Measure Theory

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Lecture 24

Riesez representation theorem

So in the last lecture we just looked at locally compact Hausdorff spaces and we saw that the space of continuous functions on such a space is a rich class. In fact, if X is large enough it has an infinite dimensional vector space and now we will look at certain linear functionals on this vector space and connected with measures. So this is a rather measure theorm due to f Riesz. It is called Riesz representation theorem.

So, I will not prove this theorem this because the proof is somewhat long and it takes more time. What we do is we will state the theorem in full detail, look at some examples and look at it is consequences and then we will use the theorem to construct the Lebesgue measure on Rn again. So we have already done one construction of the Lebesgue measure starting with cubes and finding measurability using open sets and thing like that.

Instead we will use a linear functional and thus the linear functional is going to be the Riemann integral of f and that will give a measure according to the Riesz representation theorem where certain properties which we have seen earlier like completeness, translation invariance, etcetera, will follow from the theorem itself, and more importantly we will have certain uniqueness properties of the Lebesgue measure which we will deduce from Riesz representation theorem.

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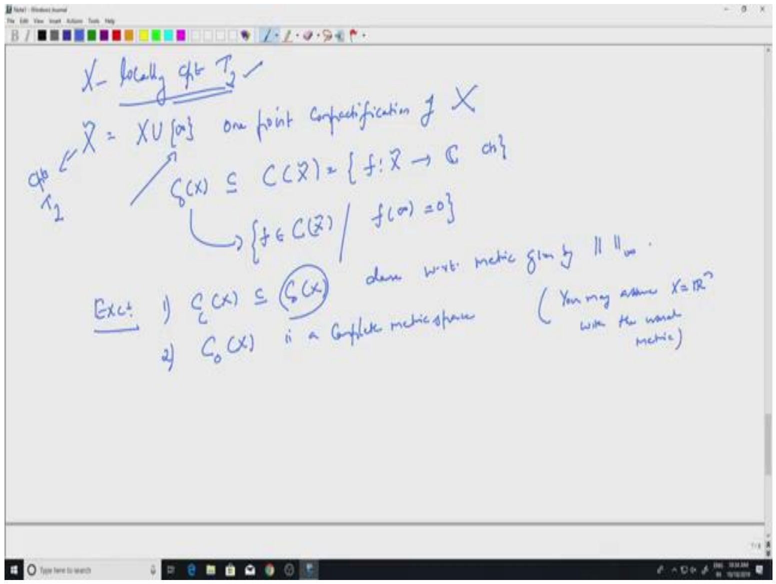

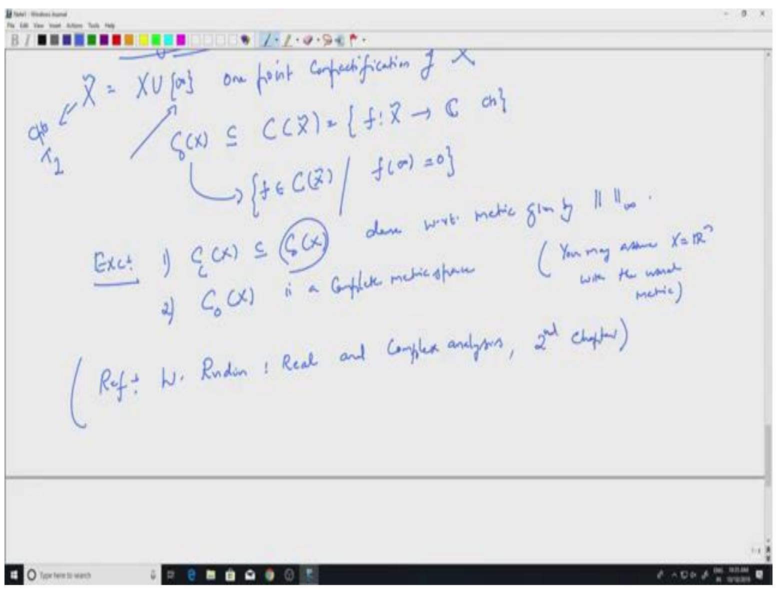

Let us start. So going back to what we have done, X is a locally compact Hausdorff space and X tilde has one point compactification and we can see C 0 X as a subspace of this. So to complete, so let me add one more exercise here to so that you get used to these spaces. These spaces are important spaces in analysis and we will be looking at C 0 X later on, where, when we study complex measures. So right now C c X is contained in C 0 of X and C 0 X is the space of continuous functions vanishing at infinity is a complete metric space.

