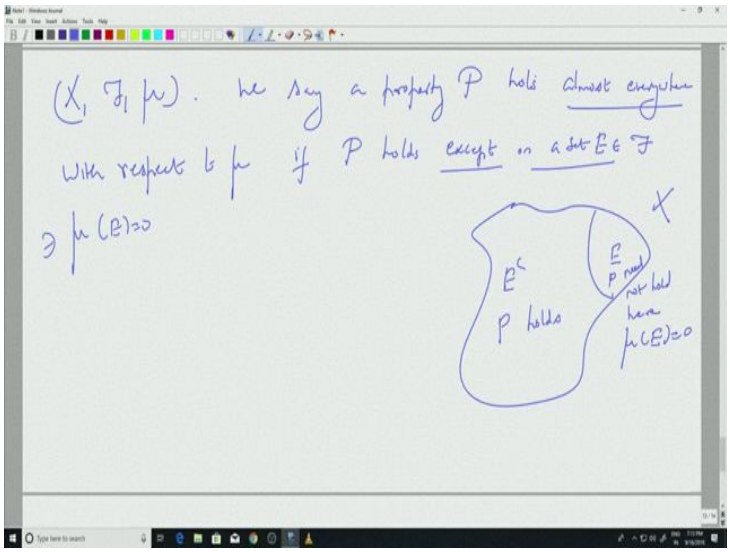
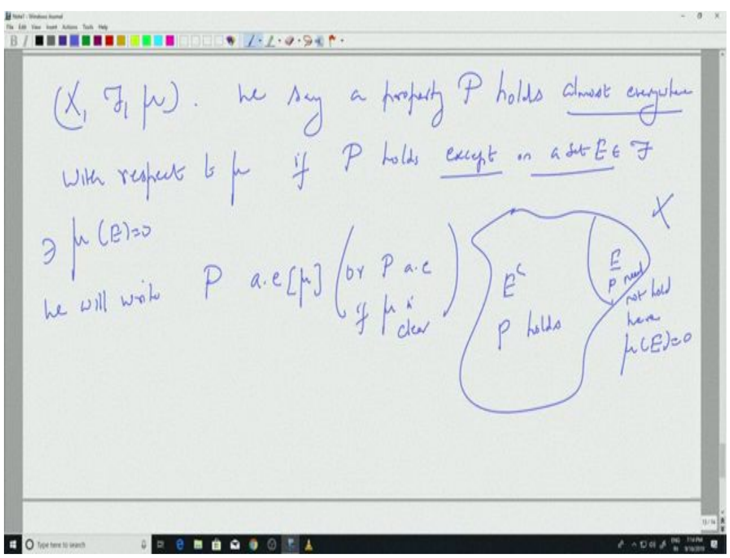
**Measures Theory**
**Professor E.K. Narayanan**
**Indian Institute of Science, Bengaluru**
**Lecture no. 10**
**Sets of measures zero and completion**

Okay, so, in this session we will slightly generalise all the theorems we have done so far by bringing in sets of measures zero. So there are sets of measures zero and those are negligible. In the sense whatever happens on the sets of measures zero, we can neglect and strengthen the statement of the results. So, let us define this formally.

(Refer Slide Time: 1:00)



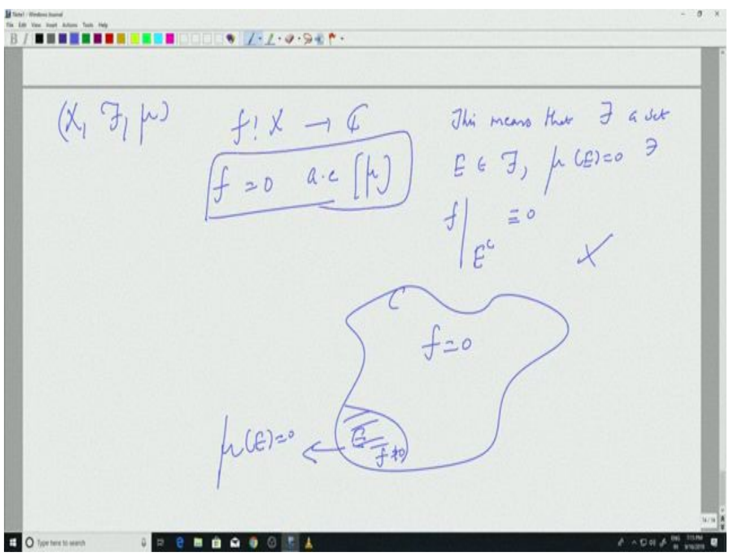


So, we say, so, we always have the triple X, F, mu okay. We say a property holds, we say a property P will give examples to say what properties we are looking at, say property P holds almost everywhere. So, it may not hold everywhere it holds almost everywhere. Well with respect to the measure mu okay. So, everything is with respect to the space we are looking at.

So, with respect to mu. If the property P holds except on a set E. So, it does not hold on E okay. When I said E, which should be measurable, such that mu of E is zero. So, what we mean is I have some space X and I have a set E outside E some P holds okay. We need not hold here, need not hold here okay. But the important part is the set where P does not hold has measure zero in that case we say P holds almost everywhere okay, so,(())(2:43) .

And we will write P that is the property almost everywhere. That is abbreviated as A dot E with respect to mu okay. So, we may not write the measure mu when it is understood. So, when, we may simply say P holds almost everywhere okay, if mu is understood, if mu is clear from the situation. There may be two measures in that case we have to specify with respect to which measure the property holds.

