Measure Theory
Professor E. K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 25
Positive Borel measures

We just saw the statement of the Riesz representation theorem, which tells you that positive linear functionals are actually given by integration against positive measures. So along with it we also get a sigma algebra. So this is going to be very similar to the Lebesgue sigma algebra and Lebesgue measure. So you will see regularity properties et cetera, as we have seen in the case of Lebesgue measure. So we will continue with that.

We will look at the space with a bit more structure. So, if we look at statement of the Riesz representation theorem there are two regularity statements. One is the outer regularity which is true for all sets, but the inner regularity is true only for sets with finite measures. This is in fact can be improved, if you assume a little bit more on the space x which is what we will do now and the Euclidean spaces and the spaces we are generally familiar with will satisfy these assumptions.

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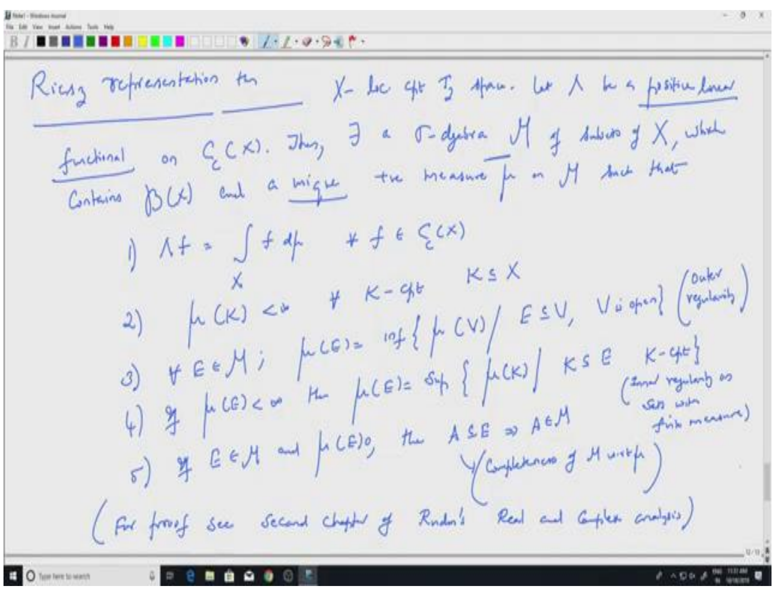

So, let us start. So this is the Riesz representation theorem we had written down earlier. So the linear functional is given by an integral that is 1, and the comeback sets are finite measures and these two are the regularity properties, along with completeness so that much we have. So, now we will look at the space with a little bit more structure or properties.

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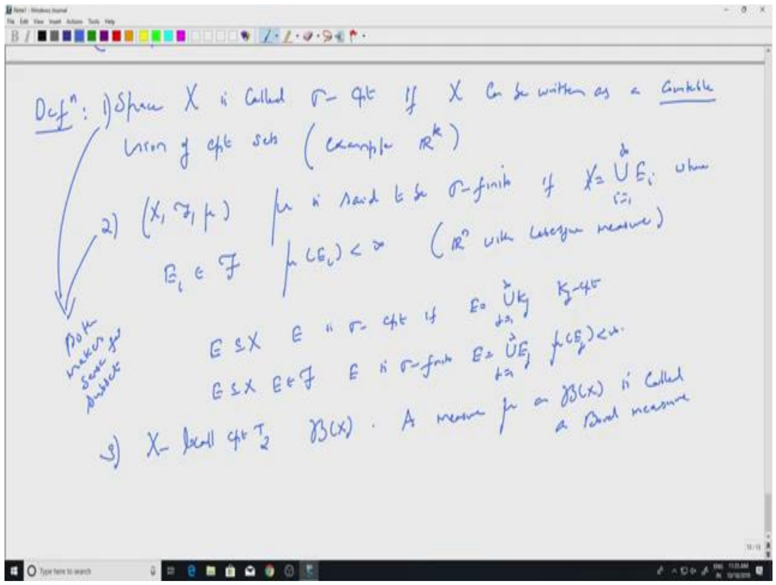