So some of these are sort of straight forward exercise but if you are having difficulty with the locally compact Hausdorff space and so on you may assume that X is the real line or Rn, X is equal to Rn with the usual metric. Whatever I say will be applicable for Rn as well. So most of this is so reference here maybe perhaps, so you can look at book by Walter Rudin: Real and Complex Analysis. This is from second chapter. So let us go ahead.

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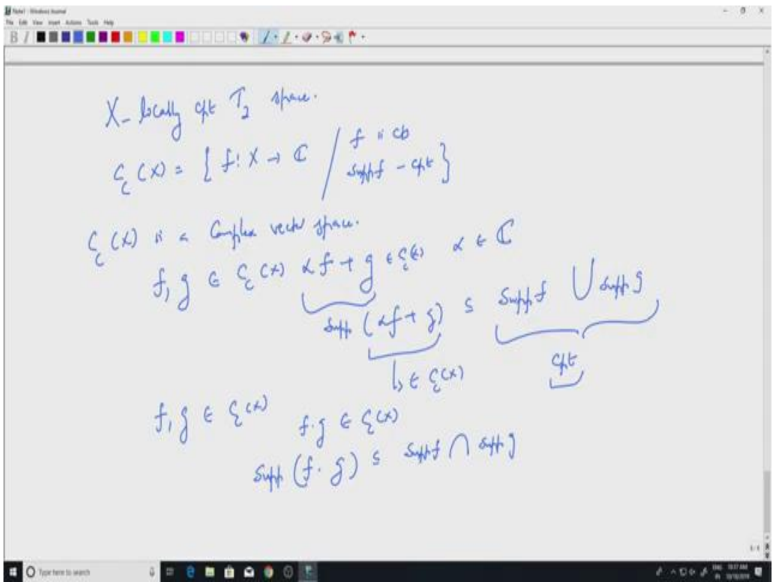

So C c X, so X is again a locally compact T2 space C c X, all those functions with compact support, f from X to C continuous and support of f is compact. So, we saw that by Urysohn's Lemma there are many such functions. So, this is generally a infinite dimensional space of X is big enough. If X is finite we saw that this is just Cn, so there we do not bother too much about such spaces because they are already taken care by linear algebra. So, here if you look at C c X, this is a vector space.

C c X is a complex vector space. Well, why is that? If I take f and g in C c X then alpha f plus g, where alpha belonging to complex plane is scalar will also be in C c X because the support of this function alpha f plus g is contained in the union of, so maybe instead of writing it in English, let me write it in Mathematics language, support of f union support of g, and this is of course compact, this is just union of two compact sets so this is a compact set.

So support of alpha f plus g is compact and so this would belong to C c X. Not just the sum you can also take the product, if I take two functions in C c X, then of course f times g is also a continuous function and will also belong to C c of X right because in this case the support is actually the intersection, support of f dot g, f times g, this is contained in the support of f because both of them should be nonzero for the product to be nonzero. So, support of f, support of g.

