Measure Theory

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Lecture 21

Measurable Functions

So, we have seen invariants properties of the Lebesgue measure, we have seen finer properties of the measurable sets, we have proved that the Lebesgue sigma algebra is the completion of the Borel sigma algebra with respect to the Lebesgue measure. Now, we will continue with the measurable functions and see some more finer property.

So, recall that we started the course with abstract integration of abstract measure theory and so on, where we had defined measurability of functions and we looked at integrals and so on. So, we will make some of those results finer using the extra structure on R n. So, that is what we will do first, and then we will compare Riemann integrals with Lebesgue integrals. Let us start.

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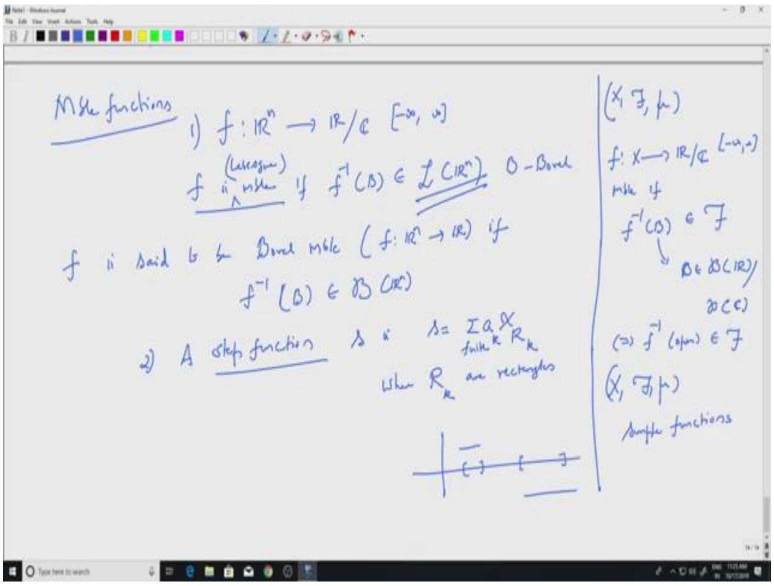

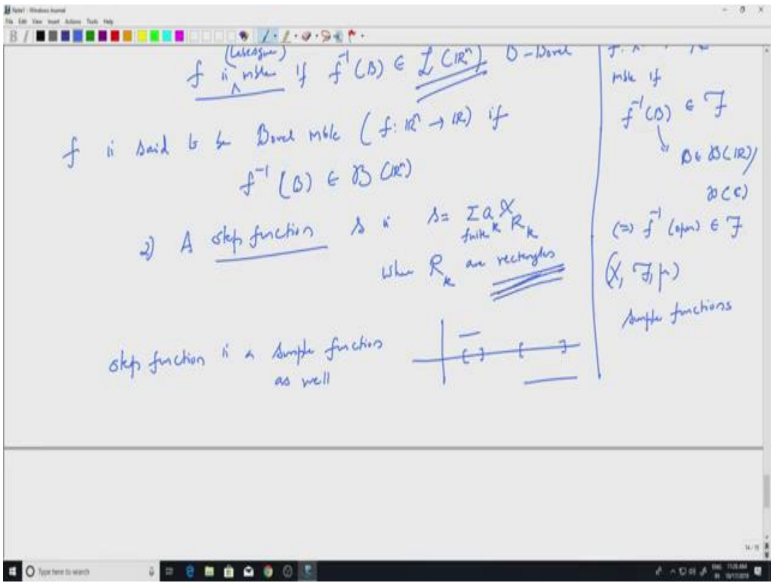

So, measurable functions, so we know what they are. So, let us recall that, so on the side. So, if I had a triple X, F, mew, this was the abstract setting. X was a space, F was a sigma algebra and mew was a measure. We said real valued function or a complex valued function or sometimes it was minus infinity to infinity and so on. This is measurable if f inverse of a Borel set was in the sigma algebra we are interested in.

