Outer Measure on Rn

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Lecture 13

Measure Theory

So, today will take the first significant step in defining the Lebesgue measure on Rn. We will do this by defining what is known as the outer measure on all subsets of Rn, so this would be a quantity defined on the power set of Rn. For each subset of Rn we define its outer measure to be a number using the cubes which cover the set. But, this is not going to be a countably additive measure will have to restrict to a sub-class of the power set which would be actually a sigma algebra, this will be called the Lebesgue sigma algebra.

On the Lebesgue sigma algebra, the restriction of the outer measure will be a countably additive measure, so this would be a genuine example of a measure, which is one of the most important missions you would come across. And the Lebesgue sigma algebra which I just mentioned will contain the borel sigma algebra which we had already defined earlier. So, we called that borel sigma algebra is the smallest sigma algebra containing all the open sets, so we have lots of sets there. The Lebesgue sigma algebra will be slightly bigger than the borel sigma algebra.

In fact, it is going to be the completion of the borel sigma algebra with respect to the Lebesgue measure which we would be defining using the outer measure. So, that is our goal in the next few lectures, we will take the first step by defining the outer measure today.

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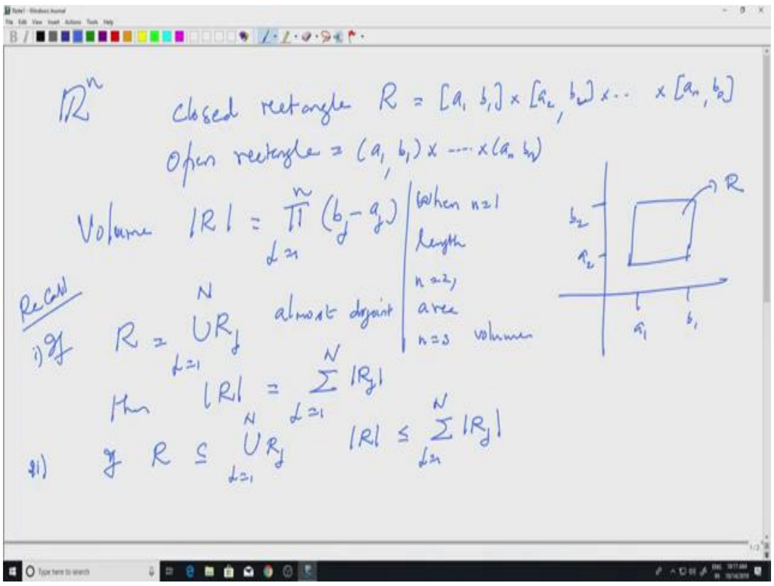

So, let us start. So, let us start by recalling rectangles, so we will be dealing with Rn, so recall that a closed rectangle was denoted by R, so this is a product of intervals so [a1, b1] cross [a2, b2] cross etc. etc. cross [an, bn]. So, product of intervals, so in R2, it would be like a1, b1 here, let us say a2, b2 here, we have the box here. So, this is a rectangle in R2.

Open rectangle would be a product of open intervals, open rectangle the product of open intervals, so we have open interval (a1, b1) dot-dot-dot an, bn so that is an open set, closed rectangle is a closed set and you may have some of them to be closed and open and so on. So, you may have rectangles of various types where certain parts were not involved. For such a rectangle we define the volume, so this is what we did in the last class. Volume of R is denoted by modulus of R, this is the product of the sides, length of the sides of the rectangle, j equal to one to n.

So, in the in dimension 1 when n equal to 1 this is the length, when n equal to 2 this is the area and when n equal to 3 we have the usual volume. So, that is what we call a volume of the rectangle and we looked at some elementary properties so recall that, if a rectangle R is the union of finitely many rectangles almost disjoint then the volumes add up. Then modulus of R is, summation j equal to 1 to N mod Rj, so this was one of the things we proved.

Second one is, sort of easy if R is content in union Rj, j equal to 1 to N, then modulus of R is less than or equal to summation j equal to 1 to N, modulus of Rj, because some of the Rj's may overlap and you may get more area so that is very easy to see.