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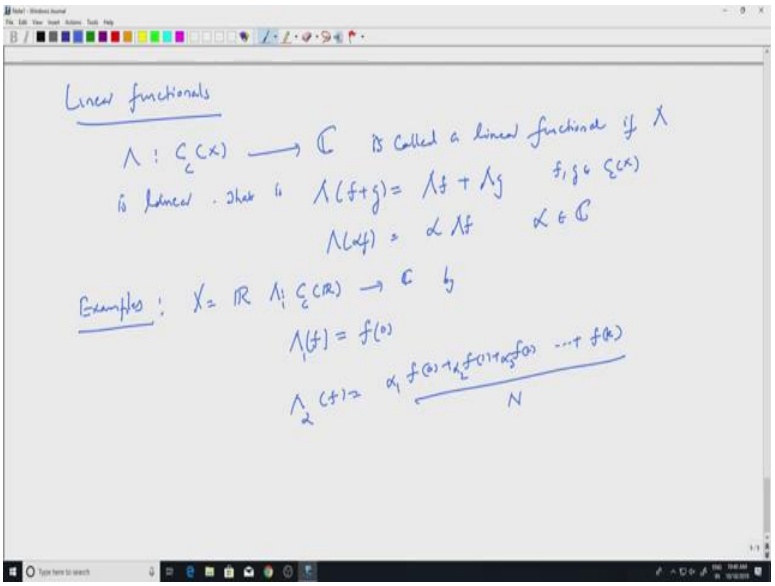

So that is about the space now the space is a linear space vector space so we can talk about linear functionals so this is what we do in linear algebra, generally we have a vector space and we have linear maps. So here we have linear functionals which means that it takes values from the real line or complex plane. So we look C c of X from C c of X to the complex plane, so we are looking at complex valued continuous functions so our basic field will be complex plane.

This is called a linear functional if it is a linear map, if lambda is linear. What does that mean? That is lambda of f plus g equal to lambda f plus lambda g and lambda of alpha f equal to alpha times lambda f, where alpha is a complex number f and g are in C c X. The vectors in the vector space are functions, such a thing is called a linear functional. Well, let us look at trivial example, just to get used to it.

So, let us take X to be the real line and I am looking C c of R, of course, there are lots of functions with compact support, you take any interval, you can draw a function which is supported there, you draw an appropriate triangle which is a continuous function. So we define lambda to be to the complex plane by f lambda of f, so lambda of f is a scalar. So. I should give you a number. Let us take f of 0.

This is obviously a linear map because f plus g will go to f 0 plus g 0 which is lambda f plus lambda g and if I multiply f with alpha, alpha comes out. So this is obviously a linear map. Of course, you can take many other points, for example, so you can take, so let us say lambda 1, lambda 2 you can take to be f of 0 plus f of 1 plus f of 2 etcetera maybe f of K, for some K. That is also a linear map, you can divide by some number N or you can put constants here alpha 1, alpha 2, alpha 3 et cetera they are all linear maps.

They are all linear functionals defined on C c R. Well, what is the relationship between measures and these linear functionals? That is the Riesz representation theorem, which I will state in full detail, but before that let us look at this itself. How do you write f 0 using a measure? So define this measure, define delta naught. Let us say on B of R, B of R remember is the Borel sigma algebra, Borel sigma algebra of R, so delta naught is a measure. How is it defined?

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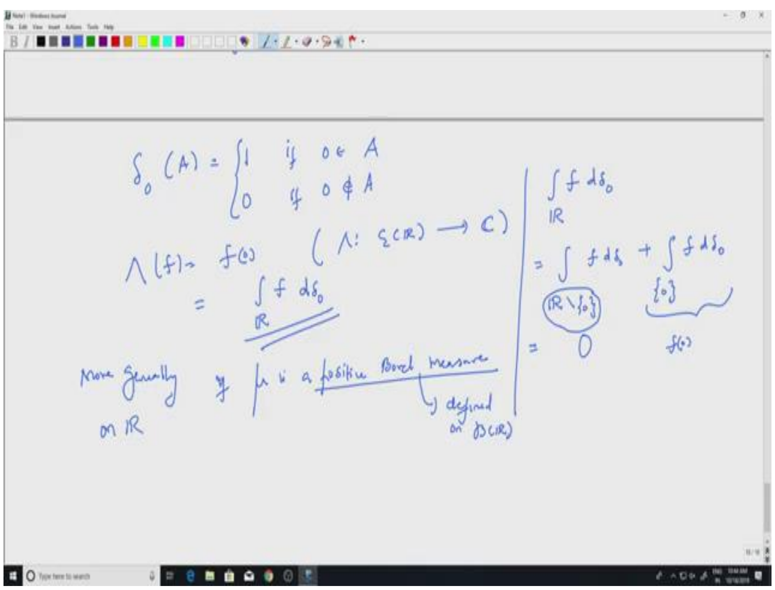