So, this was a Borel set. So, B belong to Borel sigma algebra of R or Borel sigma algebra of C and so on, which was equivalent to saying f inverse of an open set is, open set belong to the sigma algebra that we will do. So, recall that and then that is the definition of measurable functions now, so if I have a function f taking values in the real line, the complex plane or it can remember it can take minus infinity and plus infinity and so on, that is allowed.

Appropriately we change the sigma algebra then f is measurable, f is measurable if f inverse of a Borel set, f inverse of a Borel set belong to the Lebesgue sigma algebra of R n. So, this is the one you should keep in mind, because the right hand side has a Borel sigma algebra B Borel. So, whenever we pull back a Borel set we should get a Lebesgue set. So, that is what we mean by a measurable function.

If we put a restrictive condition that f inverse B actually belongs to the Borel sigma algebra instead of the Lebesgue sigma algebra, then we say f is Borel measurable. So, f is said to be Borel measurable to again have f is defined on R n, let us say to R. If f inverse of a Borel set if it was of a Borel set B belongs to the Borel sigma algebra. But here it is a when we say measurable, so you can say Lebesgue measurable if you like, but I will simply say measurable.

So, whenever we say miserable, the underlined sigma algebra is the Lebesgue sigma algebra, alright. So, that is one, a step function, so this is again using the structure of R n. In the general settings, when we looked at the triple X, F, mew, we had simple functions. We had simple functions, which makes sense here as well whether it is on R n or capital X does not matter. We have simple functions always.

But step functions use the structure of R n just like on the real line, we know what is the functions are, these are linear combination of indicator functions of intervals. So, intervals will be replaced by cubes. So, a step function S is a measurable function, well it will be always miserable because we are looking at the cubes, S is a function which is given by a linear combination of finite linear combination of rectangles or cubes to finite where r k are rectangles.

So, this is of course, measurable, r k are rectangles. So, in the real line this would be a usual measurable function, usual step function. So, if I look at the real line, a rectangle is a is an interval. So, you have one interval, you have you may have a bigger interval and so on. You fix a constant here, you fix another constant here that is your step function and this is simply the generalization to R n but that uses rectangles. So, this define only on R. So, these are of course simple function.

A step function, so step function is a simple function as well. Step function is a simple function as well. But the converse is not true, because simple functions can be linear combination of arbitrary measurable sets, indicator of arbitrary measurable sets. The measurable sets need not be rectangles. When they are all rectangles, we say it is a step function. So, that is a smaller collection of measurable functions. So, this is one of the finer properties of measurable functions on R n because we have extra structure.

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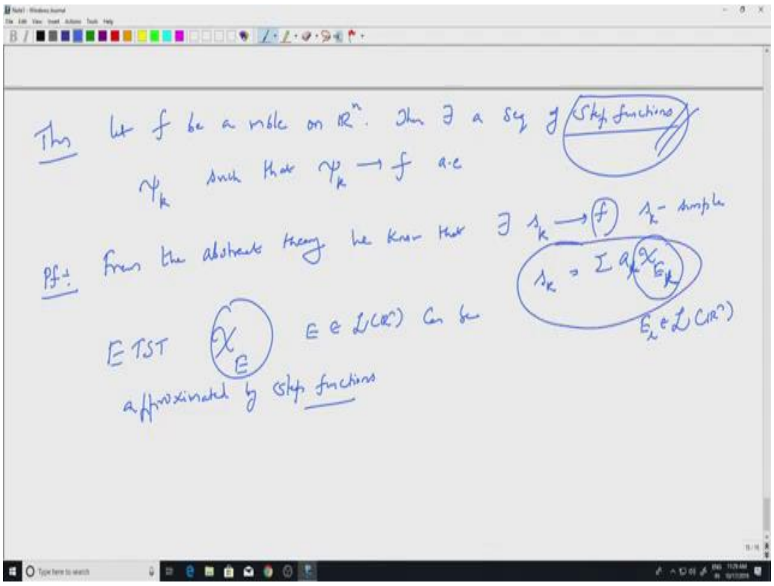

