Measure Theory
Lebesgue Measure
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Lecture 17
Lebesgue Measure

So, in the last few sessions, we define the outer measure, we saw some properties of it and we defines what are Measurable Sets. So, these are called Lebesgue Measurable sets. The collection of Lebesgue Measurable sets is what we were looking at, we proved that is it a sigma algebra meaning the empty set and the whole space Rn are Lebesgue sets that is a trivial assertion.

We proved that it is closed under countable unions and compliments. Thus, L of Rn the Lebesgue sets form a sigma algebra and we restrict the outer measure which is defined on the power set of Rn to this sub collection which is the Lebesgue sets we will proof that it is actually a measure that means we have to proof that it is countably additive, restricted to Lebesgue sigma algebra.

So, we take countably many disjoint sets from the Lebesgue sigma algebra and proof that it is outer measure restricted to Lebesgue sigma algebra is actually additive.

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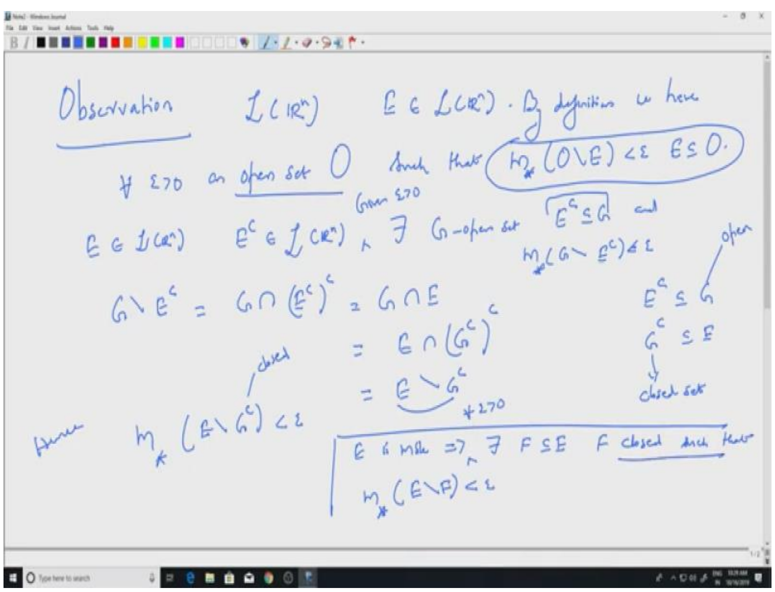

So let us start, so we will start with an observation, so observation this is a trivial observation which we will use soon. So let us, so we have this collection of sets L of Rn, so take some set measurable. So, this will belong to L of Rn, so by definition, by definition we have for every epsilon positive and open set O, O will depend on epsilon, such that. So, we have is an open set, such that the outer measure of O minus E is less than epsilon and of course E should be contained in O.

We have, we can approximate E nicely from above using open sets well this is same as saying we can approximate E from inside by close sets. So let us see why, so E belongs to L of Rn, L of Rn is a sigma algebra, this is the Lebesgue sigma algebra. So, E compliment will also be measurable and so by definition there exist. So, given epsilon positive, given epsilon positive there exist some open set. So, let us call that G an open set such that E compliment is contained in G and measure the outer measure of G minus E compliment this is less than epsilon.

Well what is G minus epsilon, so G minus E to the E compliment, this is same as G intersection E compliment, whole compliment, that is the definition of course, which is G intersection E because E compliment, compliment this. Which is equal to E intersection G compliment whole compliment. So, I am just writing trivial set theoretic operations here. So recall that E compliment is contained in G. So, E compliment is contained in G and so if I take G compliment that would be contained in E, and G compliment will be a close set. Because G is open, so G compliment is closed.

Well, now coming back to this equality, we have this is equal to E minus G compliment that is the definition and we know that the outer measure of the left hand side is less than epsilon, so we will get after measure of this is less than epsilon. So, M star of so let us say hence, M star of E minus G compliment is less than epsilon and this is a clos set.

So, that is just rewriting the definition of measurability. So, instead of this you can say that there a close set contained in E such that, so let me write this as a statement, so this same as, so what we want to say is E is measurable. Implies in fact it is equivalent to but let us not worry about it implies, there exist F contained in E, so you are approximating E form inside F closed such that the outer measure of E minus F, F is inside E is less than epsilon. So, well implies for every epsilon positive, there exist a F closed such that from inside you can approximate E by close sets. So, this a easy observation we will use that.