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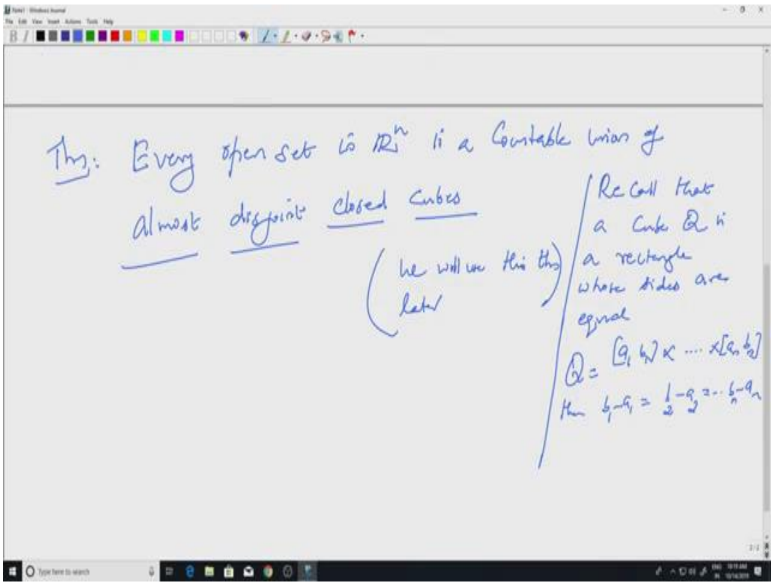

And we also proved errors of logical result a theorem we proved was, every open set in Rn is a countable union of almost disjoint closed cubes. So, what is a cube? So we call cube is, recall that a cube is a rectangle, a cube Q is a rectangle whose sides are equal. It means if Q is the product of [a1, b1] etc. etc. [an, bn] then b1 minus a1 equal to b2 minus a2 equal to bn minus an, all the sides are equal then we say Q is a cube.

So, this is something which we proved every open set in Rn is a countable union of almost disjoint a closed cube. So, this is something which we will use at some point of time, so will use it later, we will use this theorem later as of now we do not need this so will keep it as it is.

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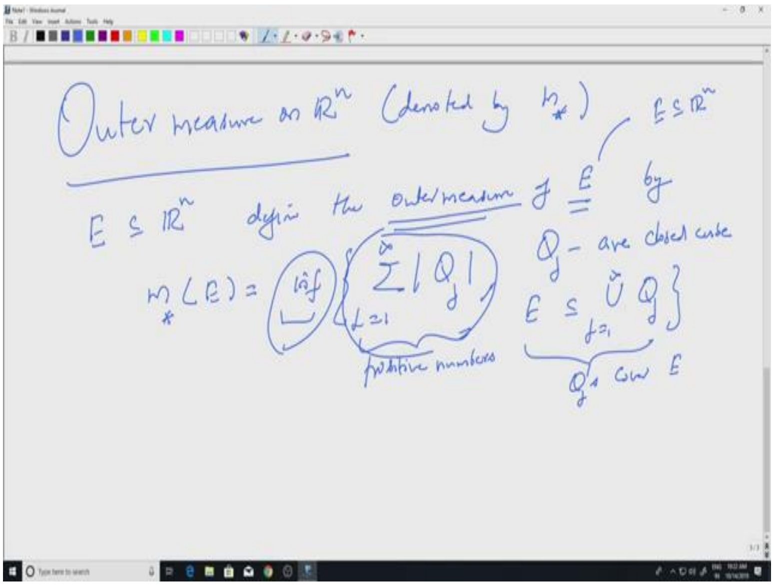

So, let us take the first significant step as, I was mentioning earlier we define outer measure. So, outer measure on Rn, this will be denoted by m star, so this is a standard notation we will use m star.

So, how is it defined? So you take any set E of Rn define the outer measure of E by well it is going to be denoted by m star E. So, it is m star of E equal to infimum of summation j equal to 1 to infinity, mod Qj, where Qj are closed cubes. So, recall that this is a closed cube is a closed rectangle whose sides are equal and what Qj is the (())(8:43) is the product of the length of the sides. It is not just arbitrary cubes, it should cover E, so E will be content in union Qj, j equal to 1 to infinity. So let us read it again.