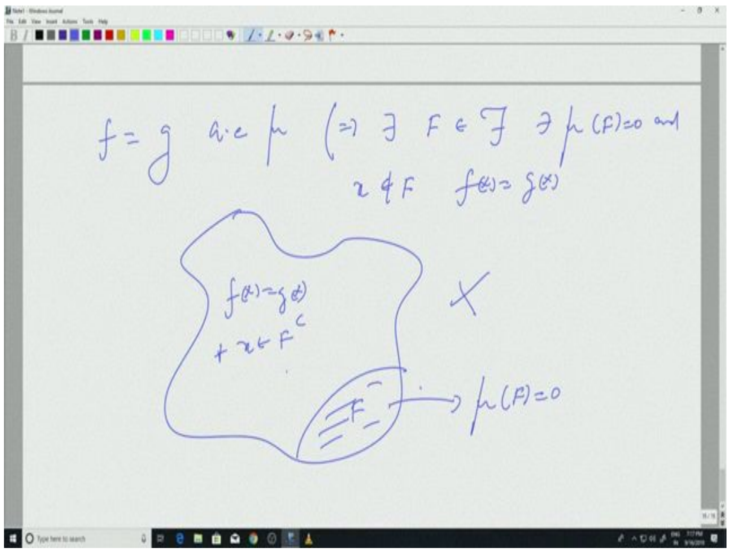
(Refer Slide Time: 3:31)



So, let us look at examples to give you an idea about properties P which we are looking at. So, again we have the measure space is mu. So, we can say a function F from X to let us say the complex plane F is equal to zero almost everywhere mu. What would this mean? This means that there exists a set E which is measurable and mu E is zero, such that F restricted to E complement is identically zero.

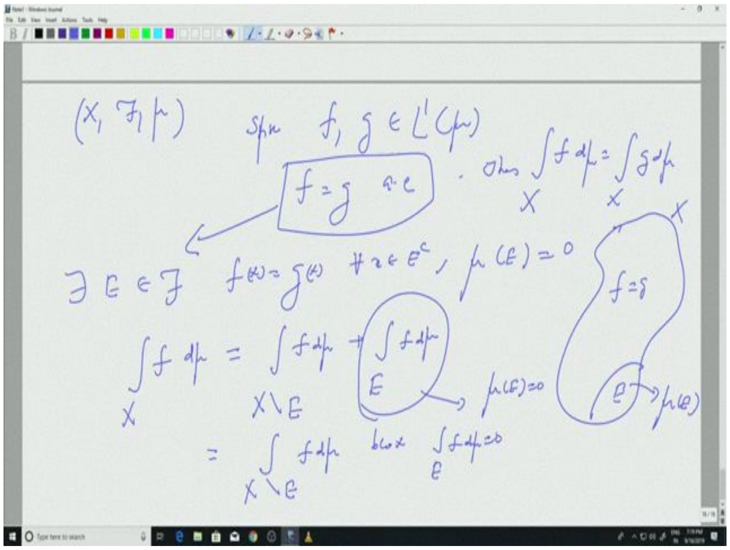
So F is a function zero almost everywhere. So there is a set. So if this is my space X, and let us say this is E, the set has measured zero okay. But my F is zero here, F need not be zero here, okay, as maybe zero at some points, but F is not identically equal to zero on this. In that case, we say F is zero almost everywhere that is one example.

(Refer Slide Time: 4:52)




Another example, F equal to G almost everywhere with respect to mu. What does that mean? That means this implies that X is some set, let us call that F in script F, such that mu F zero and outside F. If X is not in F, then F of X equal to G of X okay. So again, picture wise so I take some space X, I have some set F here, with measure zero, measure zero and here F of X equal to G of X for every X in F compliment, okay, here I do not know. But it is still in almost everywhere equality because it is happening on a set of measures zero. Well, what happens if you have sets of measures zero and you have functions almost everywhere? So let us look at that.

(Refer Slide Time: 5:59)

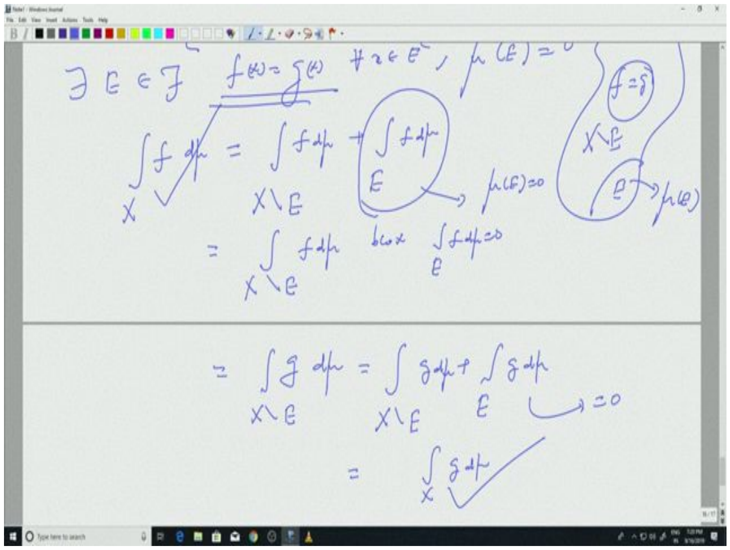


Let us say I have the space F, X mu, will see why the sets of measures zero can be neglected or discarded. So I let us take F and G, F equal to G almost everywhere. Suppose I have two functions F and G both in L1 of mu okay and F equal to G almost everywhere. Then integrals of F and integrals of G are integral of G are same.