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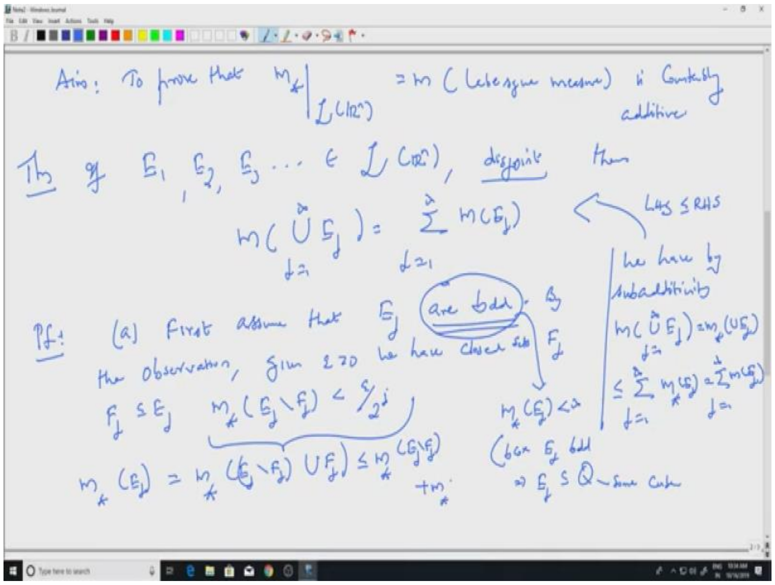

So next aim is to proof that outer measure restricted to. So, aim is to proof that, to proof that M star restricted to the Lebesgue sigma algebra. So, we denoted this by M remember that this is the Lebesgue measure is countably additive. So, actually is, (())(06:48) you when measure that the theorem we want to say, so let me write this as a theorem. This will complete the construction of the Lebesgue measure. So, if E1, E2, E2, etc, etc, they all belong to L of Rn that means they are all measurable disjoint.

Then the measure adds up, that is precisely what we want to proof. So, measure of union Ej j equal to 1 to infinity, this is summation j equal to 1 to infinity M of Ej. So, that is precisely the countable additivity of the Lebesgue Mashable. So proof, so we will use the earlier observation, so first so there are two cases so A first assume that, Ej the sets we are looking at, are bounded, are bounded Ej are bounded. So, union Ej need not be bounded, it is just the Ejs the compondance which are bounded.

Now given epsilon positive, well before that let us, let use sub additivity. So, in general forget about Ejs are bounded or not we have by sub additivity, by sub additivity M of union Ej. So, this is the measure of Ejs union Ej, well this by definition is M star, M star of the union Ej. This by sub additivity is less than or equal to summation j equal to 1 to infinity M star of Ej, which is of a summation j equal to 1 to infinity M of Ej because these are all measurable. So, each time when I write M, it is actually M star but the set would be a Lebesgue set, that is the only difference.

So we have one in equality that union left hand side less than or equal to the right hand side. So, in one way in equality is there. So, LHS is less or equal to the RHS, so that true, so we need to proof only the other way in equality to proof that they are same. So, let us start first assume that Ej are bounded, then by the earlier observation, by the earlier observation, given epsilon positive we have close sets. You can approximate the measurable sets by close sets we have close sets Fj well what are the properties of Fj, Fj are closed and is contained in Ej and the measure of the difference will be very small.

So, M star of Ej minus Fj is less than epsilon by 2 to the j. So, epsilon by 2 to the j you have seen earlier these kind of arguments because we have countable many sets, we start with epsilon by 2 to the j. What does that mean? So we are assuming that Ejs are bounded. Because Ej's are bounded M star of Ej will be finite. Because Ej are bounded because, Ej bounded implies, Ej is contained in some cube and that has finite measure by Monotonicity and so on, we have.

So bounded sets will have finite measure finite outer measure. So this tells me so from here we get M star of Ej well let us write this is as M star of Ej minus Fj union Fj, I can throw out Fj and then take the union. They are disjoint sets now, but by sub additivity I know this is less than or equal to M star of Ej minus Fj plus M star of Fj. So, this I know is small, so that I can use so what do I get I get that because they are finite all this cancellation taking something to the left right etc, will not cause any problem.