It is the outer measure of the set E, so this is an arbitrary subset of Rn right E is an arbitrary subset of Rn. m star of E is first of all infimum of some quantities, so these are positive quantities, positive numbers. What are you doing? You are covering E so this is the simply means that Qj’s cover E and you take all such Qj, all such covers construct this number summation j equal to 1 to infinity mod Qj then you take the infimum. So, what are we doing let us look at some picture.

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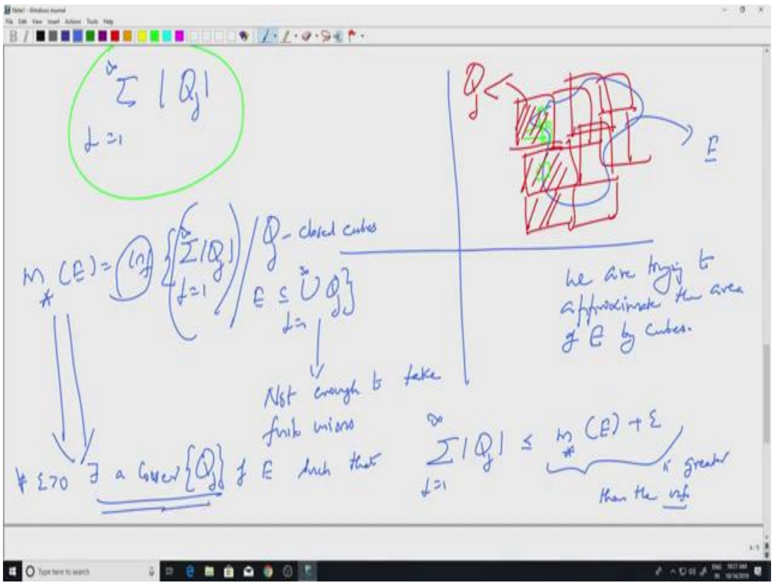

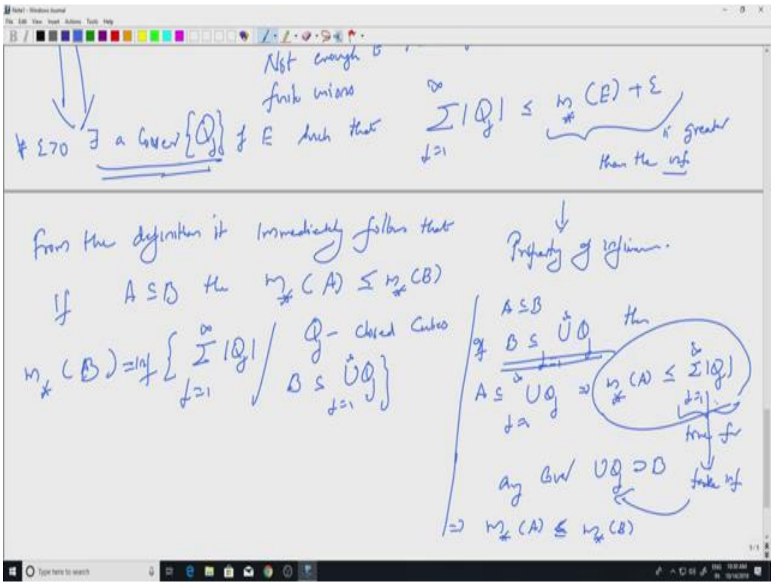

Suppose I have R2 and I have some set E like this, so let us say this is my set E, what do I do? I am covering E with cubes, so it is like this so you cover E with cubes. It may not be almost disjoint or it could be any any kind of cubes, closed cubes then you look at the union, look at the volume of these cubes, so it would be these things and then you add them. So, these are your Qj’s and then you add them so you look at mod Qj, so mod Qj is your volume of this cubes and then you sum them up j equal to 1 to infinity so you get a number for this.

Now, you take some other cube cover, so you may take some cube like this, some cube like this, some cube like this and so on. You may cover E with some other cubes in some other form and for those cubes you will get a similar quantity similar to this. And then, you take the infimum of all such things that will give you the outer measure of E. So look at the picture it will be clear to you that we are trying to approximate, so we are trying to approximate the area of E by cubes.