Well, why is that? Since its equal almost everywhere, this tells me there exists some set E in F, such that F of X is equal G of X. For every X and E complement and we have the measure of E where F may not be equal to G is equal to zero. Right. So, that is the that is a picture should keep in mind, I have some set E, I know a F is equal to G here, but here I have a set of measures.

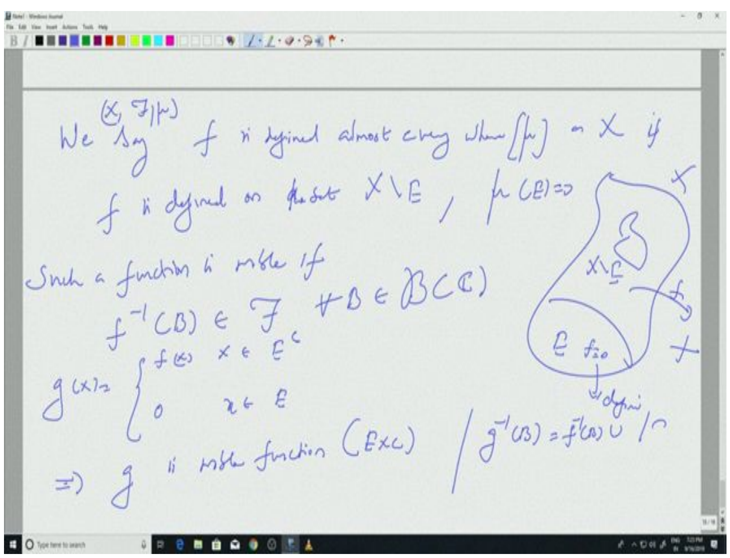
Now, if I look at the integral of F, recall E is a measurable set right. So, because of that I can write this as X minus E, F, D mu plus integral over E, F, D mu, which is equal to. Now what can you say about this? Recall that you have measure of E to be zero, for if you integrate a function over a set of measure zero, then I know that you get zero. So, you will get this is simply equal to X minus E, F, D mu. Because integral over E, F, D mu is equal to zero.

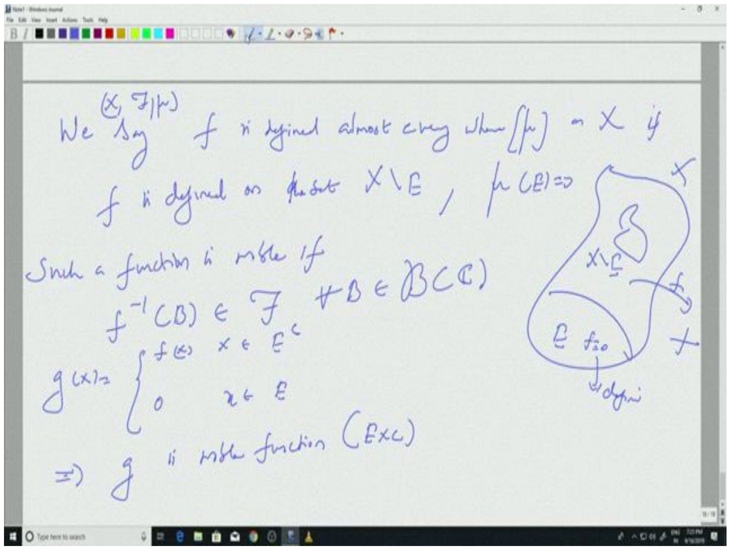
(Refer Slide Time: 8:14)



So let us go one more step. So this is equal to integral over X minus E, remember on X minus E, F is equal to G, right. X minus is E complement. So this is the X minus E part. There, F is equal to G. So I can change F to G and I will get GD. Well, this is equal to integral over X minus E, GD mu plus integral over E GD mu, right. Because this is zero. So this is nothing but integral over X GD. So I started with F, and I ended up with G they have the same integral as long as they are equal almost everywhere.

(Refer Slide Time: 9:05)





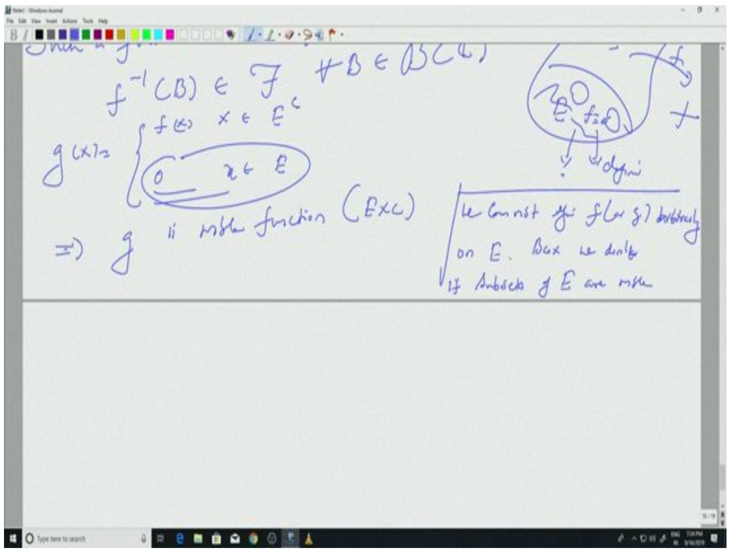
So, whenever you have sets of measure zero, that can be essentially discarded and you can concentrate on the set where the measure is concentrate, okay. In such cases what you can do is. So you can define, so let us say, we say F so I always have the triplet X as mu, we say F is defined, almost everywhere, almost everywhere with respect to mu of course, on X if F is defined on a set E compliments so write  X minus E on the set X minus E with mu of E is zero okay.