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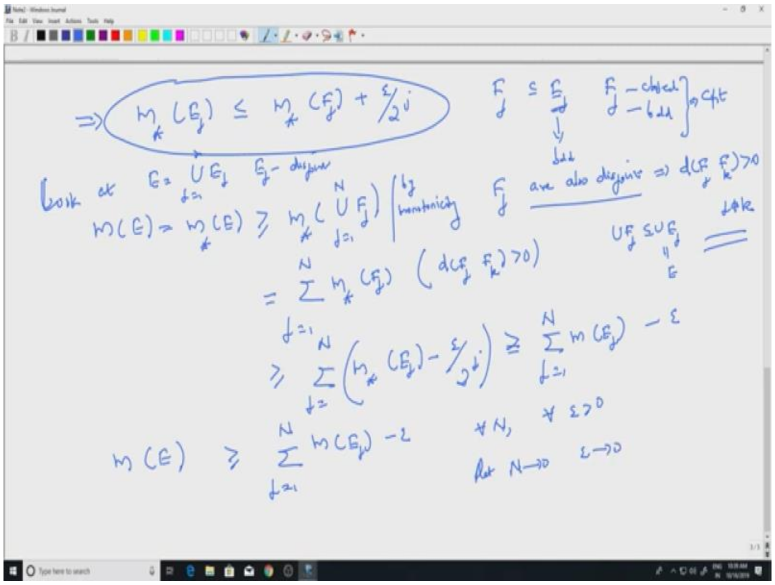

So, what we have is, so this implies that M star of Ej is less than or equal to M star of Fj plus epsilon by 2 to the j. So, remember that Fj is closed and is contained in Ej. So, Fj is close, so this is what we have got simply from the definition of the measurability.

So, Fj’s are close and Fj’s are bounded because Ej are bounded. So, this implies that Fj are compact. Not just that Fj are disjoint, Fj are also disjoint, why because Ej are disjoint and Fjs are contained in Ej. Ej are disjoint and so Fj are also disjoint. So, I have disjoint compact sets, so their distance will be positive. So, because of that if I take any two of them this distance will be positive if j is not equal to k.

And whenever you have distance positive you know that outer measure add sub. So, we will use that, so consider so let or look at M of E, which is what we want to compute M of E. Well of course this is by definition is M star of E, because is measurable E is the, well what is E is the countable union of Ej. So, maybe I did not mentioned that, so let me, did I mention, no I did not mention. So, let us, let E equal to union Ej j equal to 1 to infinity and we are trying to compute M of E, M of E so Ej is disjoint remember that as well and that is what gives Fj are also disjoint.

So M of E equals to M star of E by definition this is greater than or equal to M star of union Fj j equal to 1 to n, to some finite. Why is that? Because Fjs are contained in Ej, so union Fj will be contained in union Ej which is equal to E. So, by monotonicity, so this is simply by monotonicity, by monotonicity, we have this. But Fj are at a distance, so I can apply the property to get that this is actually summation j equal to 1 to n M star of Fj.

Because the distance between Fj and Fk are positive. So, we have that property for M star of Fj which of course is less than or equal to, greater than or equal to well what do we do? We use this inequality so I know that it is greater than or equal to summation j equal to 1 to N, M star of Ej minus epsilon by 2 to the j, each of them I can replace by whatever is in the bracket.

But Ej are measurable, so M star of Ej is simply M of Ej, so this I write as or greater than or equal to j equal to 1 to N, M of Ej and instead of taking epsilon by 2 the j I simply add up and get epsilon. So, epsilon is bigger than the sum of these things and so I am subtracting a bigger quantity so inequality goes in the same way. So what we have proved is, what we have proved is M of E, so that is the measure of E is greater than or equal summation j equal to 1 to N M of Ej minus epsilon.

So, this is true for every N and for every epsilon positive. So, I can let N go to infinity and so let N go to infinity and epsilon go to 0. Nothing will change because the inequality is absolute there are no constants depending on N and epsilon anywhere.