And we are hoping that this quantity which we have defined m star of E will give us the area or volume or length depending on which dimension you are, if the set E is nice enough. So, for example, if E is a cube to start with then you will get its volume as m star of E, so will see some examples to understand this, but before we go further there are things one should keep in mind so let us write down, m star of E again, m star of E is simply infimum of summation mod Qj, j equal to 1 to infinity. The Qj's are closed cubes and E is covered by the Qj's, j equal to 1 to infinity and you take infimum over all such covers.

So, there are things you should notice one is, it is not enough to take finite unions that will not give you a accountably additive measure, one has to take a finite and infinite covers like union Qj and then you take infimum over such things that is what we lead you to a countably additive measure. Then it is the infimum, so infimum has certain property in the sense that, if I add some epsilon to m star of E, so need to the definition implies this is something which we will use a very often. Definition implies that for every epsilon positive, there exist a cover Qj of E such that summation mod Qj, j equal to 1 to infinity is less than or equal to m star E plus epsilon.

Because m star of E plus epsilon is greater than the infimum. So, there is a number in the collection we are looking at. In this collection there is a number which would be less than or equal to m star of E plus epsilon which gives us this cover Qj, this is simply the property of infimum. So, let us this is the property of infimum. This is something which we will use all the time infimum, keep that in mind. From the definition it immediately follows, so from the definition it immediately follows that if A is content in B then m star of A is less than or equal to m star of B.

Well why is that? So recall that m star of A or let start with m star of B, so m star of B is you look at infimum of summation mod Qj, where Qj's are covers for B. So, Qj are closed cubes and B is content in union Qj, j equal to 1 to infinity, but A is content in B so if B is content in the union of Qj, j equal to 1 to infinity, then A is also content in the same Qj's, j equal to 1 to infinity. This would imply that m star of A is less than or equal to summation mod Qj, j equal to 1 to infinity, but, this is true for any cover union Qj containing B.

We started with cover containing B, cover of B and we have proved that m star of A is less than or equal to this quantity. So, I can take infimum over all such covers on the right hand side, take infimum over this collection, that will give me that m star of A is less than or equal to m star of B. So, as the set becomes bigger and bigger, the outer measure also becomes bigger.

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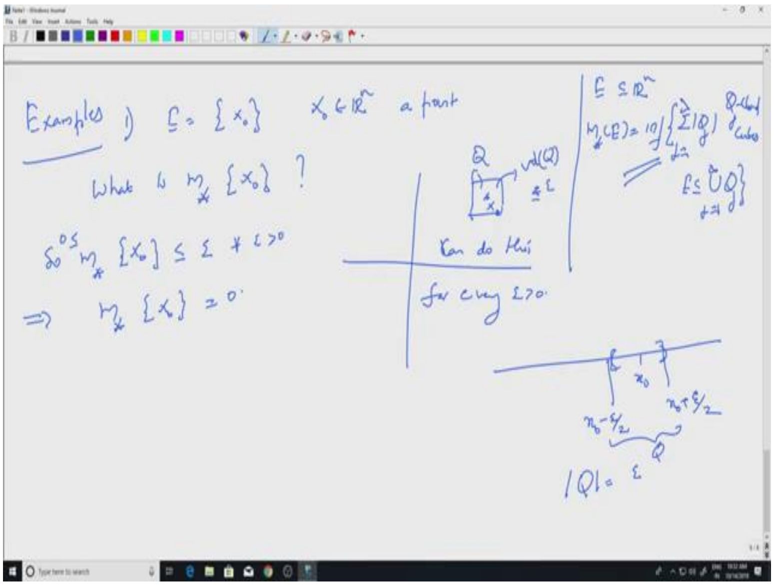

So, let us look at examples so will keep the definition of m star here, so if I take E is a subset of Rnn, m star of E is infimum of certain quantities.

So, infimum of summation mod Qj, j equal to 1 to infinity, Qj closed cubes and E is content in union Qj. So with this, let us look at examples, easy examples. So, take a point so take E to be a singleton some point, x not a point in Rn. What is the outer measure of E? So what is m star of E? The point does not have any volume, so it should be 0. So, let us see why it is? This is because so if I take any point X not in Rn , I can cover it with a cube of volume less than epsilon, so let us call this Q where volume of Q is equal to epsilon or less than or equal to epsilon.

