Measure Theory
Professor E. K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture No 22
Riemann and Lebesgue integrals

So, we have seen the Lebesgue measure its invariants properties with respect to translation dilation and reflection. We have seen an example of an unmeasurable set and we looked at measurable functions. Now, we will be comparing the Riemann Integration and Lebesgue Integration in this lecture and our aim will be to show that Lebesgue integral is more general than Riemann integral.

More importantly when function is Riemann integral will show that function is measurable, and the Riemann integral is equal to the Lebesgue integral. So, that actually gives us a genuine generalization of Riemann integral. More importantly, the, the collection of measurable functions with respect to which we can do Lebesgue integration is much larger than Riemann integrable functions.

In particular, we know that if I have a sequence of measurable functions, the limit of a measurable limit of the sequence of measurable functions is measurable, which is not true in the case of Riemann integral per say, if I have a sequence of Riemann integrable functions, the limit need not be Riemann integral to such drawbacks of Riemann integral in some sense are rectified using Lebesgue integration. Okay, let us start.

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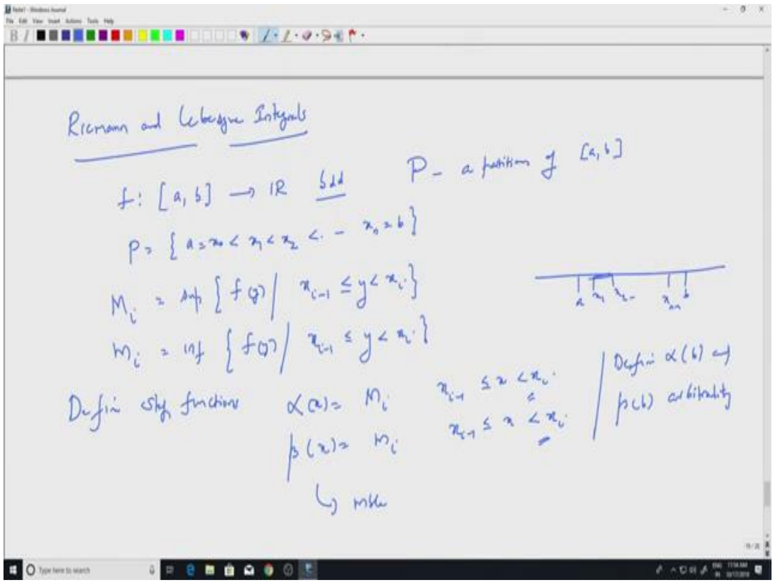

So, Riemann and Lebesgue integral. So, we will, we will recall Riemann integration but not in full detail. So, we have an interval AB and we have a function defined on AB let us say real valued and we assume that it is bounded. So, Riemann Integration is defined mainly for bounded functions and capital letter P denotes a partition of the set AB. So, we can let us denoted by x naught and of course we take things which are less in increasing order, X naught less than X1 less than X2, etc., etc., let us say Xn which is B.

So all that we are doing is we have AB, and we simply partition it. So it is X1, X2, etc., Xn minus 1 and Xn is B. Well, capital Mi. This is the supremum of the function F. Where you take Xi minus 1 less than Y less than or equal to Xi. So, on one side it is strict. Let us not, these things do not make much difference, but let us be really careful. Small mi is the infimum. Since, the function is bounded, these things exist and are finite. Less than of equal to Y strictly less than X.

So in each interval, we are looking at the supremum and the infimum of the function okay. So, we can define using this we define step functions. So define step functions, alpha and beta. So, alpha will correspond to the upper sum. So, alpha x is defined to be capital Mi that is a supremum when x falls into that interval, so, Xi minus 1, less than to X strictly less than Xi similarly, beta of X to be the infimum Xi minus 1 less than to X less than Xi.

You can define arbitrarily the, at the endpoint so, define alpha of B and beta of B arbitrarily. It does not really matter in the integration, because the strictly equality at this end point we will leave out B when we define alpha and beta, so you assign some values if you want you put 0, does not really matter. So, of course, these are measurable functions, these are Step function, Measurable Functions. Lebesgue Measurable functions. Now, if you look at so, let us have a measure space first of all.

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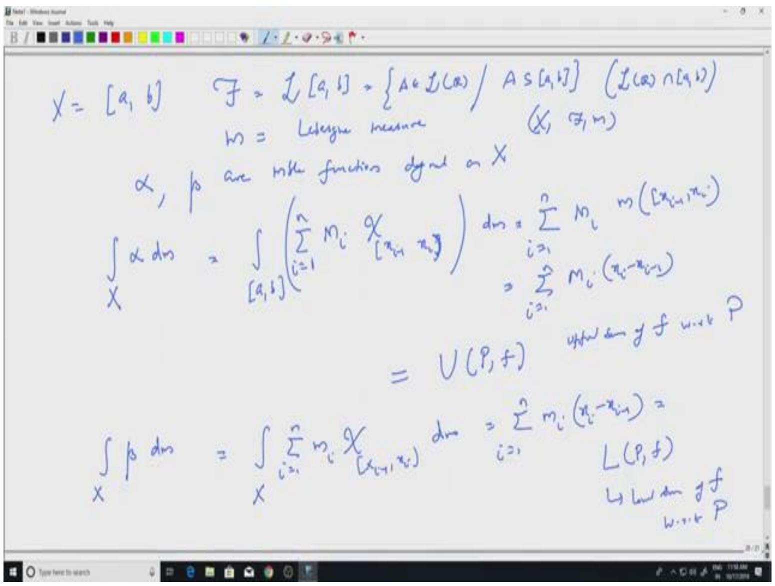