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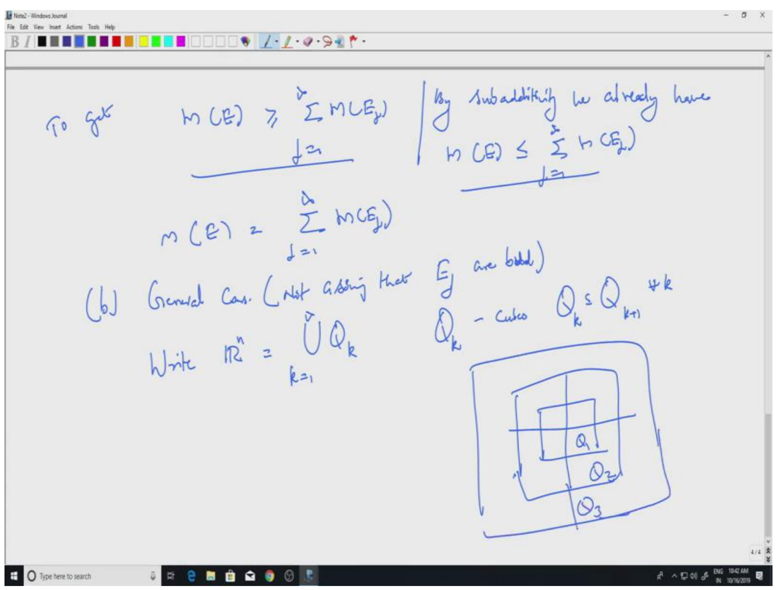

So to get, to get M of E greater than or equal to summation j equal to 1 to infinity M of Ej. So, we by sub additivity we already have the other inequality, sub additivity we already have M of E less than or equal to summation j equal to 1 to infinity M of Ej.

So, since we have this and this we get equality. So, M of E is nothing but summation j equal to 1 to infinity M of Ej. So that is countable additivity. But we assume something, so let us go back to the statement in the proof, first assume that Ej are bounded, so we assume that Ej are bounded we need to show that without that assumption we can still get countable additivity so that is the next case. So, case B general case, so that means not assuming that, not assuming that Ej are bounded.

So, what do we do? We make them bounded by intersecting with bounded sets. So, write Rn to be union Qk, k equal to 1 to infinity you can choose any appropriate bounded sets measurable bounded sets but that is not important here. Qk cubes and let us say Qk is contained in Qk plus 1 for every k, what am I doing?

So if this is your Rn you start with Q1 like this and then make something bigger than that that is your Q2 and then make something even bigger than that and so on and so forth. So until you fill up Rn so you will get Q3n things like that. So, this can be done.

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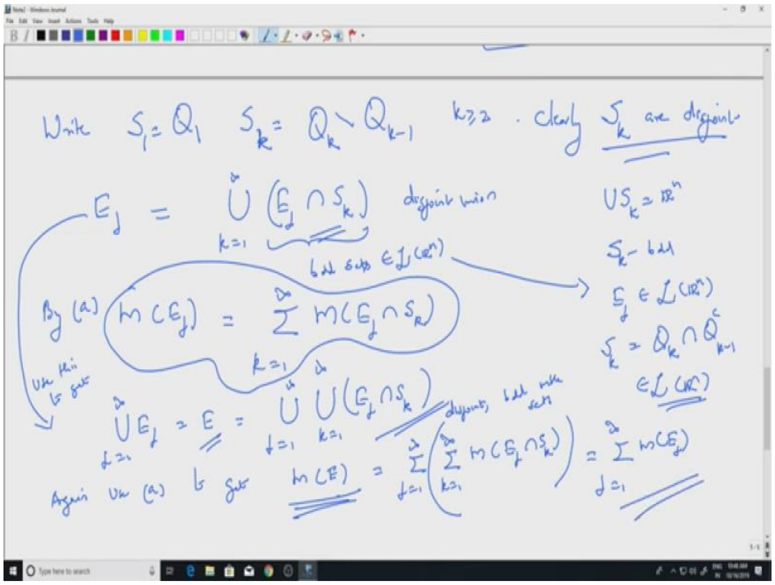