This I can do for every can do this for every epsilon, every epsilon positive. So, for example on the real line we are looking at one point x naught and you can cover it with an interval. So, here the closed cube will be a closed interval, so let us say x naught minus epsilon by 2 to x naught plus epsilon by 2, this is my Q. What is the volume of the Q? This is simply epsilon and I can do this for every epsilon, so m star of a singleton is less than or equal to epsilon for every epsilon positive, but I know that m star is also positive because of the infimum of positive quantities.

So, it is greater than or equal to 0, so this will tell me that m star of a singleton is 0. So, points have outer measure, 0 which is what one should expect.

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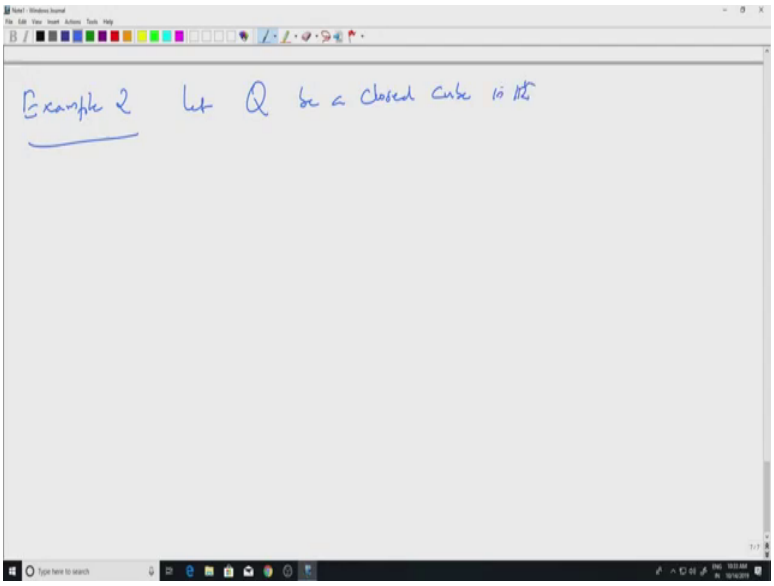