So, I have I have the space AB, this is my X if you like the sigma Algebra is the Lebesgue sigma Algebra of AB. What does that mean? This is you look at all those sets in in the Lebesgue sigma Algebra or which are contained here this is simply taking the intersection of, so this is like taking the intersection of LFR with the interval AB you look at all those measurable sets which are inside AB.

And M is the Lebesgue Measure, M is the Lebesgue Measure, so it is defined on all of our. So it is defined on any measurable set contained in R. So that is our triple X F M. And alpha and beta are the functions which we define alpha and beta are measurable functions, defined on X, defined on X. So integration makes sense as long as they exist, okay. So let us integrate. So let us compute the integral of X alpha dm.

Well, what is this? This I am integrating over the set AB, of course, so I can write like this, it is not necessary. What is the alpha? alpha summation it is defined to be Mi over this interval so, I will go from 1, 2, 3, etc., so it is a step function. So, this is simply capital Mi multiplied with the indicator function of Xi minus 1 to Xi. So, here open let us say yeah, this is the, so i going from one to some N, n or n minus one or whatever, so linear combination of dm.

Well we know how to do this because the integral is linear, so it goes inside and integral of an integrated function is simply the measure of that set. So, this is summation i equal to 1 to n capital Mi times measure of the interval Xi minus one to Xi, which is which is the length of that interval. So, that is Xi minus Xi minus 1. With, in our language this is in the Riemann integral languages this is simply the upper sum of F with respect to the partition. So, this is upper sum of F with respect to the partition P. Now, we do the same thing for the lower sum.

So, if you if you integrate the function beta Dm, well this is integral over x again summation i equal to one to n small mi these are the infimum indicator of Xi minus one to Xi dm, which if you continue this you will get i equal to one to n small mi times the length of the interval, which is this which is the lower sum, lower sum of f, lower sum of f with respect to the partition p. Good. So, both lower sum and upper sum have been returned in terms of Lebesgue integrals. So, now we take we take a sequence.

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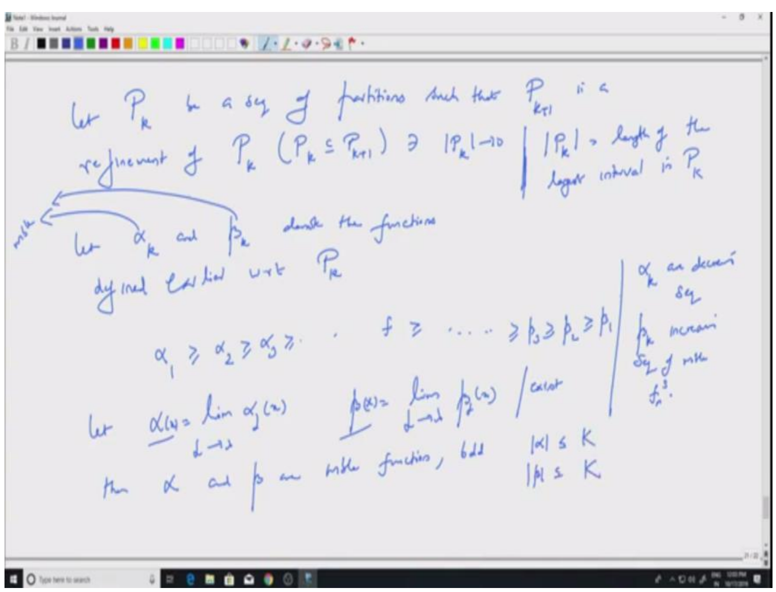

