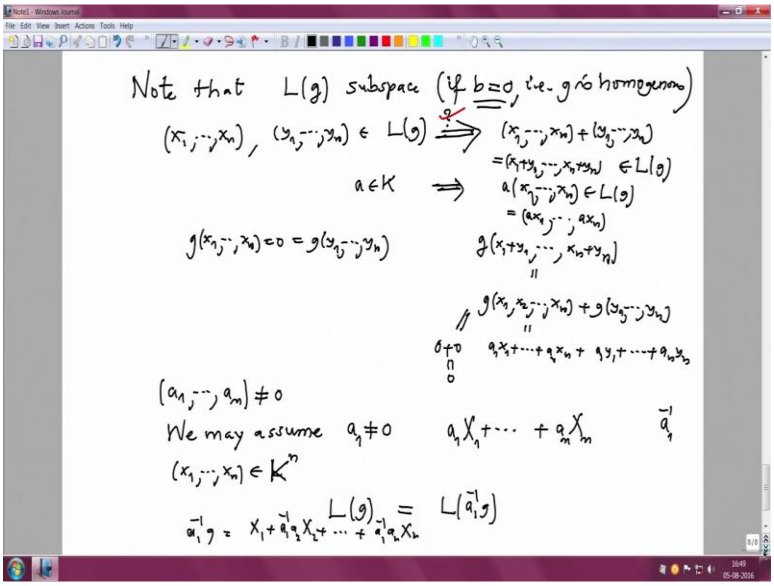

Now, first of all we note that this is note that L g is actually is subspace. So, this discussion going under this under the assumption if b 0 that is g is homogeneous, so what we have to check as we have seen in the last in lecture that to check somebody subspace we have to check 2 things that if you have 2 elements when they sum with there and also if you have a scalar and if you have one element one element in the sub subset then the scalar multiple is also there.

So, if you have x, x 1 to x n this 1 tuple and another is y 1 to y n if this belong to L g then we need to check this is what we need to check that x 1 x n plus y 1 y n also belong to L g and another thing you need to check that if you x 1 x n belong into L g and if a belong to a scalar then a times x 1 to x n also belongs to l g that will make this L g as a subspace, but this is more or less obvious because you see this then they belong to L g means g of x 1 to x n is 0 also g of y 1 to y n is 0 and what we want is g of what is the sum in K power n that is the precisely the component wise that is x 1 plus y 1 x n plus y n.

And similarly this scalar multiplication is a x 1 a x n. So, we need to check from these 2 is g of x 1 plus y 1 x n plus y n this is also 0, but g of x 1 plus y 1 etcetera x n plus y n is nothing but g of x 1 x 2 x n plus g of y 1 y n this is simply 0, this g of this you know this is what by definition remember this is a x 1 a 1 x 1 dot, dot, dot a n x n and this is a 1 y 1 plus dot, dot, dot plus a n y n and when you add them. Now use the vector space is used and combine the terms with a 1 coefficient is x 1 y 1 a 2 coefficient is a x 2 plus y 2 and so on.

So, you precisely get this, but both are 0 therefore, this is 0. So, this is 0 this is equal to 0 plus 0 which is 0. So, what we have checked is this condition. So, we have checked this similarly this similarly when you plug it in that a will be come out. So, we what we have checked it, but it was a very important your b was 0 b 0 was very very important.

So, we are check it is the subspace and the next is the; how do we find this subspace? So, one simple thing is. So, we have already noted that we are considering a 1 to a n the tuple is not 0 tuple. So, in this case we will assume we may assume without loss of generality we may assume that this the first coordinate instead of non 0 if it is 0. So, let us; let me go back and show you see if it is 0, if this a 1 is 0, there may be a 2 could be 0 or there somewhere it is non 0 in that case I will rename the variables. So, that whichever is variable as coefficient non 0, I will bring it they would be the first position.