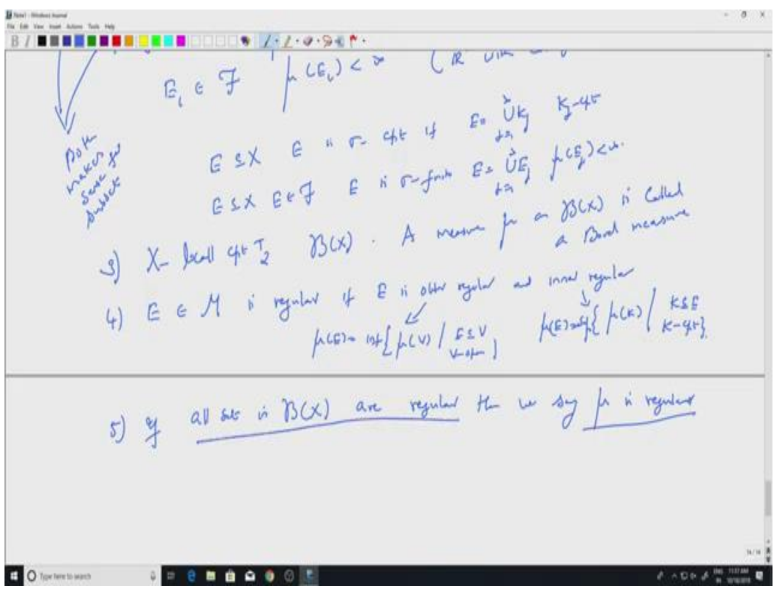

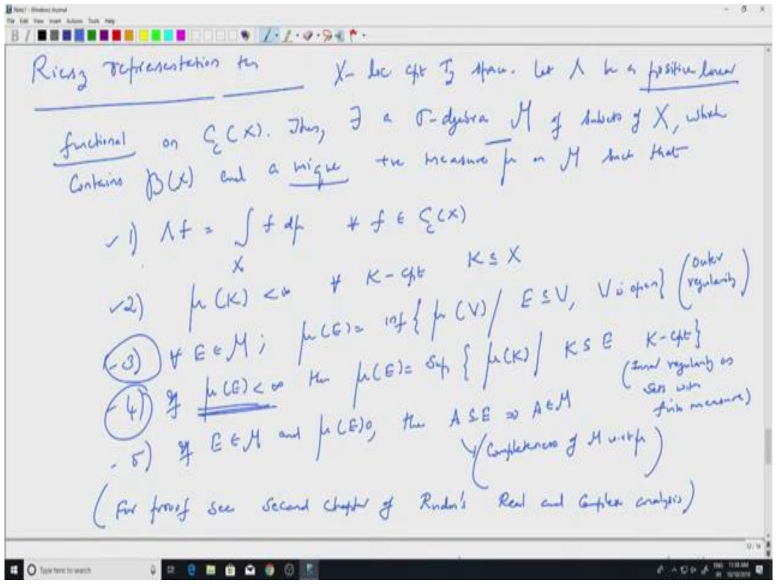

So, let me define all the so we will have certain definitions now. The space x is called sigma compact. So, sigma always denotes union countable union. So sigma compact will be that X is, X can be written as a countable, so countable is important, union of compact sets. So you should always check that these assumptions are satisfied for Rn, so example is Rk. So that is one definition. Two, a similar definition for measure spaces. So, I have x, f and mu.

Now, I am not asking anything on X, X is simply a space. Mu is said to be sigma finite. So you can guess what the definition is going to be. If you can write the whole space to be, union of Ei, i equal to 1 to infinity, where Ei are of course measurable sets and mu of Ei is finite. So there is a way of writing x as a union of sets with finite measures. So, again Rn with Lebesgue measure will be example, so Lebesgue measure is sigma finite and you can define this for subsets. So both the definitions make sense for both make sense for subsets.