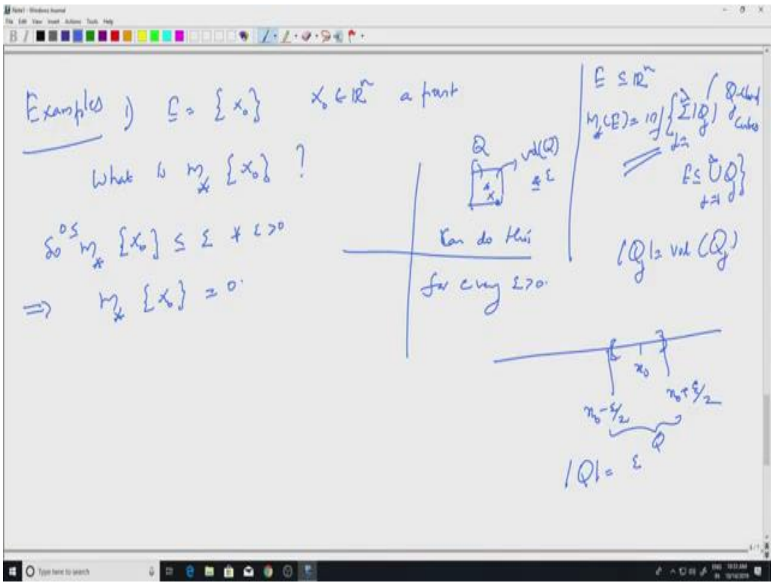

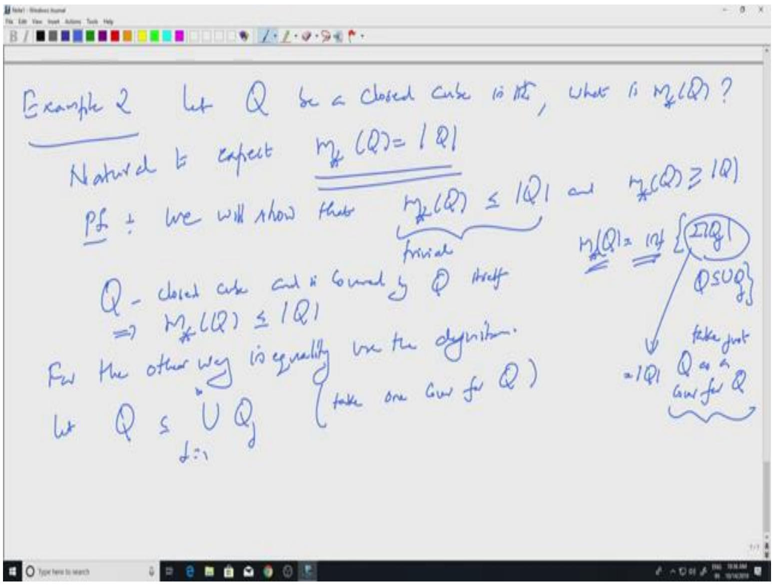

Let us look at the next example, Example 2- Let Q be a closed cube in Rn, so recall that we defined the outer measure of any set using the volume of the cubes. So, mod Qj is simply the volume of Qj, so one would want or that should be the measure of Q so if Q is a closed cube then, what is we want to find out what is m star of Q.

So, it is natural to expect m star of Q to be equal to mod Q, this is what we want for the basic objects like closed cubes, it should be the volume of Q that should be the outer measure, so let us prove this. So proof, we will prove this one that outer measure of closed cube is actually the volume of the cube, volume of the cube Q. So, recall that m star is the infimum of certain things, so we will prove that, we will show that m star of Q is less than or equal to mod Q and m star of Q is greater than or equal to mod Q. So, that will tell me that they are equal.

One of them should be trivial, this should be trivial, why? Because Q is a closed cube and is covered by Q itself, so recall that m star of Q is the infimum of various things. So, this is an infimum of summation mod Qj where Q is content in union Qj closed cubes and so on. So, I can take just Q as a cover for Q. So, this would be simply mod Q in this case and I am taking infimum of such things. So, I have mod Q as a number here and I am taking infimum of things since mod Q is here, m star of Q will be less than or equal to mod Q.

So, this implies m star of Q is less than or equal to mod Q. We want to show that other way inequality so that they are equal, so for that we use the definition. How do we use the definition? So let Q be content in union Qj, j equal to 1 to infinity so we are taking one cover for Q using closed cubes.

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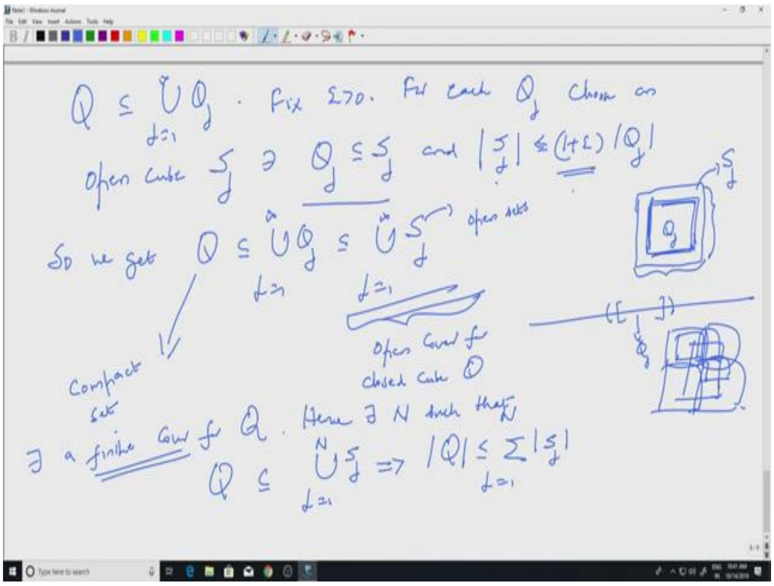

Let me write again, Q is content in union Qj, j equal to 1 to infinity for each. So, may be fix epsilon first, fix epsilon positive for each Qj choose an open cube Sj, so it should be an open cube and I write it again, choose an open cube Sj such that Qj is content in Sj and so there are two things so one is this.