So, we can assume a 1 is non 0 then when x 1 x 2 x n is the solution is a L g is a solution if it is in K power n and if it is a 1 x 1 then the best is actually look at this a 1 is non 0. So, we have this a 1 x 1 plus, plus, plus, plus a n x n this is the equation we had and this was non 0. So, I will keep; I will multiply this equation by a 1, a 1 inverse on both on this I want to multiply this equation by a 1 inverse that I am allow to do because this a 1 in non 0 element in the fields. So, inverse exists and also note that if i take a g, L g or L a 1 inverse g, this 2 things are same because if x 1 x 2 x n belongs here if and only if it belongs here because we are multiplying by a 1 inverse it is. So, in that case the equation this a 1 inverse g will become x 1 plus a 1 inverse a 2 x 2 and so on; a 1 inverse a n x n.

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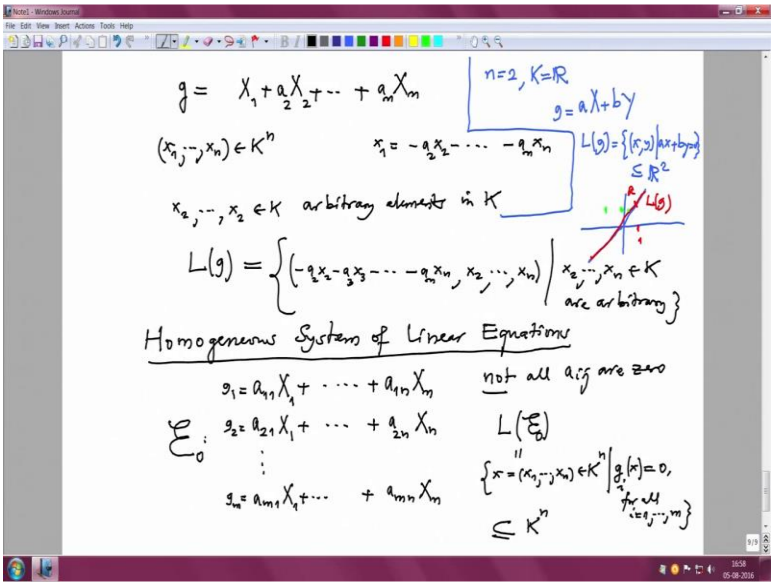

So, we without loss you may assume g looks like this g is X 1 plus now in the new coefficient I do not want to keep writing a 1 inverse x. So, this a 2 x new coefficient. So, it is enough we can solve it is equation or find the solution space, but you see if x 1 to x n in K power n if I take if I; this is the solution then what did equation, I will get in x 1 is equal to minus a 2 x 2 minus etcetera minus a n x n.

So, if I take x 2 to x n arbitrary elements in K and I will take x 1 which is determined by this equation and; that means, the solution space. So, in this case L g is therefore, first coordinate I will write it that is a 2 x 2 minus a 3 x 3 minus, minus, minus a n x n comma X 2 comma X n where X 2 X n in K are arbitrary so; that means, we found a solution space and it is it is big begin the same that the degree of freedom is n minus 1 n minus 1 coordinates are arbitrary and the same procedure we will we will continue for arbitrary number of equations.

So, first I will consider a homogeneous system of linear equations homogeneous system of linear equations. So, good thing about homogeneous system is the solution space is subspace. So, we can draw the pictures also for example, I should have drawn a picture in the 1 equation case suppose n equal to 2, let me draw a picture here in this corner if n equal to 2 if n equal to 2 a field is real numbers and we have 1 equation, there are 1 and homogeneous at 2.

So, our equation will look like a x plus b y this is our equation and we are looking at the solution space; that means, this is our g we are looking at L g, L g is now pair of it is in the plane. So, it is small x comma small y such that a x plus b y equal to 0 this is in r 2; obviously, the picture will look like this it as to go through 0, 0, 0; obviously, solution because it is a homogeneous equation and it go like this. So, it is a line passing through a origin this is L g and how do you see a and b with if y 0 and x is 1, if you take y equal to 0 x equal to 1, you get a that is if it is take this is 1, this a if you take x equal to 0 and y is a x is equal to 0 in y is 1 then it is b. So, y is 1, this is 1 in the b, b will be here, yes b will be here this.