So, again let us draw some picture. So, this is E, this is X minus E and I have F going from here to complex plane or wherever it is. Now, such a function is measurable if FN was also. So, I am taking a Borel set in the complex plane. So, FN was of B, so that would be something here, right. So intersected with, so FN was B should be inside X minus E and it is a measurable set for every B in moral set of the complex play right. We can say that.

But I can define F to be zero here okay, define. Well, what do I mean? I am defining a mu function if you like G of X equal to F of X. If X is in E compliment, I know that it exists zero if X is in E. So, now G is defined everywhere, G is defined on the whole set X. Initially F was defined only on X minus E. I am extending F to be zero on E as well, okay. E is a measurable set because of that. So, this would imply G is a measurable set, G is a measurable function that is easy to check.

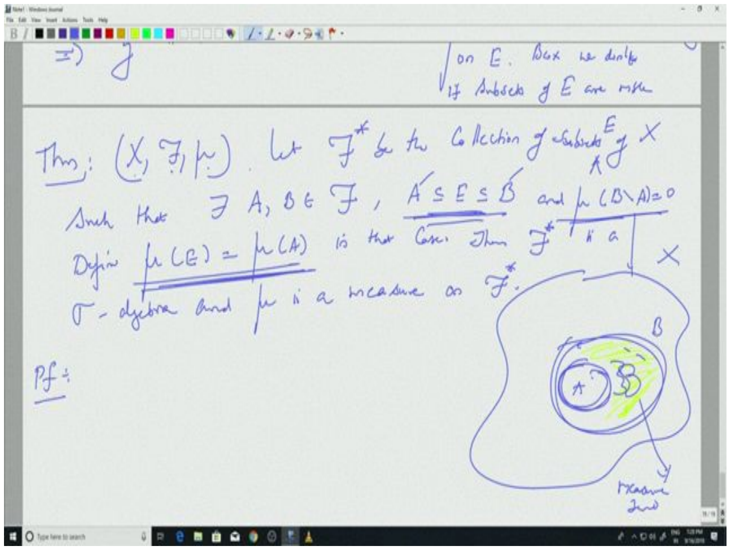
So, I will leave it as a simple exercise. What do you do is? You look at G inverse of B. Well, this would be equal to F inverse of B and something right union or intersection, depending on how you write it with something else but F is measurable. So, I know that certain part of this will go inside script F and the remaining is defined by E okay. So, I will leave it to you to figure out what exactly this inverse image is, so that you can write, you can prove that G is a measurable function. And of course, when you integrate G over E it is zero. So, only whatever is in E compliment will matter that means whatever values of F on E compliment will matter.

(Refer Slide Time: 12:31)



But in this case, I am defining it to be a constant zero on E. I cannot define, we cannot define F or G arbitrarily on E, okay. Because we do not know if subsets of E are measurable. If you take an arbitrary definition of F here, then when I take the inverse image, I will get some subsets of E and we do not know if that is measurable. So, the resulting function may not be measurable, but if I put a constant like zero or one there, then it is going to be measurable. But we can make an arbitrary definition here and make it measurable, if I know subsets of E are measurable, so we will do that okay.

(Refer Slide Time: 13:46)



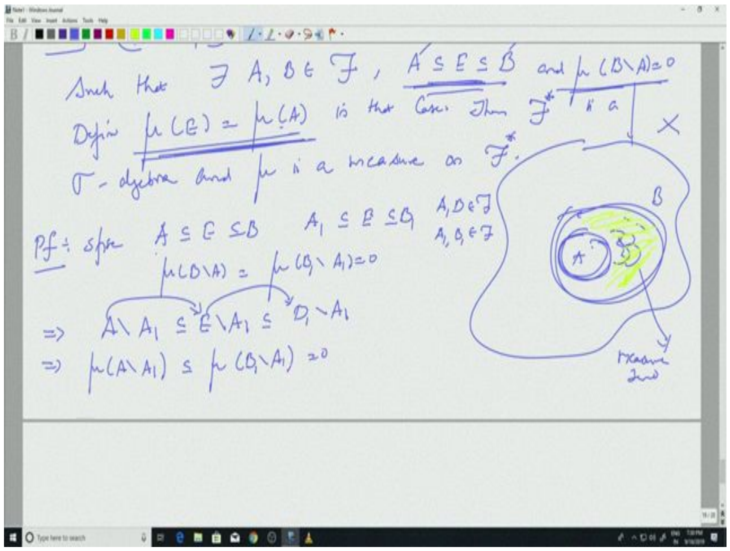
So that is the next step. Let us state, it as a theorem. So, theorem. So, I have a triple X, F and mu okay. Let F star be the collection of subsets of X, collection of subsets E of X okay, So let us write this clearly, so I am taking subsets E of X such that, there exists A and V in script F. So script F is the original sigma algebra we started with and A is contained in E, contained in B okay. And measure of B minus A is zero okay.

Define mu of E to be equal to mu of A in that case. Then F star is a sigma algebra, is the sigma algebra and mu is a measure on F star. So let us read it again. I am starting with the sigma algebra F and measure mu, so on a space X. And I am adding more sets, what kind of sets? I am taking any set E which has this property, right there are two things. So it is sandwiched between A and B such that B minus A has measure zero.

So, let us look at a situation where A and B are in pictures. So I have X. So let us say I have, I said E, A here, and something like B here. So let us draw it big. But measure of B minus A zero. So this portion has measure zero right? So this has measured zero. We are supposed to take sets E with this property. So I can take A and anything from here right? So I am putting subsets of sets of measure zero to the sigma algebra.