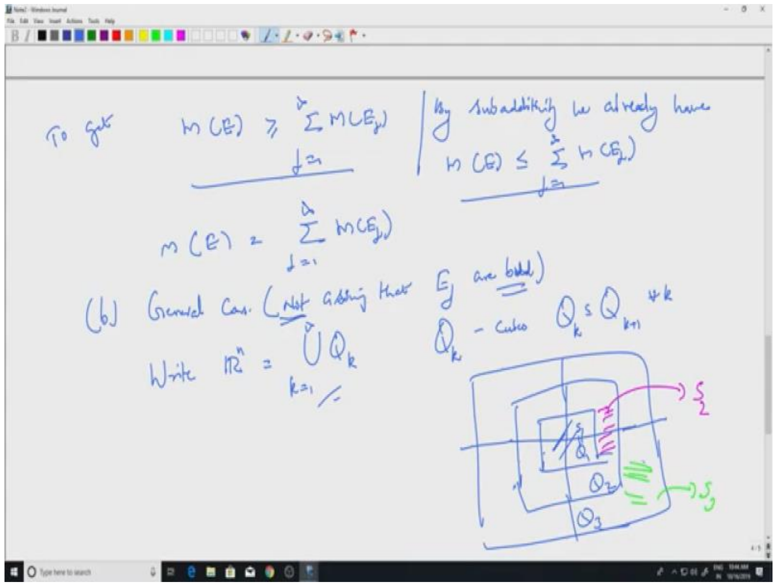

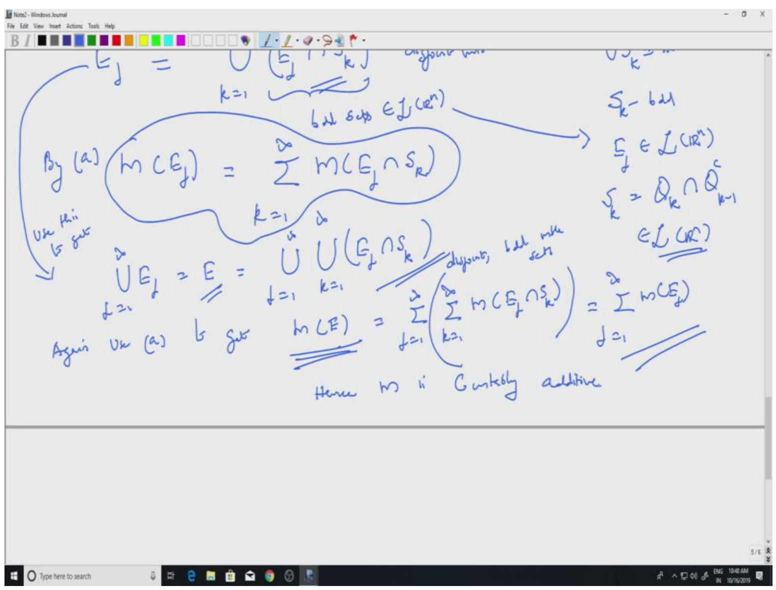

Well what is the advantage of writing Qk like that? That is because well write S1 equal to Q1 S2 equal to or Sk in general Sk equal to Qk minus Qk minus 1, k greater than or equal to 2. So, in the picture well I have Q1 is equal to S1 and whatever is here, whatever is here that is your Q2 or Q2 minus Q1 that is S2 and, whatever will be here that part is Q3 minus Q2. So, that is S3 and so on and so forth. So, well from the picture itself it is clear that they are disjoint.

So we will use that, so Sk is disjoint so clearly Sk are disjoint. So, then we can write each set Ej so remember now we are in the general case Ej are not assumed to be bounded, so we are not assuming that they are bounded. So, these are general measurable sets Ej but they are disjoint. So, I can write Ej as summation k equal to 1 to infinity Ej intersected with Sk and this would be a disjoint union. Well this is a disjoint union because Sk are disjoint and I am intersecting with Sk so whatever I get also will be disjoint union.

But why do I get equality that is because if I look at union Sk that is all of Rn, because we started with union Qk which is Rn and then we disjoint if add them. So, you get first you get this portion then you get this then you get the green portion and so on so you get Rn by taking the union. So, this is a disjoint union well it is not just disjoint Sk are bounded. So, Sk are bounded so these are bounded sets and measurable.

Well why are they measurable? Because Ej is measurable, so let us see why, Ej is measurable and Sk what is Sk? Sk is Qk intersected with Qk minus 1 compliment and so that is also measurable because each of them is measurable. Qk is a cube, so it is measurable and Qk minus 1 compliment is measurable because it is a compliment of a measurable set and this is a sigma algebra.