What does it mean? So if I look at E contained in x, E is sigma compact if E is equal to union Kj, j equal to 1 to infinity Kj compact. Similarly, E is contained in x. E is a measurable set, then E is sigma finite. If E is union Ej, j equal to 1 to infinity with mu of Ej finite. So these are natural definitions, extensions to definition to subsets. Now, if X is locally compact E2 in the whole locally compact E2 not necessary. We have the Borel sigma algebra of X.

I measure mu on mu of X is called a Borel measure. So that is just to say it is defined on the Borel sigma algebra. It can be defined on a bigger sigma algebra than the Borel set like we have seen in the statement of Riesz representation theorem. We have the script and which is bigger than the Borel sigma algebra. Now regularity, mu is regular. Well maybe let me start with a set which is regular and then we will go to mu.

So, if I take E in some sigma algebra, let us say m is regular, so you can compare it with the statement in the Riesz representation theorem. It is regular if E is outer regular, and inner regular. So what is outer regular? Mu of E equal to infimum of mu of v where E is contained in v and v open. Inner regular, you can approximate from inside by compact sets. So that would be the supremum, mu of K. K contained in E, K compact.

So this is in the context of Riesz representation theorem and if all sets in M or in B of x that is enough, all sets on B of x, the Borel sigma algebra are regular, then we say mu is regular. So a regular measure is where all sets in B of X are regular. So, the Riesz representation theorem gives you outer regularity but inner regularity is only for sets with finite measures. But this can be improved with the sigma compactness assumption.

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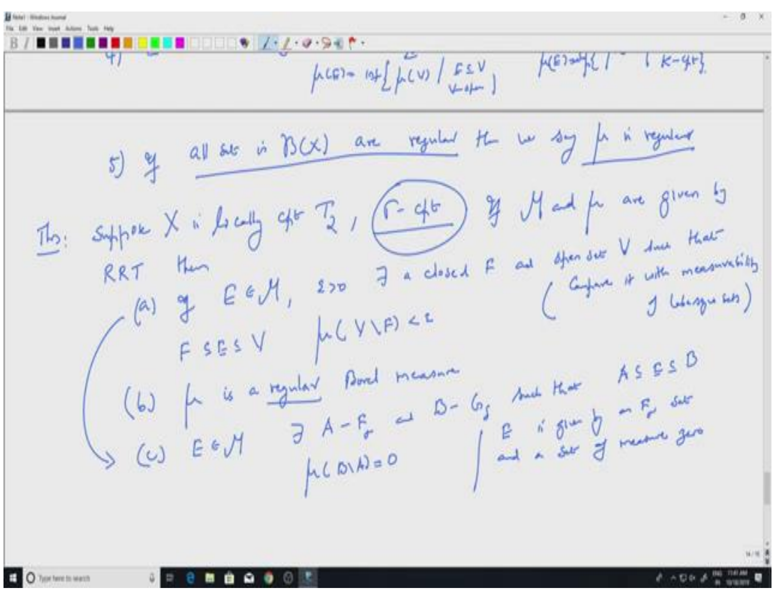