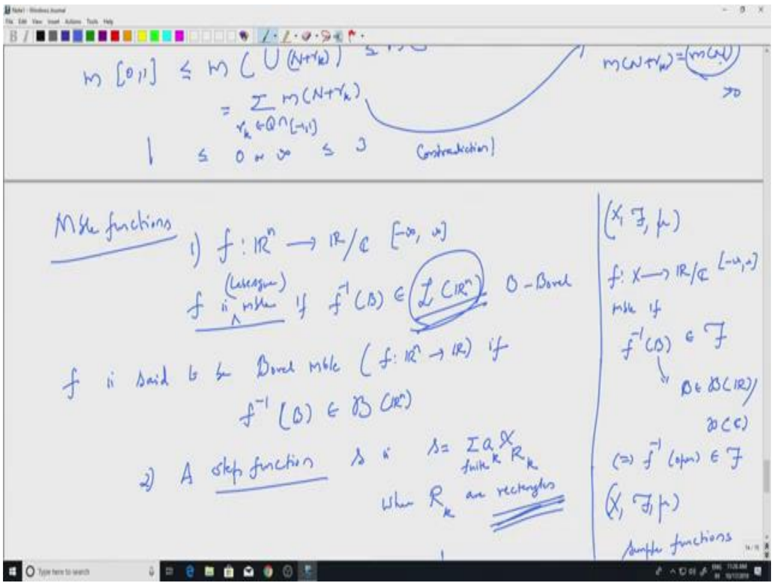

So, let f be a miserable function, so we call that whenever we say measurable, we mean Lebesgue measurable and when you pull back Borel sets you will have you will land in Lebesgue sigma algebra. So, that is what is meant by a measurable function, measurable on R n. It can take complex or real values, then there exists a sequence of step functions. We will call that psi k such that psi k converges to f, almost everywhere.

So, remember we had some in the general setting in the abstract settings. We had simple functions converging to f almost everywhere. Those simple functions can be modified to get step functions right because this is the structure of R n in which we are using. So, proof of this is not very difficult. So, from the abstract theory, from the abstract theory, we know that there exists simple functions there exist s k converging to f, s k simple, this we know.

Each s k is a linear combination of measurable sets. Well, maybe use L here this 2, where E L is a Lebesgue set. So, to show that any measurable can be approximated by step functions, it is enough to locate indicator of a measurable set and prove that it happens for such a set. So, enough to show that indicator of E, where E is an arbitrary measurable set can be approximated by approximated by step functions.

So, this is where we will use the finer properties of Lebesgue sets, which we had proved earlier. So, if you can approximate indicator function of an arbitrary measurable set by step functions, then using the general theorem we can prove this for any arbitrary measurable function, so that part I will leave it to you.

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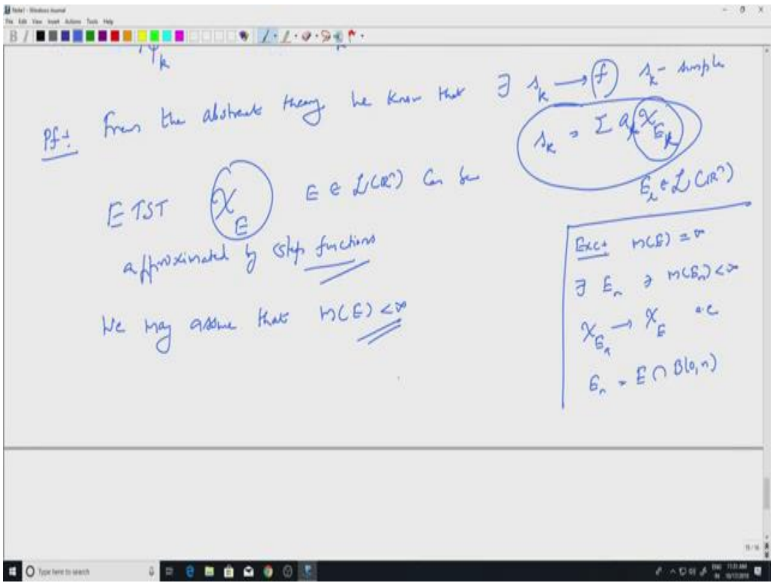