And the volume of Sj remember volume of Sj is simply the product of the length of the sides is less than or equal to 1 plus epsilon times mod Qj. While this is possible, so if I have this as Qj closed, so I have the boundary as well, so this is my Qj. I take a small open cube which is bigger than this, so it is like this so that the product of the sides that is the volume here. So, this would be my Sj so that the volume is less than or equal to 1 plus epsilon mod Qj on the real line, what is happening? So, I have Qj here, so this is my Qj. I take a slightly bigger open interval so that this inequality is true, that is trivial so that can be done.

For each Qj we get this, so we get the closed cube Q is content in union j equal to 1 to infinity Qj that I know, then I have chosen Sj's, so this would be content in j equal to 1 to infinity Sj. But, remember this is an open set so this is an open cover for closed cube Q because these are all open sets, Sj are open cubes, so they are open sets. But, Q is compact so this is a closed cube so this is compact set and I have an open cover. So, there exist a finite cover for Q hence there exist N such that, so this is may be slightly bigger than the finite cover does not really matter.

Q is content in union j equal to 1 to capital N Sj, so I am still writing j equal to 1 to N. All of them may not be there but some them will form the finite cover. So I had the closed cube Q first, then I covered it with closed cubes Qj's, I made them bigger so that I have an open set and so on and I chose finitely many from there. Because of compactness I get a finite cover for Q, so this immediately implies or remember this was one of the exercises when you cover a rectangle with other rectangles, you have the volume inequality which is summation j equal to 1 to N, mod Sj.

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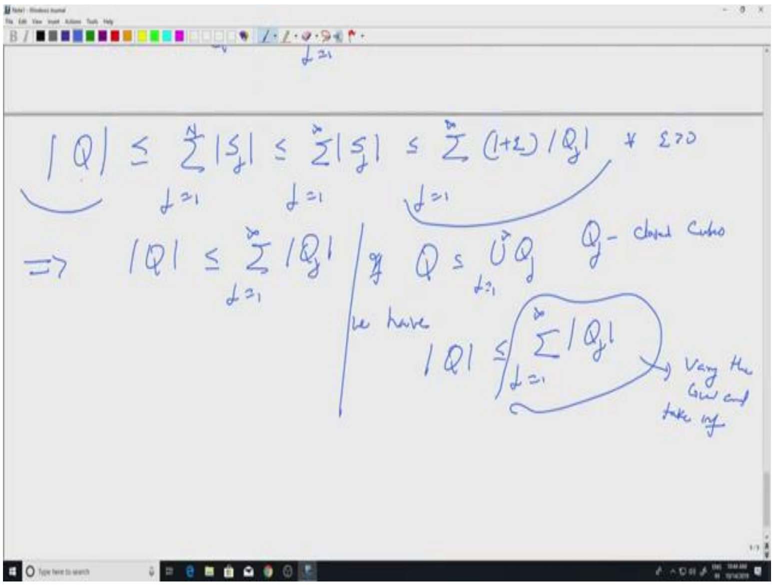

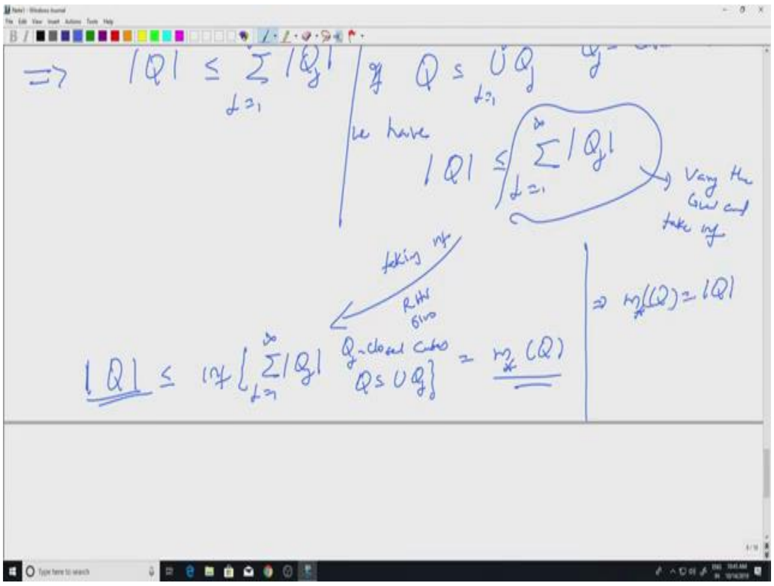

So, what did we prove? We proved that so let me keep that here so we have mod Q less than or equal to summation j equal to 1 to capital N mod Sj. Which is of course less than or equal to summation j equal to 1 to infinity mod Sj because I am adding positive quantities, but, recall that Sj's were open cubes chosen with some property, this is the property, so I replaced as mod Sj with 1 plus epsilon. So, this I can bound by j equal to 1 to infinity, 1 plus epsilon times mod Qj so, this is true for every epsilon. Now, there is no epsilon dependence so we will look at this part and this part for every, we started with an arbitrary cover.