So, I would a couple of lectures; I will device also more geometry thinking and that time you can draw more pictures which will make much more sense. So, homogeneous now I will call the system homogeneous system I denote by E, E sorry, equations and there will be bunch of linear equations number of variables are n and the number of equations are m.

So, we will have to write like that a 1 1 x 1 plus this is the first equation the first number we have denote the now the number of equation the number that it is the first second etcetera a 1 n x n if it is homogeneous it is like this otherwise i have to subtract some constant subtract or add does not matter the second equation will be a 2 1 x 1 plus, plus, plus, plus, plus a 2 n x n a m 1 x 1 plus, plus, plus, plus, plus a m n x n.

These are the linear equation and; obviously, we are considering a system where at least 1 equation makes sense. So, not all a i j are 0. So, our assumption is not all a i j s are 0 otherwise there is nothing to study and also. So, we are we want to study l of E just to remember that this homo; this is a homogeneous system I will call it absolute 0, E 0 this is just a remember on for us that it is a homogeneous system and this is the l, l of E 0 is by definition.

We are looking at the tuples x x 1 to x n in K power n such that is i call this g 1, g 2, this is g n these are the m equations. So, g; g i of x equal to 0 for all i equal to 1 to m and this is a subset of K power n and our aim is to determine this and later when we introduce the concept of dimension then we will try to what is dimension of this. So, first of all it is clear I want to check.

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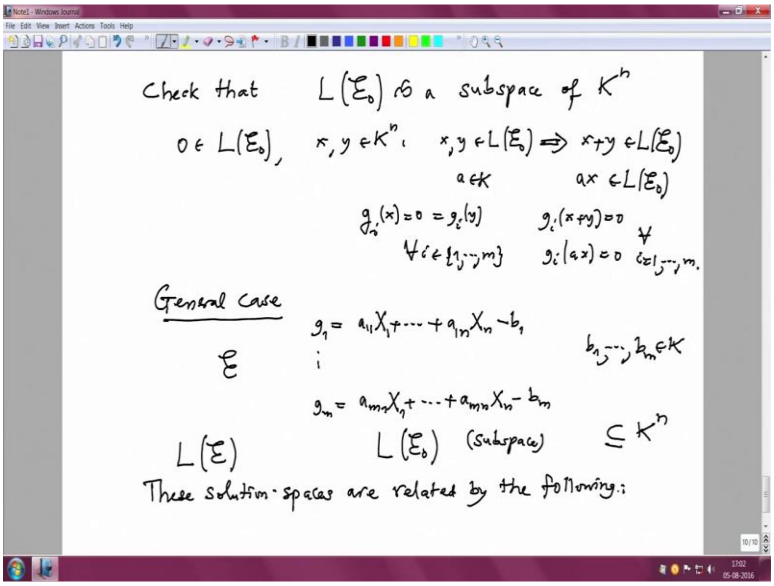

Let us check, so first we want to check that L E 0 is the subspace is the subspace of K power n. So, what do you have to check again it is; obviously, 0 belong there 0 is clearly there and we need to check that if x y are in K power n then x plus y is also you know if x y are in K power n if x and y both are in L E 0 then we need to check that x plus y is also L E 0.

And similarly we have to check that is if a is scalar then we have to check that a x is also L E 0, but the checking is very simple we have to check that for each g i given that each g i x is 0 each g i y is 0 and in the first case we just noted that g i of x plus y is also 0 and g i of a x is also 0 and this is true for every I so; that means, we have to check this is 2 for every i in I, I from 1 to n 1 to m. So, this is true for all I, I equal to 1 to m. So, it is the subspace.