So, we will just prove that any set E with any if I take an arbitrary E in L of R n, chi E can be approximated by step functions. Well how do we do this? We use the, we use one of the results, so let me recall those result recall that there exist. So, we may assume, before we go further, we may assume that measure of E is finite. So that is a that is easy exercise let me. Suppose measure of E is finite, infinite.

Then I can cut it down by balls. so there exist E n such that measure of E n is finite and chi E n will converge to chi E almost everywhere. What do you do? You simply take E n to be E intersected with ball of radius n. So then it becomes bounded and it has finite measure. So, I can assume measure of E to be finite because that is enough because of the exercise. Now, once measure of is E finite, we know that it can be approximated by finitely many cubes.

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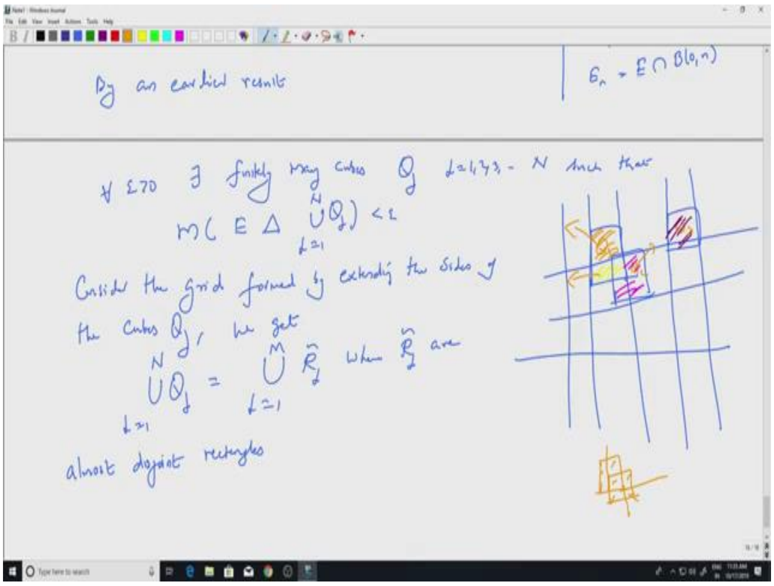