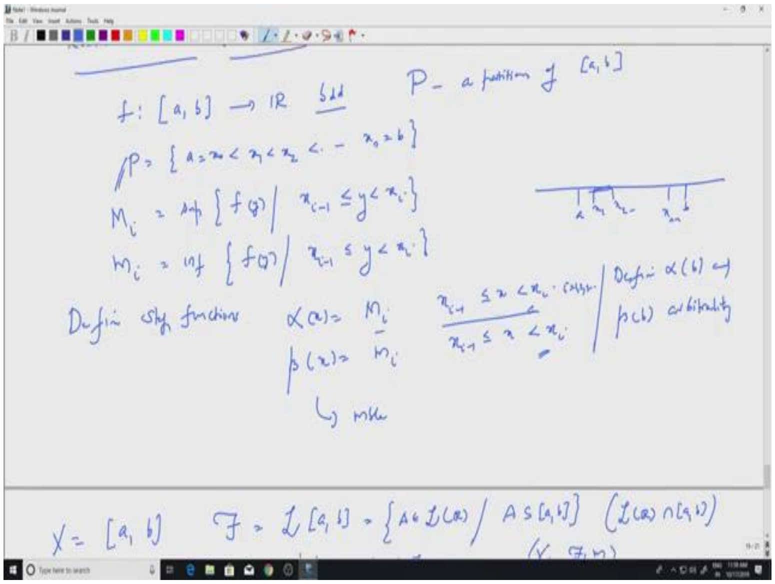

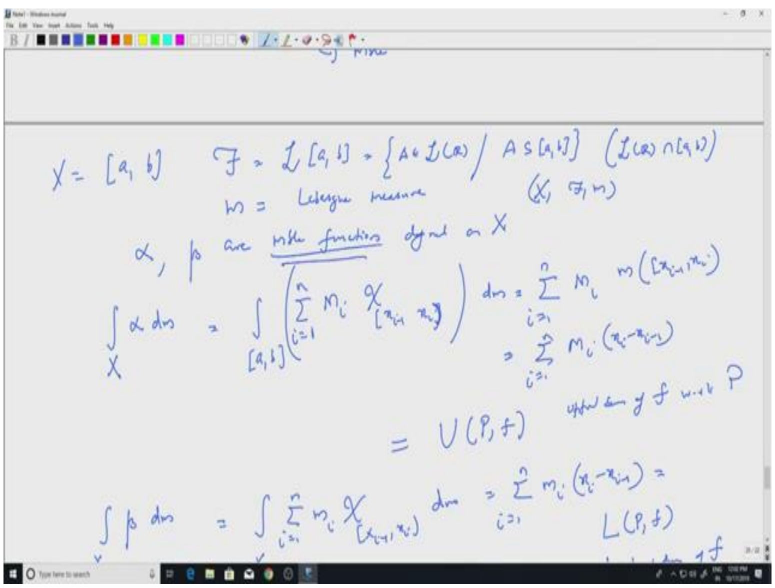

So, let Pk be a sequence of partition such that Pk plus 1 is a refinement of, refinement of Pk, which is am I saying Pk is contained in Pk plus 1, and you add more divisions in Pk, that is how you get PK plus 1. And such that modulus of PK goes to 0 so modulus of Pk is the, so that is a standard notation in Riemann integral mods PK is the length of the largest interval. Length of the largest interval in Pk, or given by defined by Pk.

So, you want the lengths of the intervals to go to 0. So, you make finer and finer partitions and the length of the intervals should go to 0. And then you take upper sums, lower sums and see what happens to the limit okay. So let alpha k and beta k so, remember if I we defined alpha and beta, given a partition p given a partition p we have the functions alpha and beta. So, if I take PK, I have alpha k and beta k.

So, let alpha k and beta k denote, denote the functions defined earlier, defined earlier with respect to the partition PK. So, these are step functions defined by supremum and infimum inside the intervals defined by PK. Well then what do we know? If you look at the definition of alpha and beta k, you will immediately see that alpha 1 is greater than or equal to alpha 2, greater than or equal to alpha 3and so on.

And you have f in the middle and greater than or equal to beta 1, greater than or equal to beta 2 and so on. Well, I should write this in a slightly better form. So, it is not decreasing sequence here it is an increasing sequence here. So, I have beta 1, smaller than beta 2, smaller than beta 3 etc. So, the alphas alpha case decrease beta case increase that is what it is saying. So, alpha k are decreasing functions, alpha k decreasing sequence, decreasing sequence, beta k increasing sequence of functions, increasing sequence of measurable functions.

And f is sandwiched in between. Since, they decrease and increase the limit will exceed. So let alpha equal to limit of alpha j, j going to infinity these are functions. So, what I mean is alpha of x alpha of x is equal to alpha limit of alpha J of x, alpha J of x is a decreasing sequence of real numbers bounded below by f of x, so it will converse, similarly, beta of x to be limit of the betas, limit j going to infinity beta J x.

So, they exist, so these alpha and beta are they exist, are nice functions and since alpha j and beta j are measurable remember these are step functions the alpha and beta are step functions for any partition we have alpha and beta which are measurable functions. So, so the alpha k and beta k are these are measurable functions. These are measurable functions and alpha and beta being the limit of measurable functions are measurable.

Then alpha and beta are measurable functions and of course bounded. So, mod alpha will be less than or equal to some M and similarly, or let us say some use letter K and similarly, mod beta is less than to K, so you choose K large enough. So their integrals will make sense. Okay, so I have measurable functions bounded on a, on a interval AB.