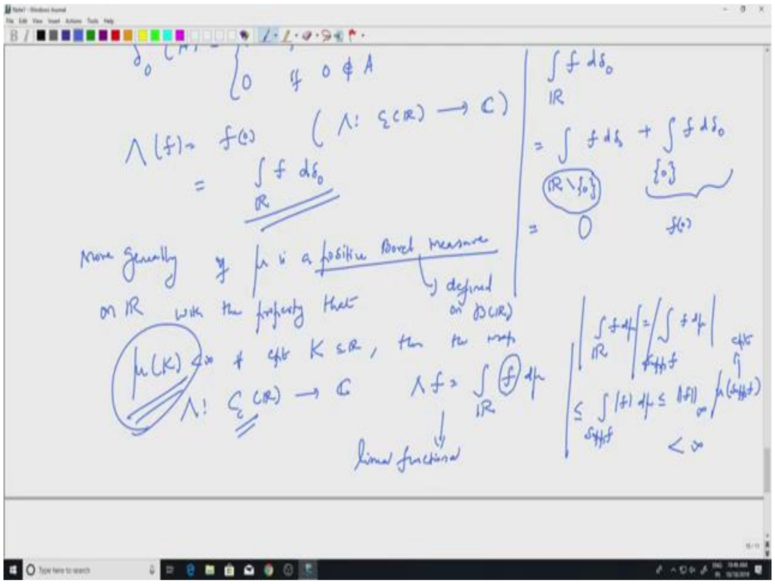

So delta naught of A equal to 1 if 0 is in A so this is the direct delta support at 0, 0 if the point 0 is not in A. So anything which intersects at A will have measure 1 otherwise it is 0. So, now if you look at the linear functional lambda, lambda of f to be a 0, this is my linear functional defined on the continuous functions on compact support right to the complex plane. I can write f0 as integral over R f d delta naught, delta naught remember is the measure. So we know how to integrate this.

Well, how do we compute something like this? We have done this several times. What is the delta naught outside the point 0 is 0. So, I can discard that. So let us compute integral over R f d delta naught well this is equal to integral over R minus the point 0 f d delta naught plus integral over the point 0 f d delta naught. This is because they are disjoint and so integration adds up, but this is the first term is 0 because this set has measure 0.

It does not intersect the point 0, so delta naught of R minus 0 is 0 and this is just a single point. So, I will have f0 as a scalar coming out and the measure of the point 0 which is 1. So that is just f0. So f going to f0 is actually given by an integral. More generally whatever examples we have seen here, these are all direct delta at 0, 1, 2 and so on and then you are adding up. So if you integrate again that measure you will get that particular linear function.

More generally if mu is a positive Borel measure on, let us say R, it can be on a locally compact Hausdorff space. So, what is a positive Borel measure? Borel measure meaning defined on the

Borel sigma algebra, then with some property. So positive Borel measure on R with the property that mu of K is finite for every compact K in contained in R. Then the map let us say lambda from C c of R to the complex plane how is it defined lambda f to be equal to integral over R f d mu.

Now, see this makes sense because f is a continuous function so it is Borel measurable, if I pull back a Borel, if I pull back an open set I am going to get an open set which is in the Borel Sigma algebra. So f is measurable, so I can integrate f with respect to the measure d mu and this is finite because of this condition, because support of f, so let us digress a little bit, integral over R f d mu, I can write as integral over support of f, f d mu plus whatever else is 0 because outside support of f, f is 0.

So, I can forgive this part and so if I take modulus, this is less than or equal to support of f, mod f d mu, mu is positive and this is less than or equal to, I can take the L infinity norm of f of outside, at the supremum norm of f outside because that is bigger than f and what remains is the measure of support of f, but that is finite because this is compact and this condition. So this is a perfectly well defined map and it is clearly linear because it is a integral Riesz linear, so this is a linear functional. So that is a general definition of linear, well general collection of linear functionals.