Then I get a bigger sigma algebra and I can extend the measure to success. Because that measure of A measure of B are same. So, anything inside has measure zero. So, any set if I add to this, it should not change the measure of the set. So, that is why those definition right. Whatever you are adding to E is actually a part of set of measure zero. So, it will not change the measure okay. So, we have to prove that this, this is well defined and this defines a measure. So let us prove that okay.

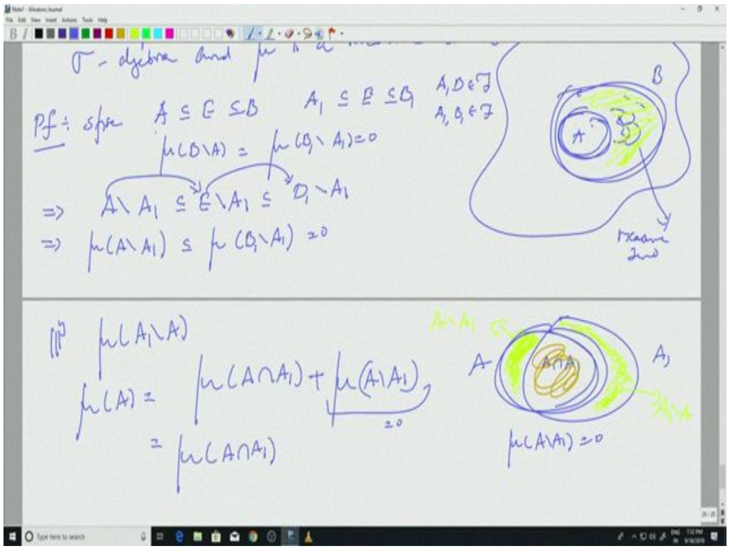
(Refer Slide Time: 17:39)



So first is well define us suppose. Well, what does well define us? Suppose I have two sets situation for the same set E okay, I have A1 contained in E contained in B1 that can happen, right? Where are the sets A and B? So A, B, A1 and B1 they are all measurable okay. And E may be sandwiched between A and B and similarly the same set E is sandwich between A1 and B1. Well not just this, we also know that measure of B minus A equal to measure of B1 minus A1 equal to zero right. So, we are taking sets E with that property.

Now, this gives me two definitions for measure of E, one is measure of A, mu of A and mu have A1. I want to know if they are same okay that is easy. So, we already have A minus A1 is contained in E minus A1. Well why is that? Because A is contained in E okay. Which is of course contained in B1 minus A1 right? Because E is contained in B1. So this tells me measure of A minus A1, remember all these are measurable sets because they are all from the sigma algebra. This is of course less than or equal to measure of B1 minus A1 by monotonicity, but this is zero.

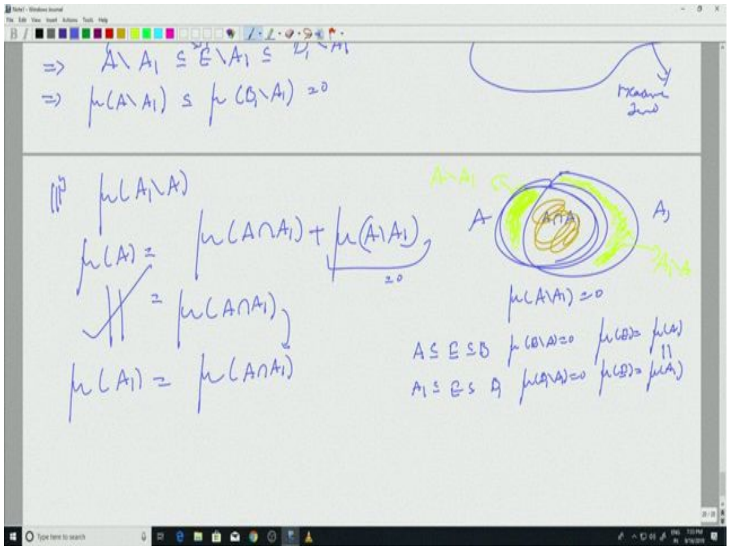
(Refer Slide Time: 19:28)



So, what did we prove? I have two sets A and A1. So, it may be A maybe like this, and let us say A1 is like this. So, this is A, this is A1, this is the intersection, A intersection A. What we proved was? A minus A1. So, A minus A1 would be this portion right. So, this portion is A minus A1 that has measured zero okay? You do the same trick to prove that.

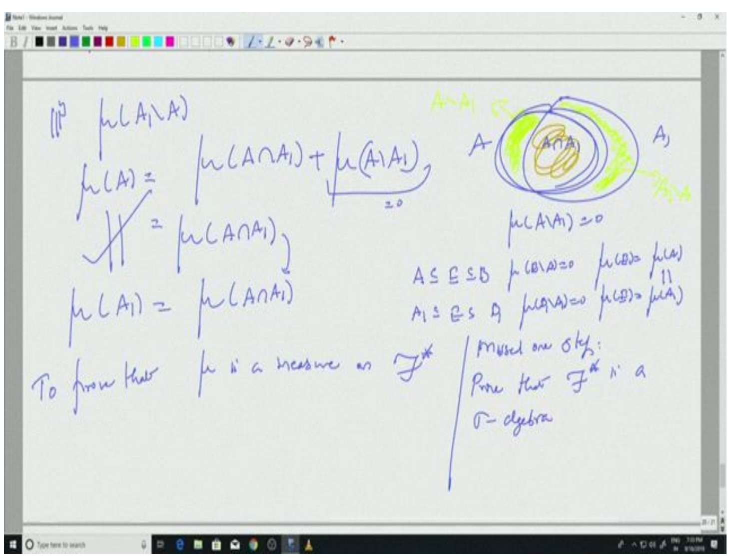
Well, which is this area? This is the area A1 minus A, this is A1minus A, use the same trick to prove that mu of A1 minus A. So, the measure of A, well what is the measure of A? So measure of A is simply, measure of A intersection A1. So, that is this area, this is this area plus measure of whatever is remaining which is A minus A1, right. But this is zero. So, this is simply mu of A intersection A1 okay.