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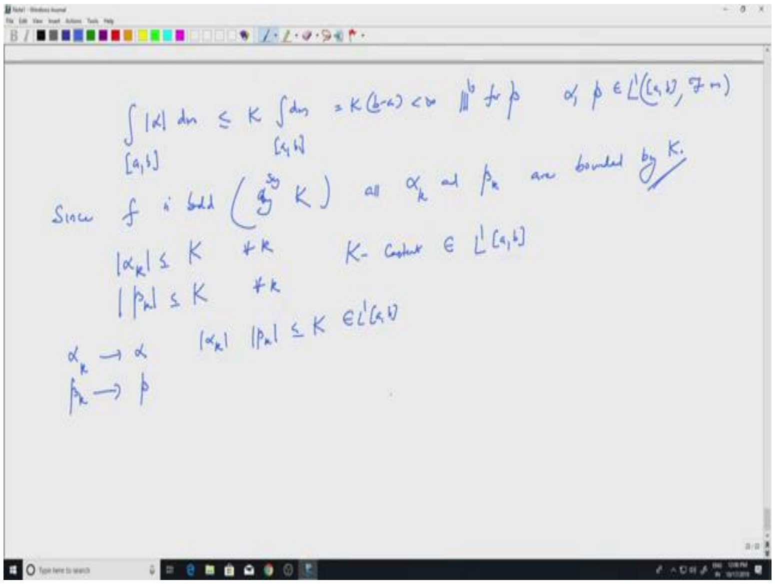

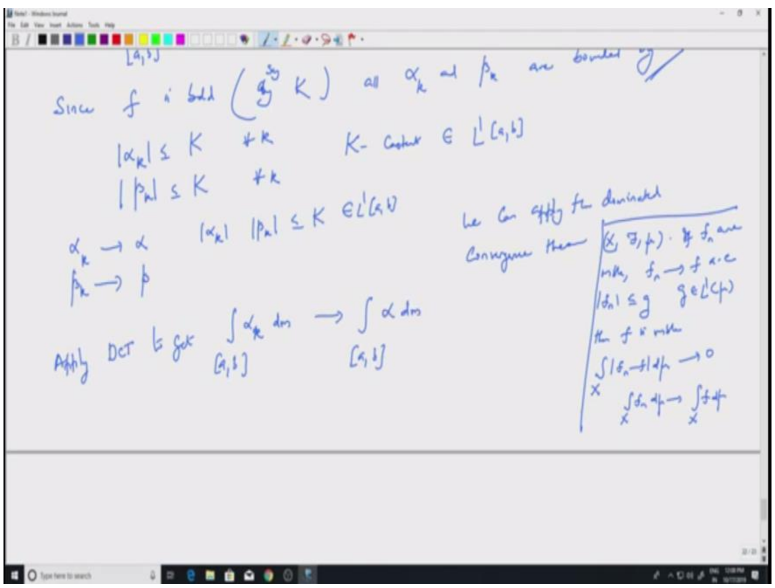

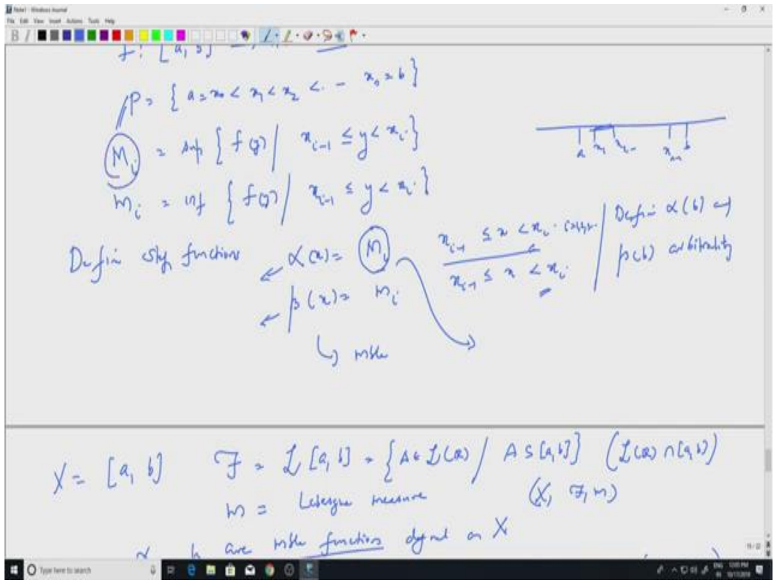

So, integral of over AB mod alpha dm, this always exist because this is a positive function we are integrating, but that is finite because I know it is less than to K integral over AB dm which is the measure of AB, which is B minus A, whic is fine. Similarly for similarly for beta, so both alpha and beta belong to L 1 of AV with respect to the sigma Algebra and the measure. So, we remember we have defined the space L1 of mu where mu was a measure.

And since f is bounded by every alpha k, so since let me write it down this as a statement, since f is bounded let us say by capital K, all the alpha k and beta K, these are made up of supremum of f and infimum of f on each intervals. So, they are also bounded by k. So, all alpha k and beta k are bounded by k. So let us, let us go back to the definition of alpha and beta in the beginning that will be clear.

So, the Mi here is the supremum of okay, I did not write down that. So, let us write this is ok. So the supremum of Mi is here, so, supremum of f in that interval. So, the values of alpha are supremum of f in various intervals, but f is bounded by k. So, the Mi is also less than or equal to k. So, that corresponds to any partition, the alpha k and beta k are bounded by the same bound for f.