And when you intersect two measurable sets you will get a measurable set because you are still in the sigma algebra. So, these are bounded sets and they are inside Lebesgue sigma algebra. So, by case A, so by A well what is A? If you assume that you have a countable disjoint union so assume that you have bounded sets disjoint then you have proved that they add up. The measure adds up.

So, we can apply this case to whatever we are doing now. So, we have written Ej as union of bounded disjoint sets. So, by that case m of Ej that is the measure of Ej Lebesgue measure of Ej is equal to summation k equal to 1 to infinity measure of Ej intersection Sk. That is what we have. Now using this so use this to get union Ej. So, remember this is my E, I can write E also in the same manner well I know how to intersect Ejs with Sk and get Ej and then take another union.

So, j equal to 1 to infinity and union k equal to 1 to infinity Ej intersection Sk that is also fine you will get E and this are all again bounded measurable sets. So, again use A, again use A to get measure of E, now instead of Ej I take measure of E, I am writing E as union of bounded measurable sets, disjoint so this is still a disjoint bounded measurable sets.

So, measure of E just by the case we have proved is the sum of them measure of the sum of components. So, k equal to 1 to infinity measure of Ej intersection Sk. We just proved this. But now you can apply whatever we have just seen that is m of Ej is the sum with respect to k. So, if I look at this portion this is simply m of Vj. So, what we have just proved is summation j equal to 1 to infinity m of Ej which is the countable additivity. So, m of E equal to m of Ej, so that finishes the proof. Hence, Lebesgue measure is, so hence m is countably additive.

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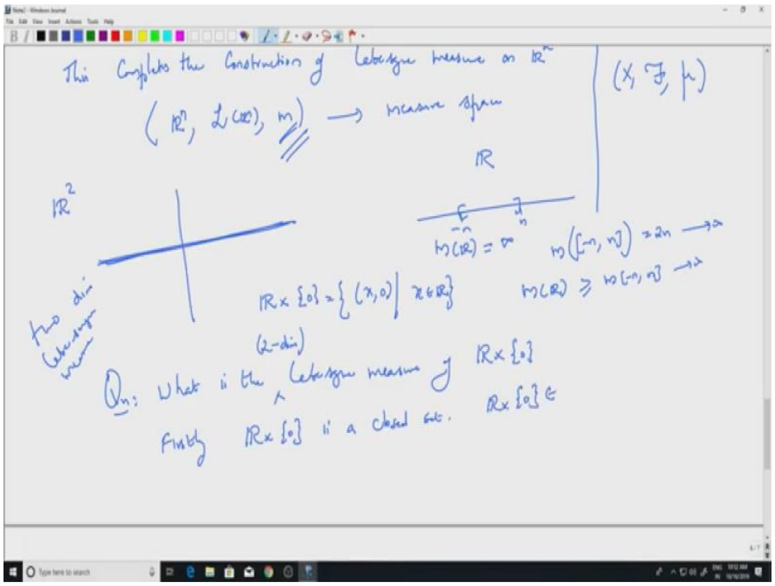

So what, so this sort of completes the conception of, so let me let me state this, this completes the construction of Lebesgue measure, the construction of Lebesgue measure, Lebesgue measure on Rn. So, what we have is if you look at the triple Rn the Lebesgue sigma algebra and the measure M. So, this is my triple this is a measure space. So, if you recall we started with abstract measure theory where we had a space X, we had a sigma algebra of subsets of X and a measure my.

So this was the triple we had, this is the measure space and we had various theorems all those theorems will be applicable, so all the abstract measure theory theorems will be applicable for this triple aspect. So, let us use that compute the Lebesgue measure of something. So, there are several things to be done we will look at finer properties and all that later on.

But let us ask some simple questions and then see how the theorems we proved in the abstract settings can be used to prove theorems in this concrete case. So, let us look at R2 as a simple example R2 look at the real line like this

We also have the real line so that is dimension 1. What is the measure of the real line in 1 in dimension 1 when I took at the Lebesgue measure of the real line I will get infinity? Because you can take the intervals minus n to n that is those are the closed cubes in R1 and measure of those intervals minus n to n, this is 2n and that goes to infinity and measure of the real line is of course greater than or equal to measure of these intervals by monotonicity and this goes to infinity.