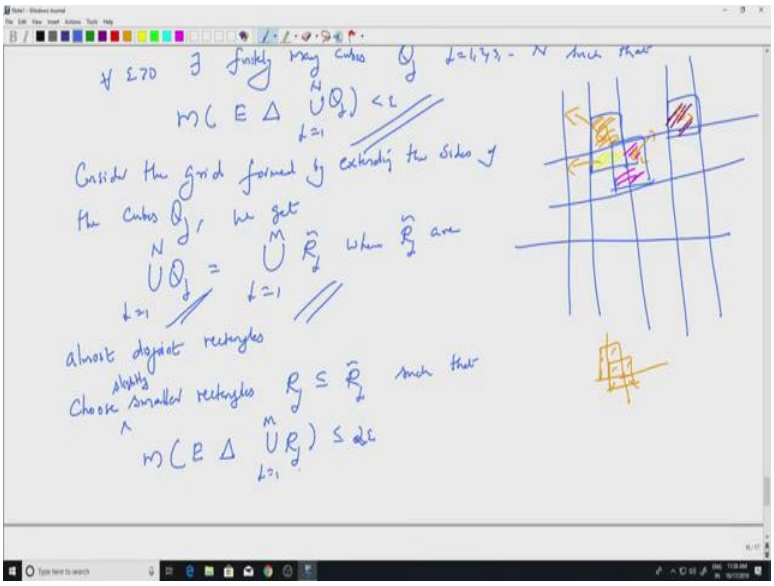

So, by an earlier result with an earlier result where an earlier result for every epsilon positive that exist finitely many cubes Q j. So, let us say j equal to 1, 2, 3 et cetera, capital N such that E can be approximated by these cubes which is a same as saying the measure of E delta the symmetric difference, union j equal to 1 to N Q j is less than epsilon. So, this we had proved this was one of the finer properties of the measurable sets.

We had proved if the measure of E is finite. So, these cubes, well, so let us say these cubes are like this, we do not how they look like so some cubes. So, this maybe one may be here and so on. So, this is how the picture looks like in let us say R 2. What you do is you extend the sides of the cubes, like this. So, you get a grid on R n, where we had the, so the cubes are here. So, let us say this is Q 1, this is Q 2 and this is Q 3 and so on.

This we have these cubes you form a grid by extending the sides. So, consider the grid form by extending the sides of the cubes to j. So, then if you look at these grids, you see that you can you get one rectangle here, you get another rectangle here and so on. Here another rectangle, here another rectangle and such here another rectangle and so on. So, each of these cubes can be written as union of rectangles and they are almost disjoint.

So, if you look at the picture this will be very clear, but let me write it down, we get that the union of Q j, j equal to one to N equal to union of j equal to 1, 2 some other number M, but finite R j tilde where R j tilde are almost disjoint rectangles. So, the R j tildes are here. So, this would be one of them, this would be another one, this is another one and so on. So, if you have cubes and you, if the cubes are already, almost disjoint than the extending the grid does not divide the cubes again.

But the cubes may be overlapping. So, if I have 2 cubes like this and if I extend the grid, I am going to get something like this. So, that will give me this rectangle, this rectangle, this one, this one, this one, this one and so on. So, several of these rectangles, but almost disjoint and the original cubes are union of almost disjoint rectangles which we have just construct. Choose smaller rectangles, smaller rectangles R j contained in R j tilde, so when I say smaller, I do not mean very small.

So, this would be let us say slightly smaller, slightly smaller rectangles R j contained in R j tilde such that the measure of E symmetric difference between R j, j equal to 1 to M, this is result 2 epsilon that is possible. Because the union Q j is same as union R j tilde and union Q j give me an epsilon difference at the most. So, if I shrink the Q js or the R j tildes, then I will get R j such that maximum difference between them as of measure less than 2 epsilon that is possible.

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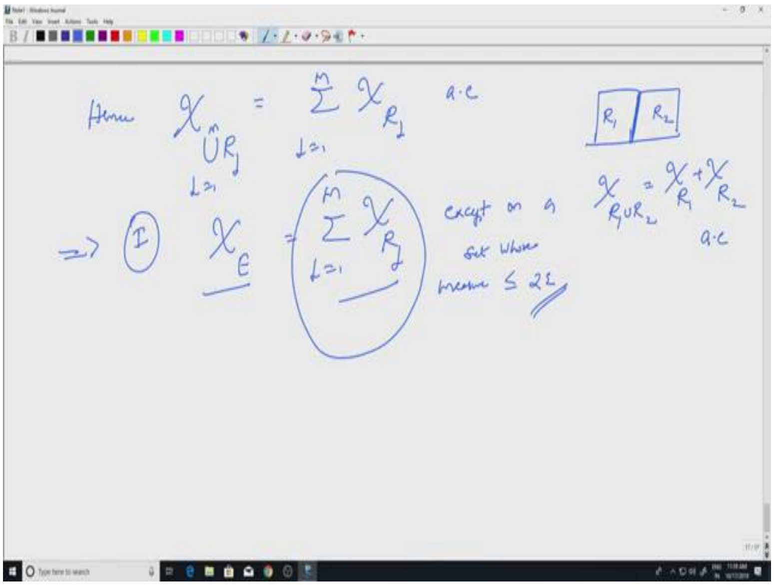