Now, let us take; I want to combine find it out how do we, I want to together do it for homogeneous and non homogeneous. So, now, consider a general system. So, general case now, I will call it instead V i call it E, i instead of E 0, I will call it E and E, it is incognizant with g 1 equal to a 1 1 x 1 plus, plus, plus, plus, plus a 1 n x n minus b 1 equal to minus b 1 plus b 1 does not matter etcetera g m equal to a m 1 x 1 plus, plus, plus, plus, plus a m n x n minus b m where b 1 to b m, there are some scalars they may be 0, may not be 0 now.

Now again; so we have L 1 n and we have L of E and we also have L of v 0 when I say E 0 means is I simply take b 0 and we would like to have a good relation between these 2 this is the subspace this in general it is not a subspace, but it is not too far away from subspace also. So, the really; these solution spaces this solution spaces both are subsets of K power n this is definitely a subspace both these solution spaces are related by the following is relation also this is to precise relation among them we also allow us to draw a pictures.

For example let me go back and show you if here in this example if there was c constant if I will correct it by an using a different colour if there was a c here, this is L of g naught. So, g naught means g 0 then what will be L g; L g will be a line still line, but passing through whom it will not pass through origin unless c is 0. So, if; how will you draw the picture what will be what will be the x axis that line will cut and y axis that line will cut if you know that.

Then it will be uniquely determined. So, it will be something like this it will be parallel line. So, my picture is also not. So, good it will be parallel line it will be parallel to this line and this is this one will be something do with c. So, it will be translated by c. So, this idea will using this idea we will formulate this.

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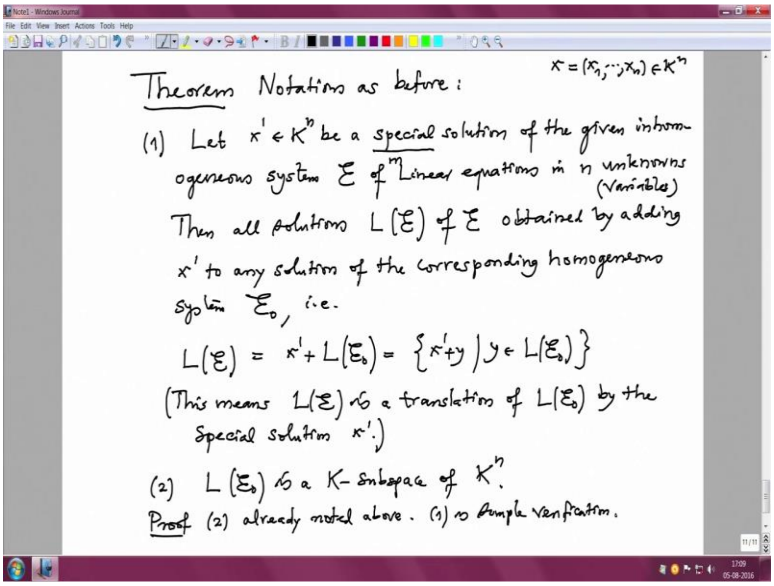

So, that let me write this as a theorem. So, notations as before one I want to write 2 statements one and remember our abbreviation, if I write x small x that is the tuple small x 1 to x n in K power n. So, let x prime in K power n be a special solution of the given in homogeneous system E of linear equations. So, m linear equations in n unknowns or I may also keep calling it variables and that is the reason I am denoting them by capital letters and special means what somehow we found it out because we have to find it out some in case of the example I will give; I we know we have found it by using our geometries we took shall see school geometry we found out intersect etcetera.

So, then all solutions one can describe L of E of E obtained by adding x prime to any solution of the corresponding homogeneous system E 0. So, what I am saying is the following. So, that is in the notation L of E equal to x prime plus L of E 0 and this plus this is clear this means x prime plus y where y is a solution of E 0.

So, this means L E, L E not E 0, L E is a translation of L E 0 by this special solution x prime that we also second one L of E 0 is a K subspace of K power n. So, I will, so 2 is already proved and proof 2 already noted above and 1 is a simple verification. So, I would leave it for you and in many case you will find this formal proof in the notes.

Thank you very much.