So, I have mod alpha k less than into capital K for a very small k. Similarly, mod beta k is less than into capital K for every k and k is a constant so constant and so it is a measurable function and belongs to L1 because measure of AB is finite. So, constant functions are there. So, what is the situation we have? We have alpha k converging to alpha, beta k converging to beta and all of them are bounded by an integrable function.

So, we have a measure we have a space we have a sigma Algebra we have a measure and we have sequence of measurable functions converging, dominated by a constant which is integrable. So, we can apply the dominated convergence theorem, we can apply the dominated convergence theorem. So, let us recall the dominated convergence theorem here. So, I have x, i have f, i have mu if fn are measurable, fn converges to f almost everywhere, mod fns are bounded by some function g, which is in L1 that is the important part.

The dominating function has to be integrable, then, well f is measurable and integral over x mod fn minus f d mu goes to 0. This was the theorem we brewed in the abstract setting which we can apply in this particular concrete setting that in particular the integral of FM d mu will converge to integral of f d mu. So, we apply that so, apply a DCT to get well what do we get? We will get that integral over x alpha k, dm we have Lebesgue measure now. So, let me change x to AB so that we keep track of the interval AB. This I know will converge to integral over AB alpha dm. alpha is the limit of alpha case.

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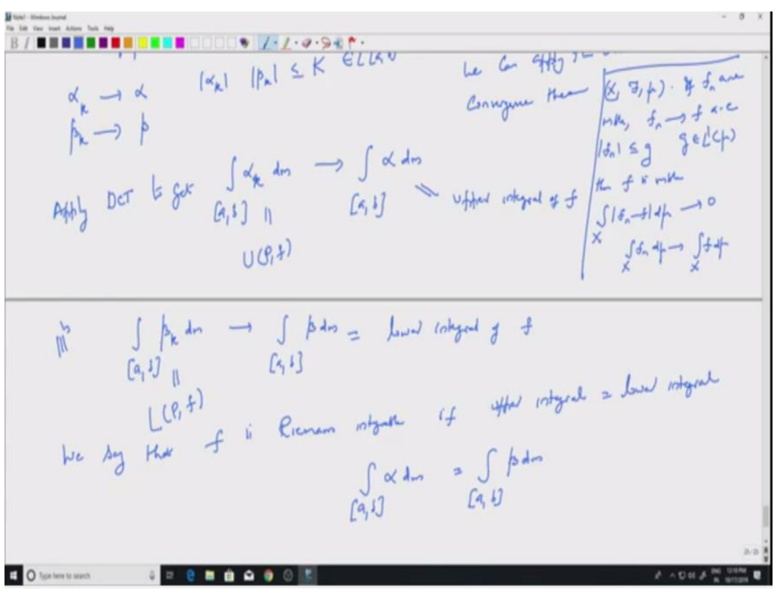

Similarly, for beta similarly, similarly, integral over AB, these are all Lebesgue integrals remember that beta k dm converges to integral over AB beta d. So, we have this much, whenever we have a Riemann integral function, we can partition, we get alpha and beta and we get the upper integral and lower integral as the usual Riemann integral. So, recall that the left hand side here is the upper sum of and this is the lower sum of f. And if they converge, they will converge to the upper sum and the lower sum of f.

So, the limit here we get is the upper sum of f or upper integral of f that is the correct terminology, upper integral of f and this is the lower integral of f and we say that f is Riemann integrable, Riemann integrable if the upper some upper integral and lower integral are same. So, upper integral is equal to lower integral, which is same as integral so upper integral is given by a Lebesgue integral now, that is alpha dm is same as integral over AB, beta dm. So, all this is just discussion starting from a Riemann integrable function, we have this much. So, now we can state the theorem.

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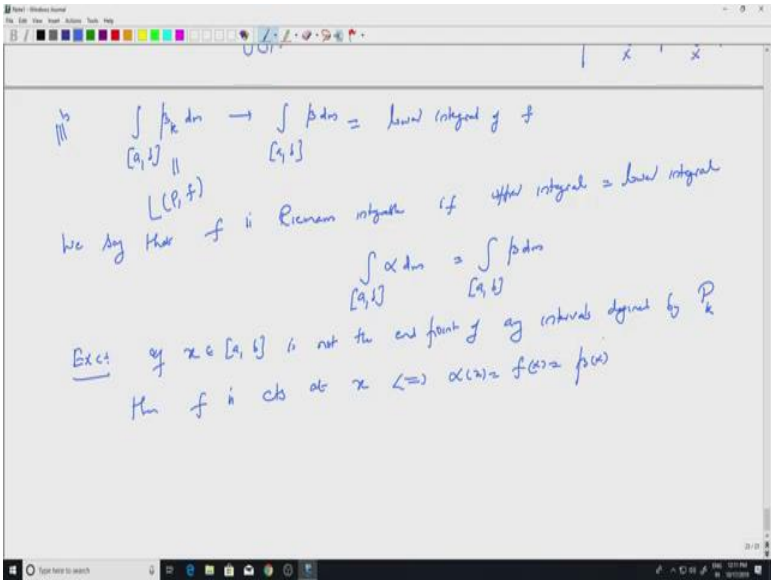

So, I will leave an exercise here. So, a simple exercise. If x and AB is not the endpoint of any intervals in Pk, any intervals defined by the partitions Pk then, so that is countably many points. Pk is our finitely mini points, you take union of Pks you will get countably mini points which as measure 0. Then f is continuous at X if and only if alpha x equal to f of x equal to beta x. This is the usual epsilon delta argument, remember what was alpha and beta, alpha Ks were decreasing and beta Js were increasing.