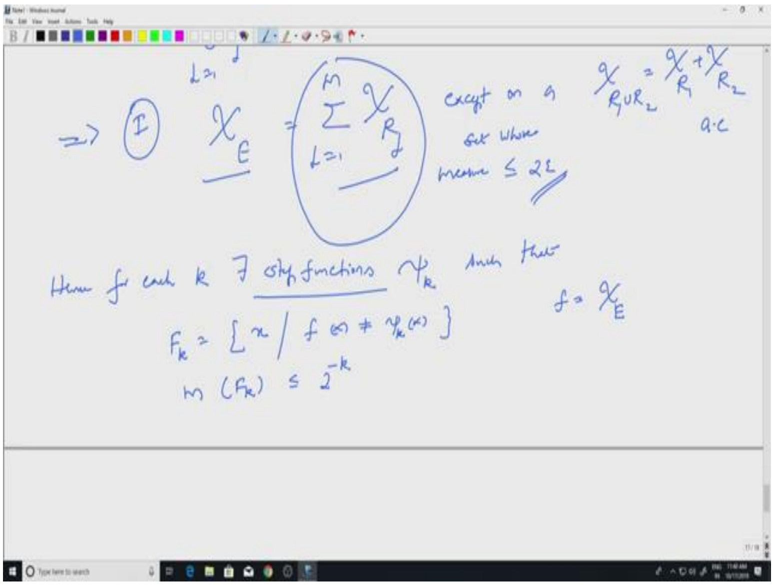

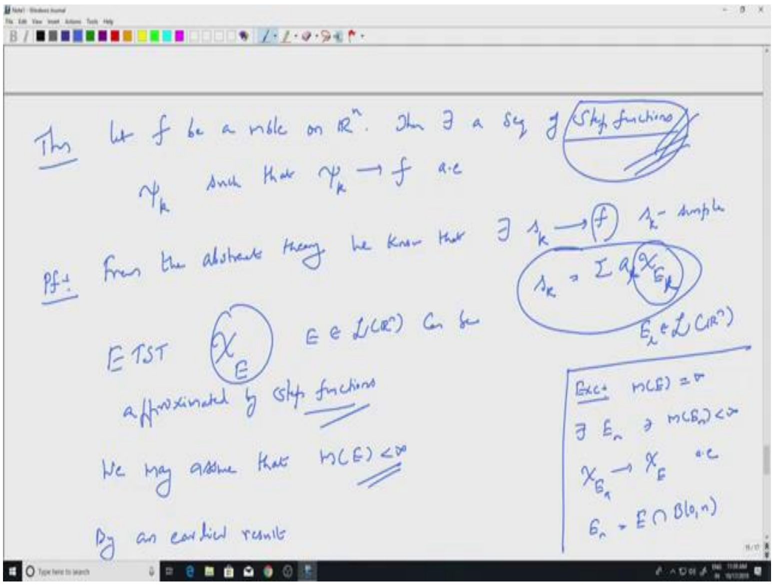

So, hence, if you look at indicator of union R j, j equal to 1 to M, remember they are disjoint, they are almost disjoint. So, this is same as summation, if I have to always disjoint rectangles, let us say R 1 and R 2, the indicator of R 1 union R 2 is equal to indicator of R 1 plus indicator of R 2. Well, of course, if they are equal if R 1 and R 2 are disjoint, but they are not disjoint on this, this line, you add twice, but that line has measured 0. So, this is equality almost everywhere.

So, the sum would be indicator of R j, j equal to 1 to M, but this may hold only almost everywhere, if they are truly disjoint of course, the sum is an equality, but otherwise they agree on the boundary but the boundaries have measured 0. So, this is something you should convince yourself. I had done these examples of computing the measure of lines and so on which gave me 0. Similarly, the boundary of the cubes will have measures 0.