(Refer Slide Time: 21:07)



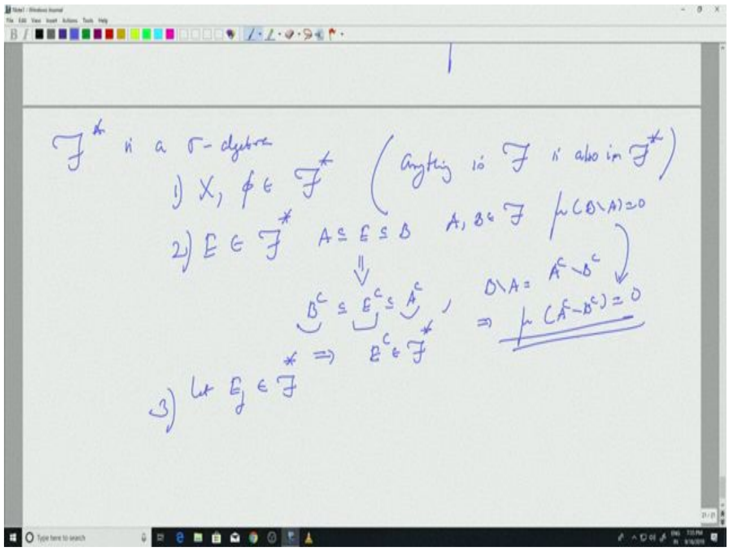
Similarly, the same argument works for measure of A1 as well right. So, measure of A1 will also be equal to measure of A intersection A. So, these two are equal, so, these two are equal. So, remember the definition for E, measure of E. So, I had A contained in E contained in B mu of B minus A is zero, mu of E was defined to be mu of A right. And I have A1 contained in E contained in B1, mu of B1 minus A1 equal to zero. And then mu of E will be, mu of A1. But these two are same, so these two are same. So, that is a well-defined property of mu. So, any such extension is well defined, we need to show that it is actually a measure.

(Refer Slide Time: 22:02)



So, to prove that mu is a measure on F star okay. So, I missed one step. So missed one step. So we need to prove that, prove that F star is a sigma algebra, F star is a sigma algebra okay. So we will start with the sigma algebra part and then so that is easy.

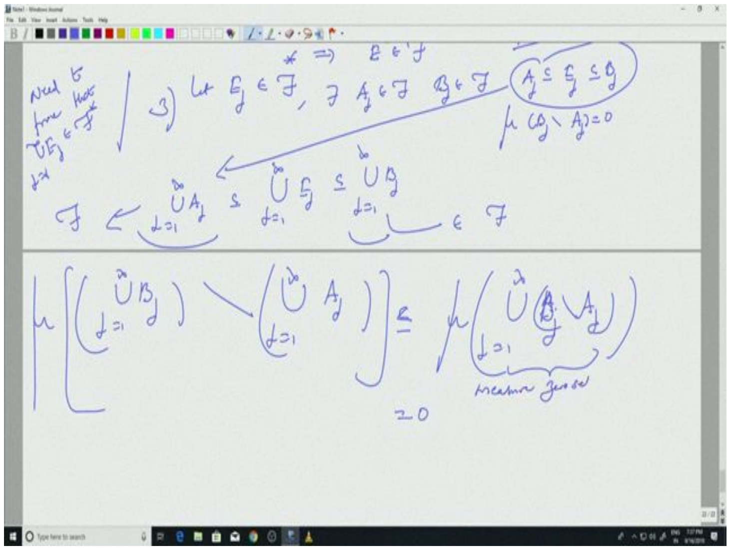
(Refer Slide Time: 22:46)