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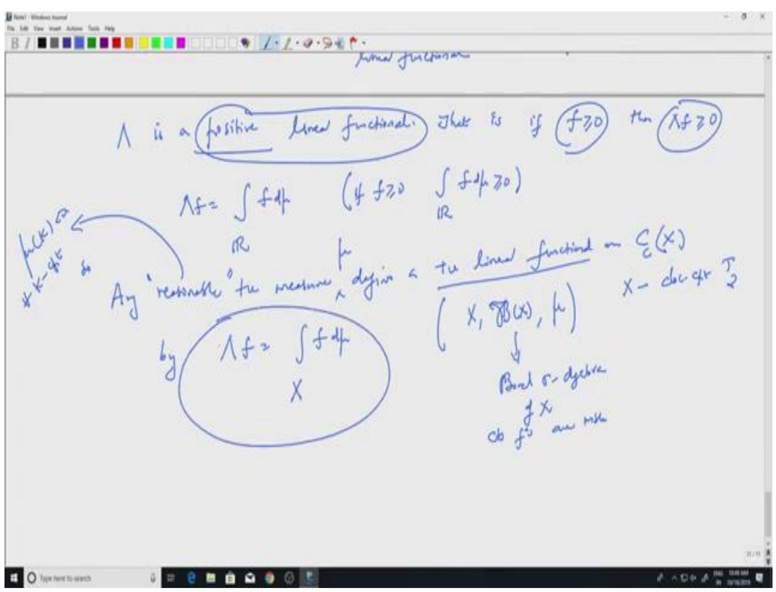

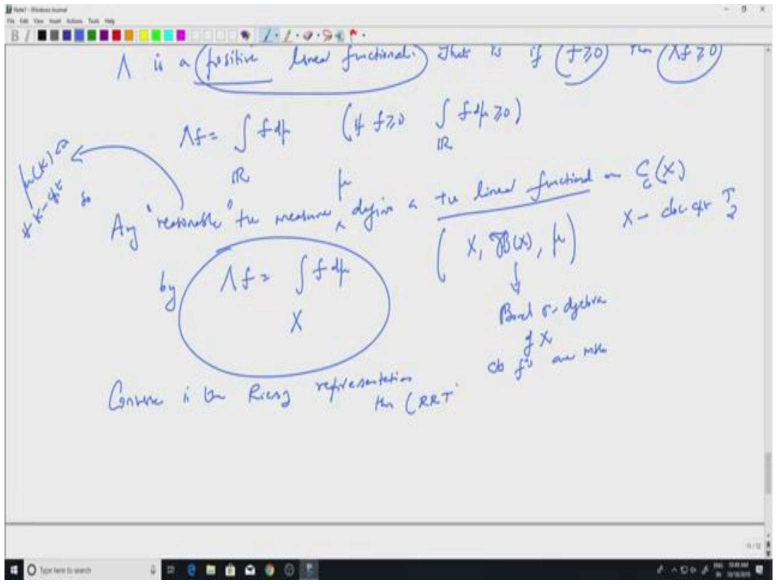

Now, what is important is that lambda is a positive linear functional. What does that mean? That is if f is non-negative, then lambda f is also non-negative, lambda f is nothing but the integral over R f d mu and mu is the positive measure, so if f is positive its integral is also positive, f d mu is positive. So positive functions are mapped to positive numbers and that thing is called as positive linear functional.

So, any positive measures defines a positive linear function, any positive measure with this property that is important. So on fine, on compact sets it has finite measures. The converse is called so, let me write down this as statement, so any reasonable positive measure, so reasonable here means that mu of K is finite for every K compact. That is what reasonable here means that any reasonable positive measure defines a positive linear functional on C c R.

Well, I can change this to C c X where X is a locally compact Hausdorff space. Just like what it defines on C c R, it defines on C c X as well. So by integration, so any reasonable positive measure mu defines a positive linear functional by lambda of f equal to integral over X f d mu. So, what is the triple we are looking at? We have this space X which is locally compact Hausdorff and the Borel sigma algebra of x. What is the Borel Sigma algebra of X? This is the sigma algebra generated by open sets of X.