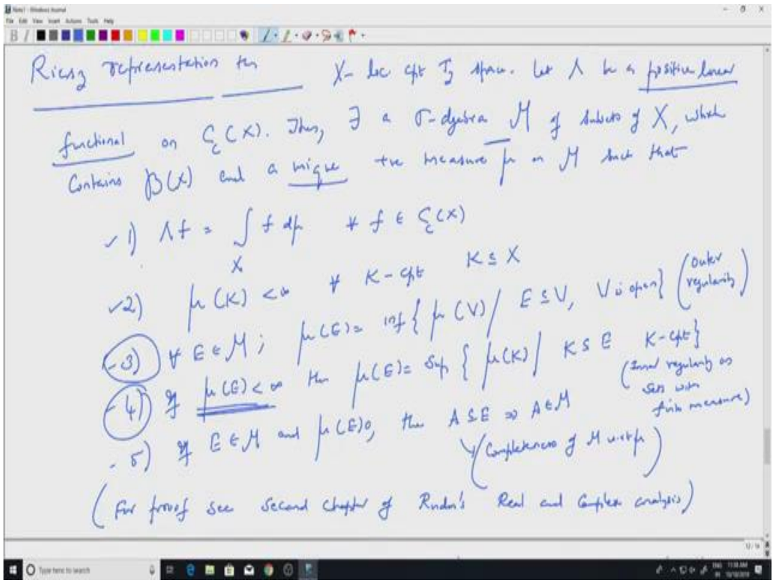

So that is the first result we will do. So, let us state a theorem. Suppose, X is locally compact t2 and sigma compact. So that is an extra assumption. If script m and mu are given by Riesz representation theorem so RRT, so remember we have five statements there. All of them are true. So that means that we have a locally compact Hausdorff space, that is a linear functional and that is given by mu. So, we have the sigma algebra m and unique positive measure mu.

So we have these two. Given by Riesz representation theorem then we can say a little bit more. So what are the assertions? One if E is in M and epsilon positive, then it is a closed set f and open set v such that f is contained in E, contained in V. So you have seen in for the Lebesgue measure and it is exactly the same statement written in one go instead of writing 2-3 statements. Mu of v minus f is less than epsilon..

So from the top you can approximate by open sets. From the bottom you can approximate by closed sets. This was the definitions of measurability for, so you can compare it with measurability of Lebesgue measure or Lebesgue sets. That is precisely the statement here. V mu is a regular, so now you have both inner and outer regularity because the space is sigma compact. So, mu is a regular Borel measure. C, well from A you know how to deduce this.

This actually implies you have seen in the case of Lebesgue measure. If I take a set in m and I can approximate it with f sigma and G Delta. So there exist a set A, f sigma and a set B which is G Delta such that A is contained in E, contained in B and of course, the difference between B and A is 0. So this means that E is given by and f sigma set and a set of measure 0. This is what we did for Lebesgue measure and that is precisely what happens here also. So let us see the proof.

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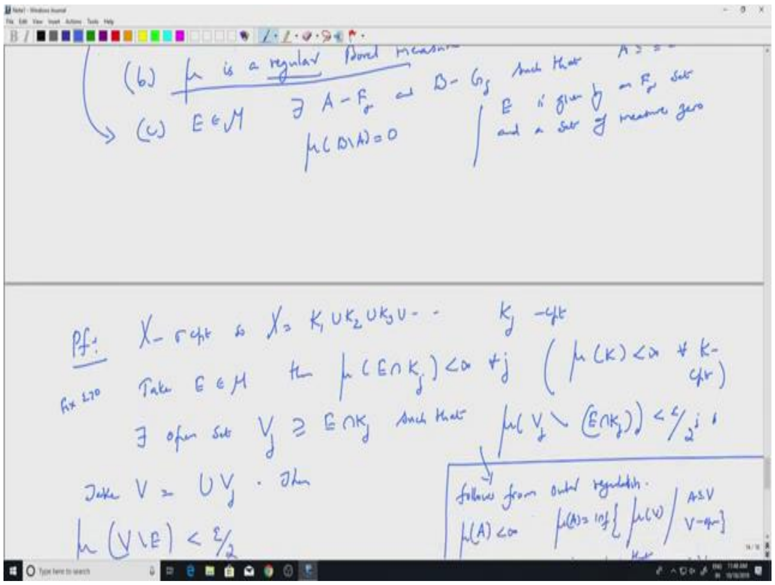