We started with an arbitrary cover of Q and for a fixed epsilon, we have proved that we have this inequality. Q and Qj's are fixed now and this is true for every epsilon. Since it is true for every epsilon I can let epsilon go to 0, so this implies mod Q will be less than or equal to summation j equal to 1 to infinity mod Qj because it is true for every epsilon. So, what did we prove? So let us rewrite this, if Q is content in union Qj, j equal to one to infinity, Qj closed cubes. We have the volume of Q is less than or equal to summation j equal to 1 to infinity mod Qj, this is always true.

Now, the left hand side is a fix number mod Q, right hand side I can vary the cover and take infimum. Left hand side does not change because it is a fix number, so I can do this for all the covers Qj's and take the infimum. So, taking infimum on the right hand side gives mod Q is less than or equal to infimum of summation j equal to 1s to infinity mod Qj, Qj closed cubes and Q is content in union Qj that how we started with. So, we are taking infimum over all such collections which is this is the definition of m star of Q. So, let us recall the first inequality which was easy to prove showed us that, m star of Q is less than or equal to mod Q and then we showed that mod Q is less than or equal to m star of Q.

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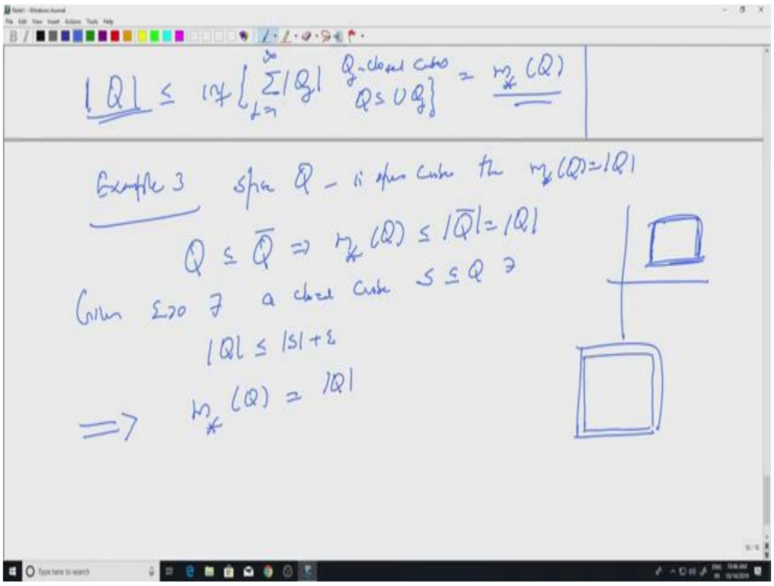

So, all this will imply that m star of Q is equal to mod Q. So, in this example we proved that, if I have a closed cube which volume is the outer measure, so one more example. Example 3- Suppose Q is an open cube then also m star Q is mod Q, that is easy because Q is content in Q closure, so open cube in without the boundary if I take the closure I will get the boundary and that is a closed cube and this would imply that m star of Q is less than or equal to mod Q bar which is mod Q. And for any open cube I can approximated by small closed cubes inside it.

So, given epsilon positive there exist a closed cube as content in Q such that, mod Q is less than or equal to mod S plus epsilon. Put together these two so I will let you complete this will imply that, m star of Q is equal to mod Q, so will stop for now. We have just seen, what is the outer measure? Which is defined for all subsets of Rn and we have seen that it coincides with the notion of volume when the set is a closed cube or an opened cube. Now, we will look at properties of the outer measure and then we will see that it turns out to be a countably additive measure, when we restrict it to an appropriate sigma algebra.