That makes sense and we have the measure mu then this is the triple and integration is well defined, f is a continuous function so it is measurable right continuous functions are measurable with respect to Borel sigma algebra because when I pull back an open set, I am going to get an open set. So any reasonable positive measure mu defines a positive linear functional on C c X by integration. The converse of this statement is called the Riesz representation theorem. Converse is the Riesz representation theorem. So we will denote it by RRT, Riesz representation theorem.

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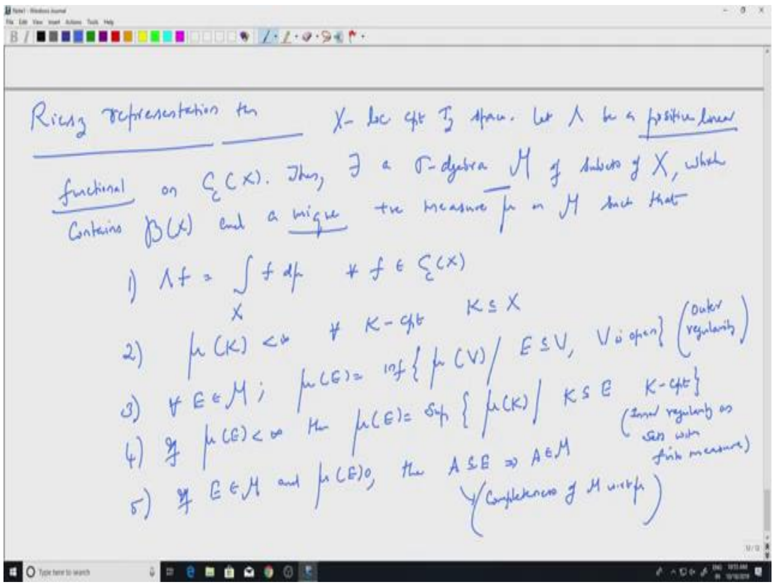

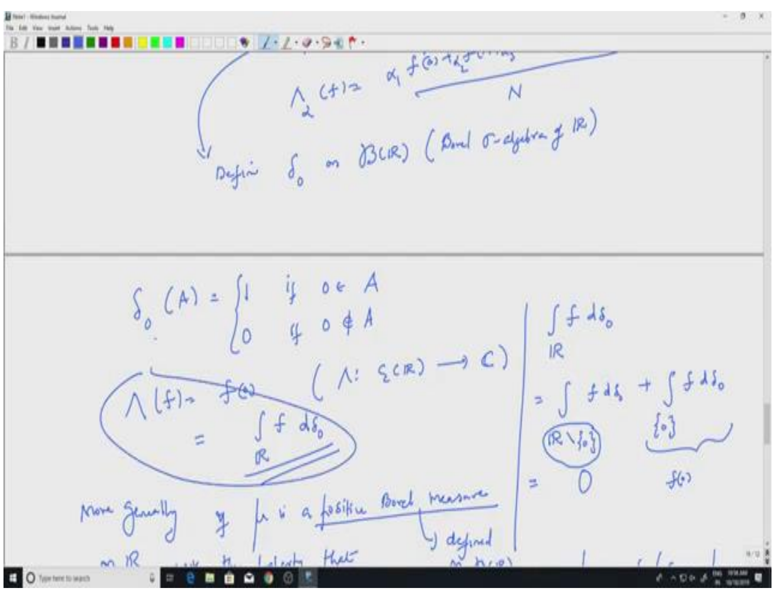

So, let me state this in full details, it is a slightly long statement. Riesz representation theorem. So, we will start with a locally compact Hausdorff space, locally compact T2 space and our object of interest is C c X. So let lambda be a positive linear functional so that is, as of now that is very important. We are only looking at positive linear functional on C c X. Then there exist a sigma algebra M, script M of subsets of X, so it is a sigma algebra consisting of subsets of X.

How big it is? It contains, which contains the Borel sigma algebra of X and a unique positive measure mu on script M such that now comes the statement of the theorem. So, we get a sigma algebra M and a measure mu which satisfies the following. So 1, lambda of f equal to integral over X f d mu for every f in C c X, so that is the important part the most important part of the theorem is the first statement that any positive linear functional on C c X is given by integration against the positive Borel measure mu.