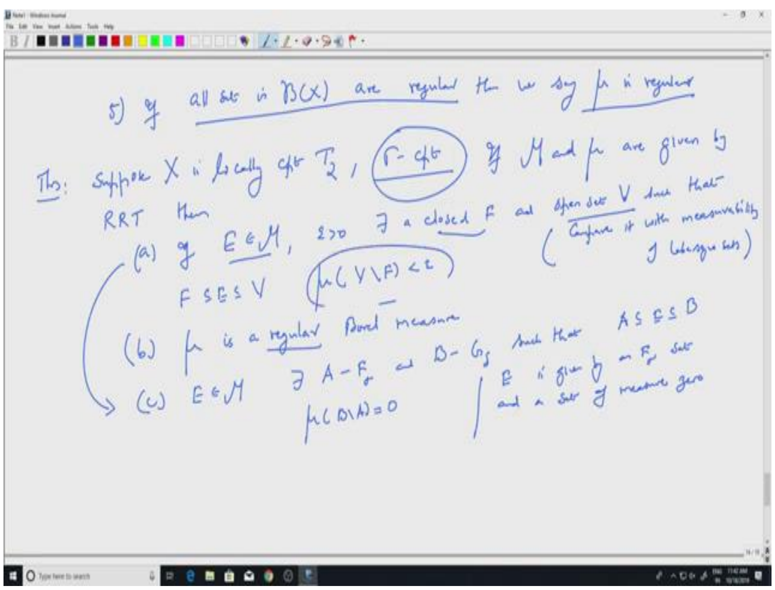

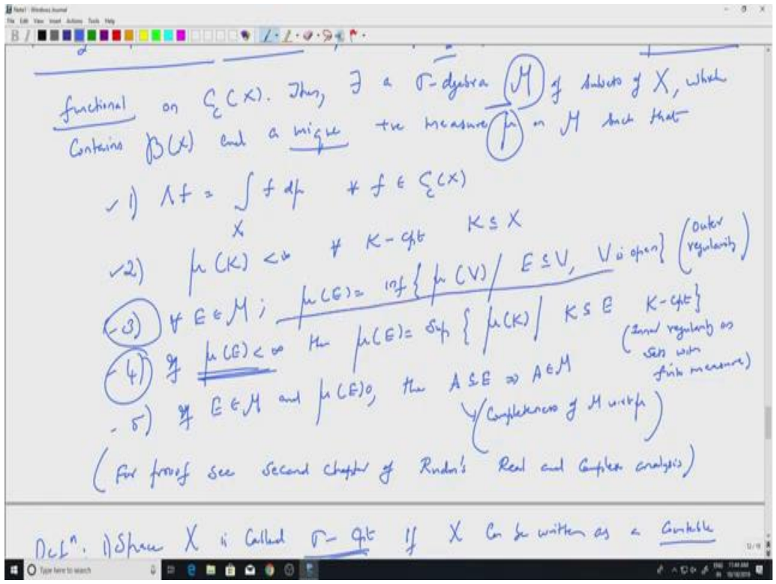

So proof follow from whatever properties we know about the measure from the Riesz representation theorem. So, first of all X is sigma compact. So I can write x as K1 union K2 union k3 et cetera, or where each Kj is compact. That is the sigma compact measure. Now, if I take a set E in M. So let us, let us see what the theorem says. The first assertion if I take a set E, I should be able to it a closed set f and an open set f so that this happens.

So take E in M, then mu of E intersection Kj this is finite for every j. Why? Because mu has a property that, mu of K is finite for every K which is a compact set. Remember that was part of the Riesz representation theorem. So, if I intersect E with Kj I am going to get something which is contained in Kj and Kj is compact. So that is why. Now because of this is there exist, well there exist open set Vj containing E intersection Kj, such that mu of Vj minus E intersection Kj is less than epsilon by 2 to the j.

So of course you start with fix epsilon positive and all that. You want to say that there are open sets and closed sets which will do something. So well, why is this true? This is simply the definition, the regularity property. So, let us go back to our regularity property. So you can approximate any set from outside. So the third, sorry, letus go back to the Riesz representation theorem.

So this property that infimum of mu V, V is open, will give me E. So that is the outer regularity. So this follows from outer regularity. Let us justify this. This follows from outer regularity. Why is that? So, let us do this for a set which has finite measures. So if I take A such that mu of A is finite, then I know mu of A is, well by outer regularity I know this is a infimum of mu v. A contained in V. V open. But these are finite ones.