So, this tells me that, so using the equality one, the indicator of E will be equal to summation j equal to 1 to M indicator of R j except on a set whose measure is less than or equal to 2 epsilon. So, given any epsilon, I can find rectangles, almost disjoint rectangles and the sum of their indicators like this. So, that the indicator function is same as the sum of the indicator function of rectangles except on a set whose measure is less than or equal to 2 epsilon.

So, we use this to produce the convergence. So, remember we want to prove that there is a sequence of there is a sequence of step functions converging to the measurable function. So, we need to get step functions, what we have done is to approximate this up to 2 epsilon. Now, you run epsilon over 1 by k. So, hence for each k there exist step functions psi k such that, so now we have step function because we are taking in indicator of rectangles and the linear combinations of that.

Such that the set F k where they differ. x is that f of x is not equal to psi k of x. So, remember f is the, we are assuming f to be the indicator function of E. Where they are not equal has measure small. So, measure is less than equal to let us say 2 to the minus k. So, because we can do this for every epsilon, we do this for epsilon equal to 2 to the minus k as k runs over 1, 2, 3, N, so on .

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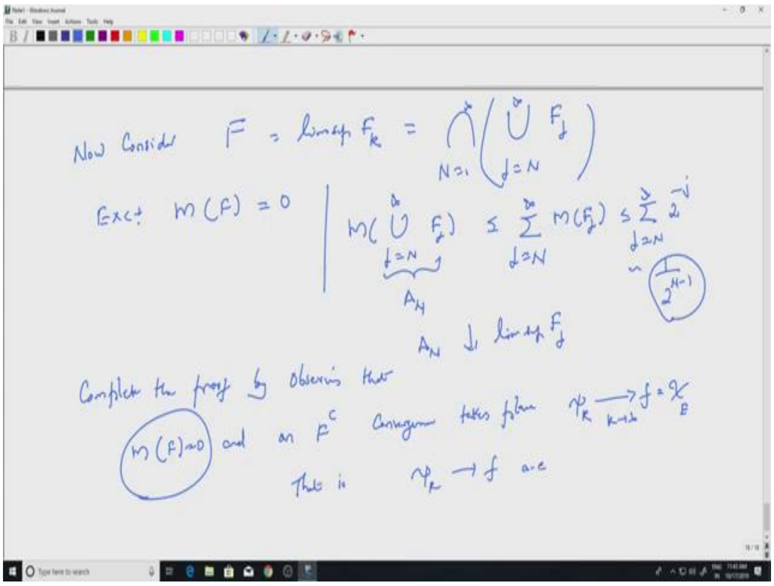