So, we have these functions alpha and beta and if alpha x equal to f of x equal to beta x then x the function f is continuous at x. So, except on a set, which is countable, countable sets of measures 0 because Singleton's are measure 0 and additivity property will tell me that countable sets have measure 0. So, except on a set of measure 0 this continuity property happens. So, now we can state the theorem. So, we have the alpha we have the beta and we have continuity if an only if alpha x equal to f of x equal to beta x.

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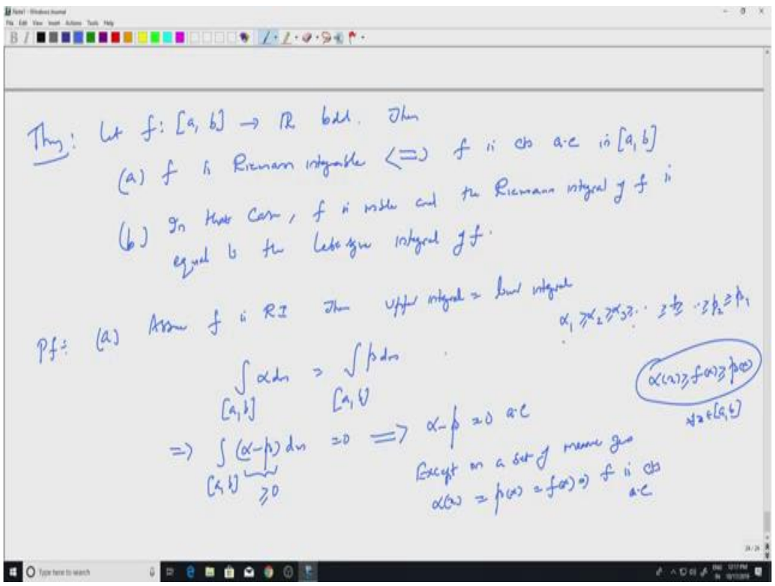