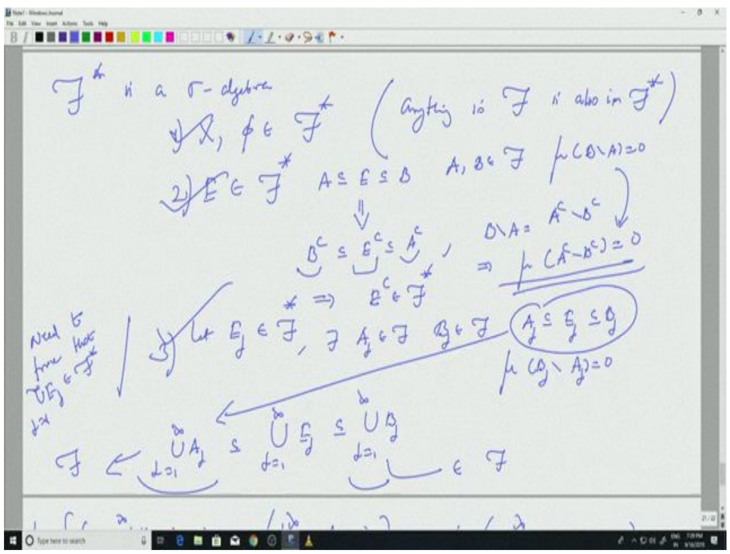


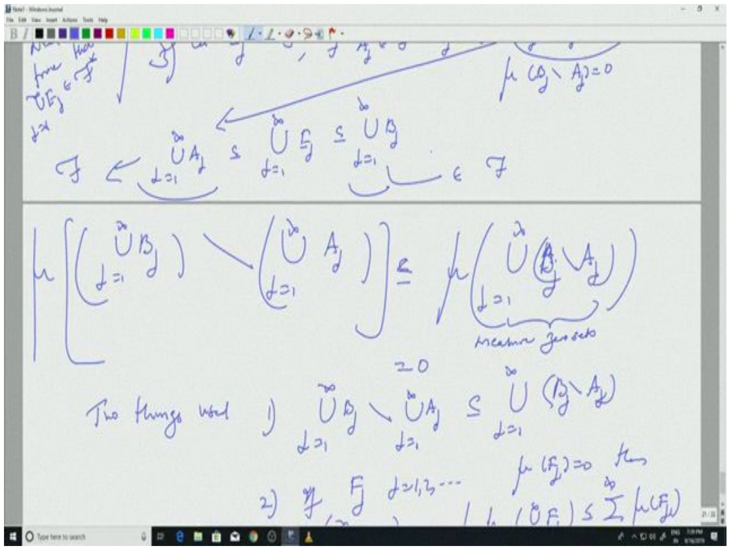
So F star is a sigma algebra, well why is that one? I have the whole space and this F star right. So recall that anything in F, anything in F is also in F star okay that is a trivial thing. So, if I have E in F star I need to prove E compliment is an F star, but E is an F star means, I have A contained in E contained in B, such that A and B are in script F and mu of B minus A is zero okay.

Now, A contained in E contained in B implies. B compliment is contained in E compliment contained in A compliment and B minus A. This is same as A compliment minus B compliment. So, this implies that measure of A complement minus B complement equal to zero because of this. So, the set E compliment is sandwiched between two sets in script F whose difference has measured zero. So, this tells me that E complement is in F star right that is how F star is defined? All those sets which are sandwiched between two sets whose difference has measured zero. Three, third property is the countable unit. So, let EJ belong to F star.

(Refer Slide Time: 24:40)







Well, if I take EJ belonging to F star, then I know that there exists AJ in script F, BJ in script F such that AJ is contained in EJ contained in BJ right there sandwich between two sets from script F, such  that measure of BJ minus AJ is equal to zero. So, from here we get that, what do we want to prove? we want to prove union. So we are trying to prove, need to prove that union EJ belong to script F start, right one to infinite.

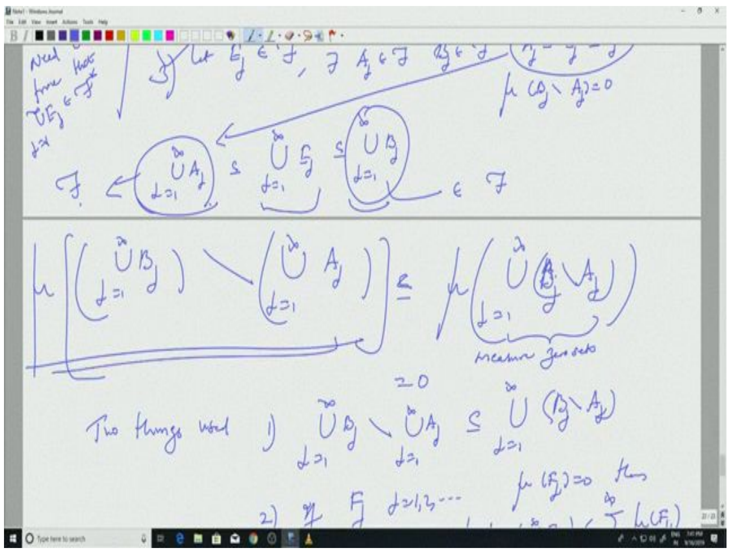
Well, so, from here we have union AJ J equal to one to infinity. So, of course contained in Union EJ, J equals one to infinity of course contained in union BJ, J equals one to infinity. I know that this endpoint is in script F, this is in script F. Because script F is a sigma algebra. What I need to prove is the middle one is in script F star. That means the difference of this and this has measured zero. So, let us look at that union of BJ. J equal to one to infinity minus union AJ J equals one to infinity, mu of that right. I want to say this is zero. So, this is nothing but this less than or equal to mu of union BJ minus AJ equal to 1 to infinity and equal to zero because these have measures zeroes right, these are measures zero sets measure zero sets.

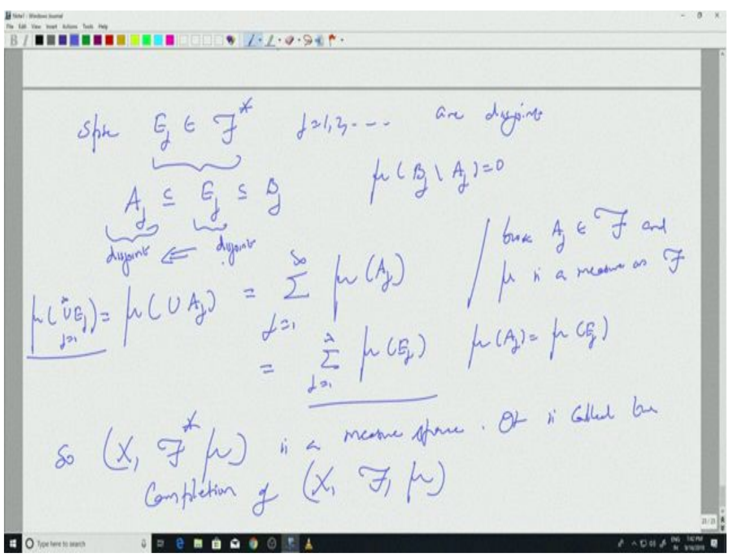
And if I take countable union of measures zero sets I will get measured zero. So, I use two things here, what are the two things? So, two steps or two identities used here, two things used here, things used, well what is one? First one is union BJ J equals one to infinity minus union AJ J equal to one to infinity is contained in Union BJ minus AJ J equal one to infinity.