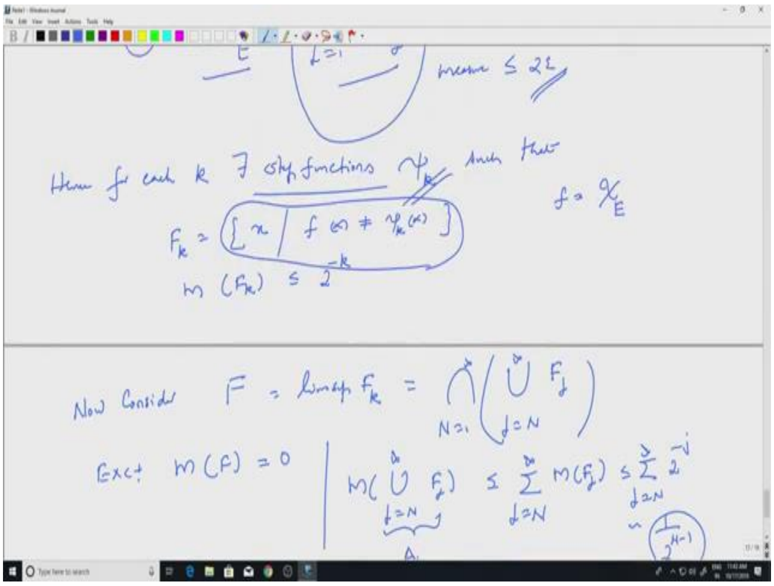

Now, consider f to be the limsup of these F k's limsup of F k, what are they, what is limsup of F k? You look at the intersection N equal to one to infinity union j equal to N to infinity F j. We look at the union from N space on nth place on words and take the intersection. This is called the limsup. Well exercise, this is pretty easy, measure of F is 0 it is easy because measure of union j equal to N to infinity of j this I know is less than 2 by subadditivity j equal to N to infinity, measure of F j but measure of F j is less than to 2 to the minus j.

So, I have j equal to N to infinity, 2 to the minus j and those you can add by geometrics sum. So, this is like 1 by 2 to the N minus 1 or something like that. And then you are taking the intersection. So, intersection gives me a decreasing sequence of sets and the measure theoretic results you already know will tell me that there is convergence. So, if I call these sets A N then A N's decrease to limsup of F j. So, measure of A N should decrease to measure of this.

But measure of A N is bounded by 1 by 2 to the N minus one which goes to 0. So, that is easy. Well, so, this is the sets, what are F k's? F k's are the sets where it is whatever we want is not happening. But the limsup of that is 0. So, outside of, so this is you can complete the proof by observing that, by observing that the measure of F equal to 0 and on F compliment, on F compliment that is the compliment of the limsup set, we have just constructed.

Convergence takes place, convergence take place, what is convergence? So, this psi k, which we have constructed psi k are step functions. Psi k which is not equal to the indicator function that set is F k, but we are outside F k when we look at F compliment, convergence takes place so psi k becomes f. So, f remember is chi as k goes to infinity, but f is such that the set capital F is has measure 0. So, this is same as seeing psi k converges to f almost everywhere.

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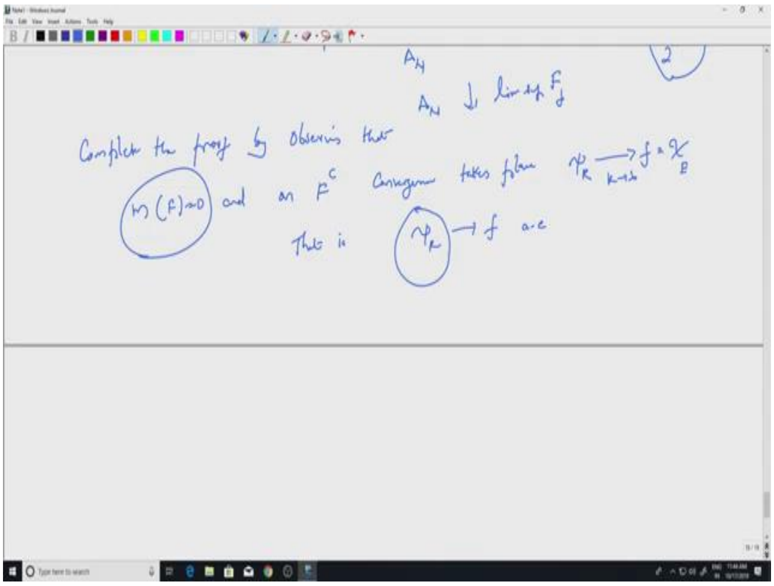

So, any Lebesgue measurable function can be approximated by step function. So, approximating by simple functions is the general theory. Because of these properties we have step functions. So, that concludes the finer properties of Lebesgue measure, Lebesgue sets and Lebesgue functions. But now, so now we have the Lebesgue theory of integration, we want to compare it with Riemann integration. We want to say that Lebesgue integrable is more general than Riemann integral.

In the sense that if the function is Riemann integral, then it is also Lebesgue integrable and the integrals are same. So, that calls for a comparison between Lebesgue integration and

Riemann integration. So, let us start with the preliminaries and in the next session, we will complete the proof.