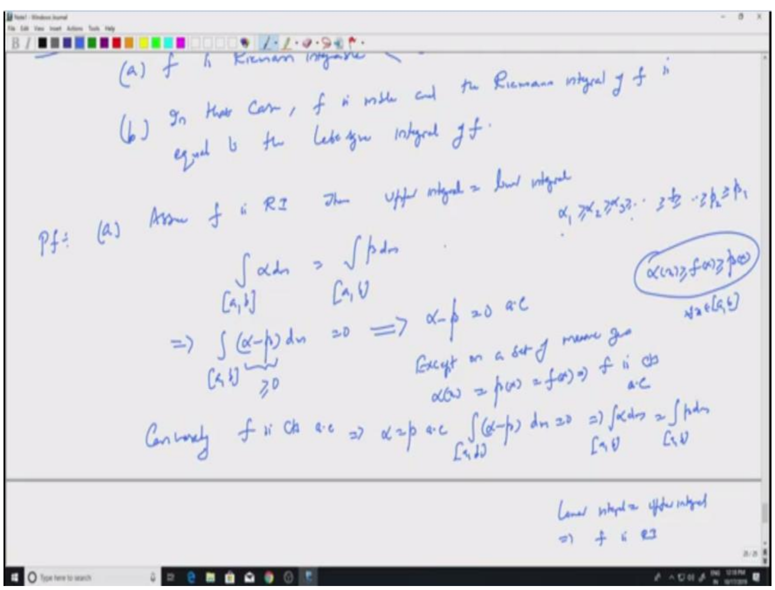

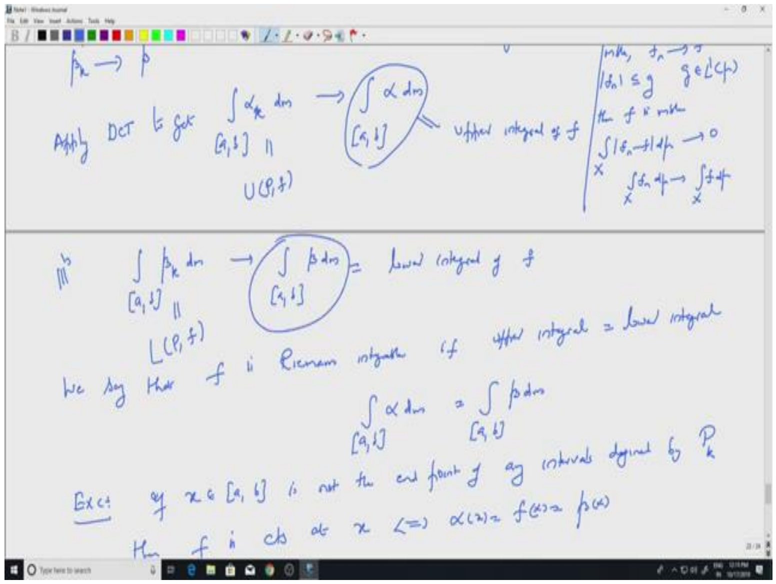

So, Theorem, so this is the comparison between Riemann and Lebesgue integral. So, let f from A to B to R it can be complex valued also but linearity of the integral will take care of all that we want, bounded then A f is Riemann integral, integrable if and only if f is continuous almost everywhere in AB. So, remember the concept of almost everywhere something is continuous almost everywhere meaning there is a set of measure 0. So that outside that it is continued, outside that meaning at all the points outside that it is continues.

We in that case that means if f is Riemann integrable, then the integrals are same, then that case f is measurable and the Riemann integral of f is equal to the Lebesgue integral of f, Lebesgue integral of f. So, if the function is Riemann integral, then it is measurable and the Riemann integral of the function f is same as the Lebesgue integral. So, let us let us try to prove A, so, assume f is Riemann integrable, f is Riemann integrable so I write Ri for that well, then what happens then the upper integral and the lower integral should be same.

So, what is the upper integral that is the integral of alpha remember the alpha be constructed using the limit of alphas K and the lower integrals are given by beta. So, they will have to be same. So, upper integral is equal to lower integral. That is same as the integral over AB alpha dm is equal to integral lower AB, beta d. These 2 will be same. So remember how alpha and beta are constructed so let me write down that alphas were decreasing etc., f was middle and betas were increasing, beta 1 smaller than beta 2 smaller than etc.

So f was in between, so the alpha so, because alpha all the alphas are bigger than all the betas, we the limit will also have this property. So not just that because f is in between we actually have that alpha x is greater or equal to f of x greater than to beta x for every x in. So alpha is greater than beta, but the integrals are same. So let me re-write this implies integral over AB alpha minus beta, dm is 0. But alpha minus beta is a non-negative function.

So, if I have a non-negative function it is and it is integral is 0, then it is 0 almost everywhere. So this is one of the theorems we proved in the abstract settings. This tells me that alpha minus beta equal to 0, almost everywhere which means except on a set of measure 0, except on a set of measure 0, alpha x is equal to beta x. But then because of this, it will also have to be equal to f of x, okay. That means f is continuous almost everywhere, f is continuous almost everywhere.

So, that is the first thing and conversely converses to L conversely f is continuous almost everywhere implies alpha equal to beta almost everywhere. So, integral over AB alpha minus beta dm equal to 0 which implies integral over alpha, integral of alpha is same as integral over beta which is same as lower integral is same as lower integral is equal to upper integral, which is same as saying f is Riemann integral. So, that part is trivial. So, that proves A.

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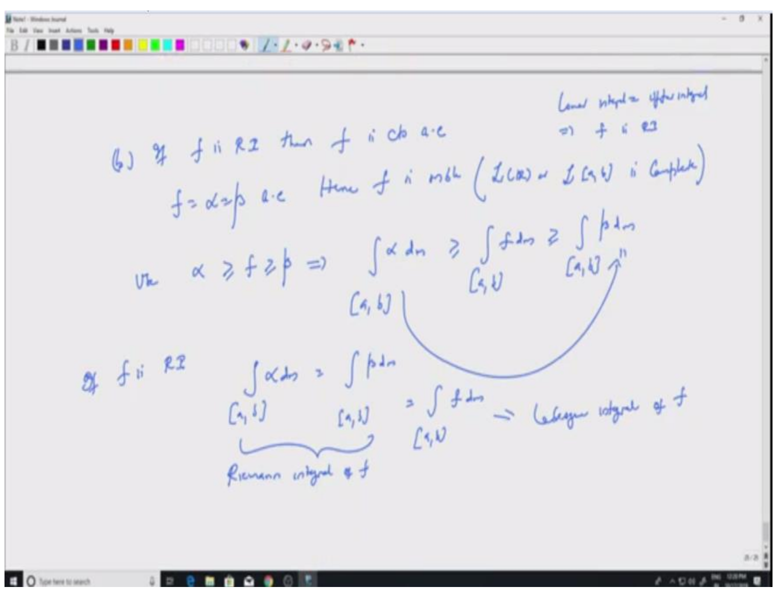

Well, what about B? B tells me that if f is Riemann integral then I know it is continuous almost everywhere then f is continuous almost everywhere. And f is equal to alpha equal to beta almost everywhere because alpha equal to beta almost everywhere so, because of this so that is what we have written here. So, hence f is measurable, hence f is measurable because alpha and beta are measurable and f is equal to alpha or beta almost everywhere.