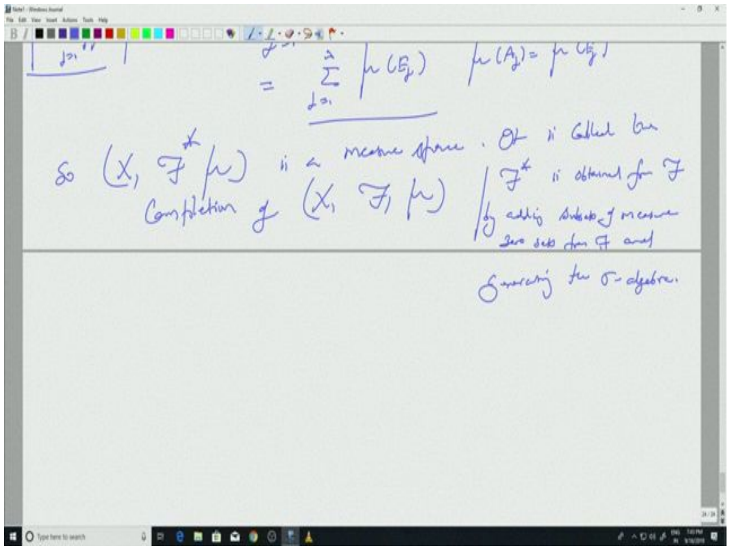
Two if I have sets FJ J equal to 1, 2, 3 etc. measure of FJ zero. Then union FJ also has measure zero. Well why is that? Because mu of union FJ, J equals one to infinity is less than or equal to J equals one to infinity, mu of FJ. Remember the monotonicity property sensitivity  property of the measure. So, all this is zero, so the sum is zero and so the union has measures zero.

Countable union of sets of measure zero is a set of measure zero. So, what we have proved is that script F is a sigma algebra, so, these three properties are fine. So, we need to show that measure mu is a, to prove that mu is a measure on F star right. Now, this is sigma algebra I want to prove that it is a measure okay.

(Refer Slide Time: 28:27)







So, suppose EJ are disjoint, suppose EJ, now they are in script F star remember not in script F, J equal to 1, 2, 3 etc. are disjoint okay. So, because of this EJ being in F star as earlier, we have AJ contained in EJ contained in BJ and mu of BJ minus AJ is zero okay, EJ are disjoint. So, that would imply that AJ are disjoint because AJs are contained in EJ. So, because of that, mu of union AJ will be summation mu of AJ right. Because AJ are in, because AJ belong to script F not script F star. I actually belongs to script F the smaller sigma algebra and mu is a measure and mu is a measure on script,

But this is same as summation J equal to one to infinity, mu of EJ right because mu of AJ is same as mu of EJ that is the definition. And this is of course, same as mu of union EJ right, J equals one to infinity. And because this was the smaller set contained in union EJ. So let us go back here, yeah. So here we had a set, here in script F and another set here in script F, whose difference had a measure zero, right.

So measure of the middle one was defined to be the measure of the smallest. So this tells me that if I start with EJ disjoined, then measure of the union is the sum of the measures. So, we have proved that X, F star mu is a measure space and it is called the completion of X, F mu, okay? When you complete the measure space, you are actually adding sets of measures zero. So, F star is obtained from F by adding subsets of measure zero sets from F and generating the sigma algebra okay.

So, we stop the session by recalling what we have just done. We started with dominated convergence theorem, which allowed us to interchange limits and integral in a rather general situation, where the sequence of measurable functions is bounded by a dominating function, which is an L1 that was enough to take, enough to interchange the integral and limit.

And then we introduce the property called almost everywhere something is to some property is supposed to hold almost everywhere, if it is true on a set, except on a set which has measures zero. So, there is negligible set where this is, this may not hold, but outside that it will hold. What we have just seen is we can complete the measure space by adding subsets of sets of measures zero. So, that we have more measurable sets and so, more measurable functions, in particular functions, defined almost everywhere can be made into, can be defined to be everywhere by appropriately defining the function on the set subsets of measure zero.

So we will see that this some measures which we will construct will be complete automatically, but there are cases especially when we look at product spaces and things like that later on, we will see that there are spaces which are not complete, it does not cause serious problem, we can always complete it by adding sets of subsets of measure zero. So, in the next class, we will we will look at some more abstract properties like this, and then start the construction of Lebesgue measure on RN.

So, that would be one of the, one of the concrete constructions you will see starting from open sets and how to define? How to define the Lebesgue measure of a set, of a class of sets, which we will call the Lebesgue sigma algebra, which will be bigger than the Boral sigma algebra which we have already introduce. And Lebesgue, the concept of Lebesgue measure will actually generalise the usual notions of length of the interval or area of the rectangle in R2, or the volume of cubes and the balls in R3 and so okay.