So, it is an infinite measure space but less you get R2. Now I am looking at 2 dimensional Lebesgue measure 2 dimensional Lebesgue measure and look at the real line here, well how do I see the real line? So the real line will be identified with R cross 0. So, this is all those points x, 0 where x belongs to R, question so question is, what is the Lebesgue measure of R cross 0? So when I say Lebesgue measure this is the 2 dimensional Lebesgue measure, I am looking at Lebesgue measure in R2 in R there is a Lebesgue measure in R2 there is a Lebesgue measure in all Rn we have constructed the Lebesgue measure.

Well how will I compute this? First of all I need to know if it is measurable. So, firstly R cross 0 if I look at this is a closed set, it is closed set. So, R cross 0 is measurable. All closed sets are measurable, all open sets are measurable, all closed sets are measurable.

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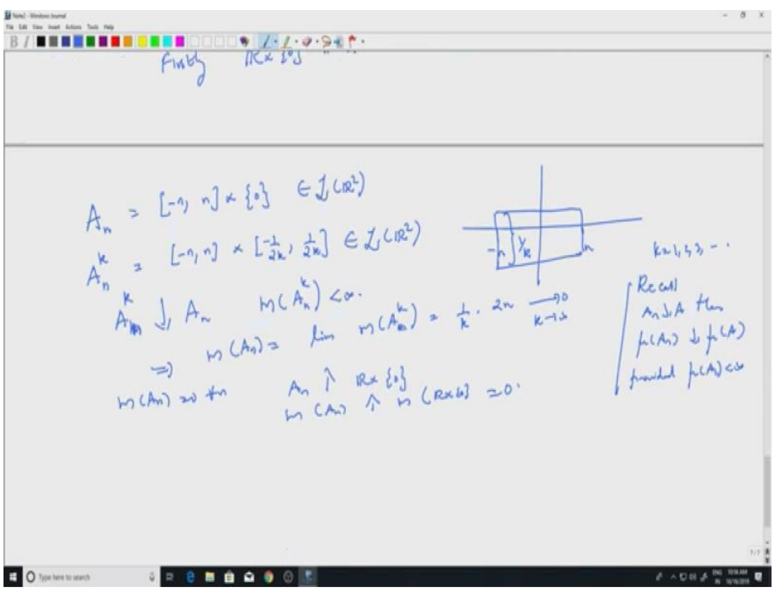

Now I want to compute the 2 dimensional Lebesgue measure of the real line. Well what we do is, we look at whatever we did earlier, so look at interval from minus n to n inside R2. So, let us call that An the set An, An is minus n to n cross 0. This is a cube with side length 0, but let us not worry about too much about it.

This is also this is a closed set so this belongs to L of R2. How do I compute its measure? Well that is easy, so what you do is you can look at a small rectangle like this. So, let us say the side length is 1 by k, k equal to 1, 2, 3, etc as, so that said so we have to denote that set by something. So, let us say A and k, A and k is of course in the x axis it is minus n to n and in the y axis I take the interval minus 1 by 2 k to 1 by 2 k. I can take anything which is close to 0 it does not really matter but let us take this.

What is the, so this is also measurable because it is a closed set. Well what is the advantage? I know that A and k as sets will converge to. So, it actually converges, so these are decreasing sets when k becomes bigger and bigger this becomes smaller n k decreases to An and measure of A and k they are all finite because they are cubes. So, we can apply the theorem, so recall the theorem, recall An decreases to A, then mu of An will decrease to mu of A. Whenever you have a measure this happens provided one of them is finite, provide mu of A1 is finite.

So, apply that this would imply that M of An equal to limit of M of Ank. But M of Ank I know because it is the cube, so it is the area, area is simply 1 by k times 2M and that goes to 0 as k goes to infinity. So, M of An is 0 for every n and Ans increase to R cross 0. So, we apply the thereon again. So, M of An will also increase to M of R cross 0 and so we get this as 0.

So, R cross 0 has measure 0 in R2. So this so let us stop here, we have, we have completed the construction of the Lebesgue measure and I have just shown you an example to compute the Lebesgue measure of the real line in R2. You can look at other lines for example the y axis or some curves some other lines all those will have measure 0 because there is no area. So such things you should compute to get a hang of Lebesgue measure in higher dimensions.