Remember our sigma Algebra is complete. So, this is crucial here. L of r or Lebesgue of AB is complete. So, if I have a function which is equal to almost everywhere to a measurable function that function will have to be measurable if the sigma Algebra is complete. But now you can use the CT. So you use, so f is measurable use DCT, okay well you do not have to use DCT again we have already used it.

So, use the fact that alpha is greater than to f greater than or equal to beta. So, this would imply when I integrate, so, these are the Lebesgue integrals alpha dm will be greater than or equal to integral over AB f dm. That makes sense now, because f is now measurable f is bounded, so it is a finite quantity, greater than or equal to integral over AB beta dm. But if f is Riemann integrable these 2 are same. So, all the 3 are same.

So, if f is Riemann integral, integrable then integral over alpha that is the upper integral that is same as the lower integral. And so it will also have to be equal to the integral of f. Which is so the left hand side so this is the Riemann integral of f. And this is the Lebesgue integral of f, this is the Lebesgue integral of f. So, if f is Riemann integral we have just proved that f is measurable, and the integrals are are same.

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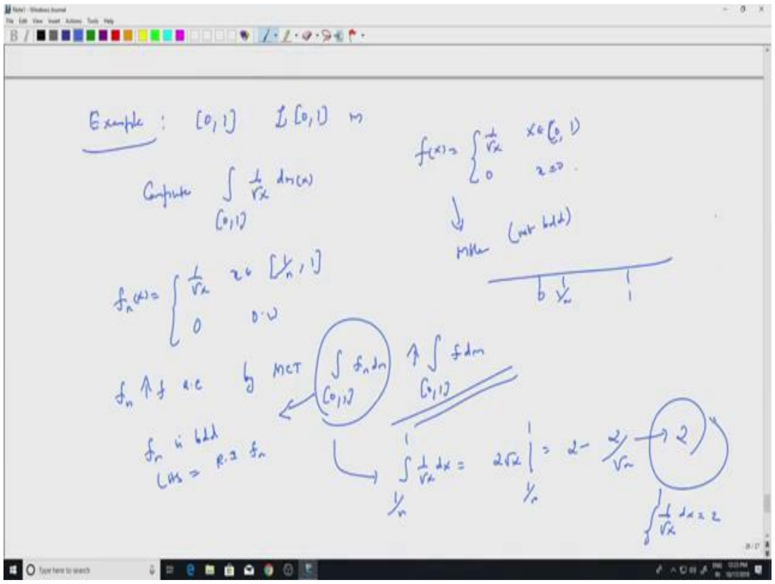

So, we will start with an example easy example. So, now if you want to integrate functions, you can do that, okay, but sometimes you may have to use various theorems. So, I will start with this example. So look at, look at 01. And you have the Lebesgue sigma Algebra, and you have the measure m, I want to compute integral over 01, let us say one by root x, dx is DMX with respect to Lebesgue integral.

So, we are looking at the function f of x to be on by root x. When x is in 0, open 01 and it could be anything at x equal 0. So, I can put 0 because this is a set of measure 0, it does not matter. So this is a measurable function. And because it is continuous, so it is measurable and I want to compute its Lebesgue measure, Lebesgue integral over 01. Well, the problem is this is not bounded.

So you can not apply the Riemann integral directly. So what you do is you cut the interval 01 at let us say 1 by n, So, you define f n x to be 1 by root x for x belonging to 1 by n to 1 and 0 otherwise. So then you see that fn increases to f almost everywhere and so by Monotone Convergence Theorem, integral over 01 fn dm will increase to integral over 01 f dm which is what we want to calculate.

But here now you can apply Riemann Aggression because our fns are bounded fn is bounded. So, this is so LHS equal to the Riemann integral of fn, fn is continuous so, it is it is Riemann integral. So how do you compute this? This is simply the usual integral from 1 by n to 1, 1 by root x dx, which is equal to 2 root x from 1 by n to 1, just 2 minus 2 by root. But MCT tells me that if I take the limit, I will get the integral of f.

So this goes to 2. That is what you generally do. If I, if I want to integrate 1 by root x, from 0 to 1, I am going to get 2. So, that is so you use MCT to justify that, if you have bounded functions there Riemann integrals are equal to the Lebesgue integral. So, we will stop here. So, we have sort of completed the construction of Lebesgue measure, we have seen finer properties of Lebesgue measure, Lebesgue sets.

We also have seen that of the function is Riemann integrable then it is measurable and the Riemann integral of the function is same as the Lebesgue integral. And the last example we discuss tells you how to deal with actual computations of Lebesgue integration. In some cases this will be very easy, in some cases you justify it using the 3 theorems you have. So, remember the 3 major theorems from Abstract Integration.

All that can be applied for Lebesgue Integration. From the next class onwards, we will go to abstract settings, we will look at locally combat Hausdorff space, so we will need some topological results. I will state all those results clearly, I will not prove any of this topological results, but I will use it in the proofs later on. So, we will be looking at my locally combat Hausdorff space and the Borel sigma Algebra.

So, remember the Borel sigma Algebra is the sigma Algebra generated by open sets and the measures defined there and we will look at certain finner properties of these positive Borel measures. So we will stop here.