So notice that everything makes sense because the right hand side is integral over X f d mu, f comes from a continuous function, f is a continuous function so it is measurable and I can integrate against mu. Well, we will see that mu has the reasonable property which we mentioned earlier so that is a second one. Mu of K is finite for every K compact subset of X. So, that tells me that the first one makes sense, the integral will be finite because of the second property that is the reasonable property we mentioned earlier.

3, now you will see somethings which are very similar to what you saw when we constructed the Lebesgue measure. So keep those properties in mind and you will see that this is purely the abstract version of those properties. For every E in M, so remember M is the Sigma algebra we have got which contains the Borel sigma algebra of X. We have mu of E to be infimum of mu of V where E is contained in V and V is open. So this is simply the outer regularity.

We had outer regularity for Lebesgue measure and remember there was an inner regularity for set and sets so that continues to hold, so let me write down that as well. If mu of E is finite, then mu of E equal to supremum over mu of K where K is now inside E and K compact. So this is true for every E, well E has to be a (())(22:42) of course, so, but you have those additional conditions that it has finite measure.

So this is the inner regularity, so inner regularity on sets with finite measures. And one more property if E is in M and measure of E is 0, then A contained in E implies A also belongs to M. So this is simply the completeness of M with respect to mu. So the Sigma algebra you get is a complete sigma algebra with respect to the measure mu. So starting from a positive linear functional one has a measure, a positive measure mu defined on a sigma algebra slightly bigger than the Borel Sigma algebra such that we have 1, 2, 3, 4, 5 these properties.

So these properties resemble the properties which you have seen for Lebesgue measure and whatever examples we have seen earlier, so look at f going to f of 0, for example. This is this gives me delta naught as the measure and of course that is a positive regular measure as you can see immediately. So this is a rather major theorem we will not prove this.

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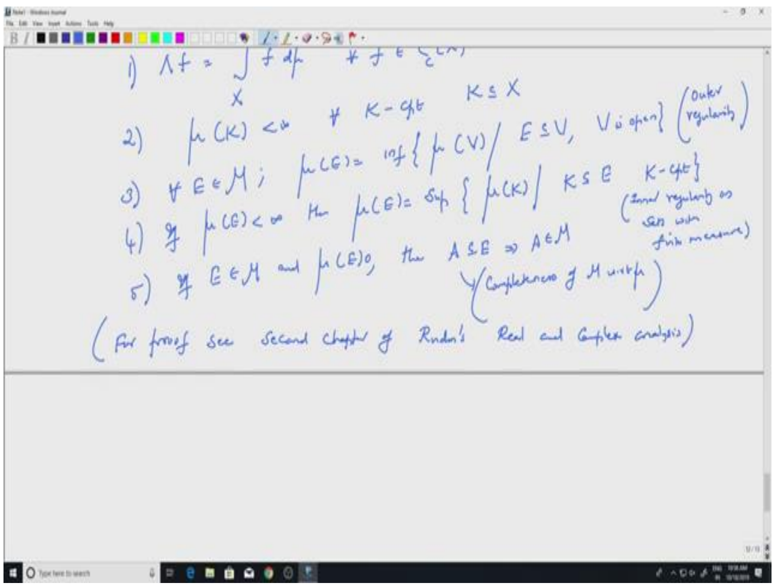

So, I will refer the, so for the proof see second chapter of Roden's real and complex analysis. It is a rather long proof. We will not prove this but we will at the consequences. So maybe let us stop here. So starting from a locally compact Hausdorff space and looking at the linear functionals there what we have done is to state the Riesz representation theorem in full which says that all positive linear functionals are given by integration against positive measures.

Now, we will look at some consequences, so looking at the properties of the measure and the sets you see similarities between Lebesgue measure, which we have encountered before. So we will continue that. So think of these results are results in abstract settings which are analogous through whatever you know for sets in Rn. We will continue this for a while and then use the Riesz representation theorem to construct the Lebesgue measure. So let us stop now.