So I have given epsilon there exist to Vj, open, such that mu Vj is less that mu of A plus epsilon by 2 the Vj. That is true because of the infimum property. But then this tells me that if we look at mu of Vj to be equal to mu of Vj minus A. Well, A will be contained in Vj plus mu of A because they are disjoint. This tells me that because of mu A, mu Vj minus mu A being less than epsilon 2 j, this implies that mu of Vj minus A is less than epsilon 2 to the Vj.

So, that is why this is true. Alright, so because of this now all that you have to do is take union. So take V to be equal to your union Vj, so this we have done for the Lebesgue measure. Then V minus E, well you can pro that this is contained in, because of the epsilon by 2 to the j et cetera, so I will leave this part to you because we have done this so many times now. So, then mu of V minus E is less than epsilon. Because of epsilon by 2 to the j and you can add them up.

Now how do you get the close set f? So we have the open set V now. You have to get the closed set, this is exactly the same proof. That is by applying, apply the same thing to E compliment. Apply the same proof to E compliment. So you will get an open set, get W open, E compliment contained in the W and mu of W minus E compliment is less than epsilon by 2. Well, so we have seen this W minus E compliment is equal to w intersection, E compliment, compliment which is W intersection E, equal to E intersection W compliment compliment.

So, take f to be equal to W compliment which is the, which is a closed set. Because W is open and this is simply E minus f so that is all we need. So, this is what we did for Lebesgue measure and that is continues to be true in this abstract setting as well. So what is the next assumption. Let us see. B says it is a regular Borel measure. So that is the next thing we have to prove. So we know that it is outer regular. We know it is inner regular for sets with finite measure. So we what we did was A was proved.

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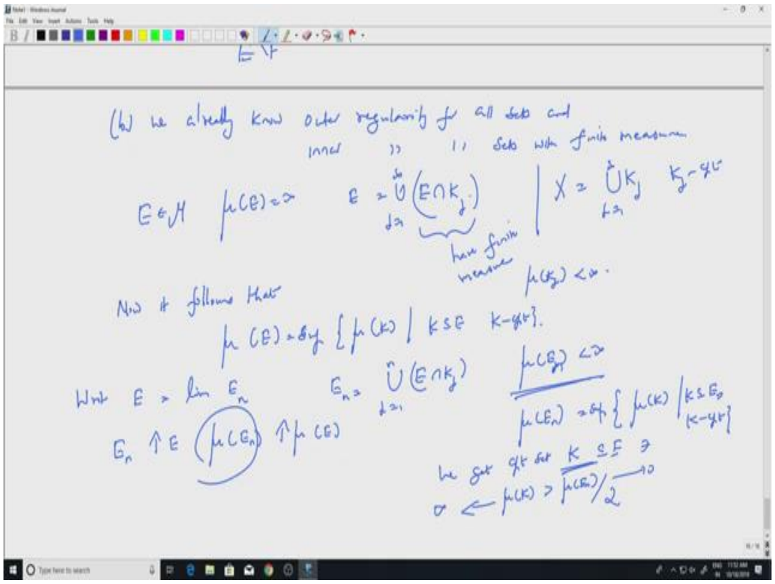

So let us look at B. We already know regularity for we know outer regularity for all sets and inner regularity for sets with finite measure. So, now if I take some set E in M, with the property that mu of E is infinity, I can write E as E intersected with Kj union j equal to 1 to infinity. Because remember my space x is sigma compact and X is written as union Kj where Kj compact. But now these are finite measure, finite measure. Well, why is that? Because mu of Kj is finite. So, then it is easy to see that mu E is supremum of such things because, so now it follows that mu E is the supremum of mu of K where K is contained in E. K is compact. Well, why is that? Let me justify this. So I can write E as so write E as the limit of the increasing union.

So what do I do? So I write E as the limit of let us say En. What is En? En is j equal to 1 to n E intersection Kj. So, all of this have finite measure. Mu of En is finite, but En increased to E, so mu En will increase to mu E, so I am taking mu E to be infinite so mu En goes to infinite. But En have measure finite so mu En for each En is the supremum of mu of K where K is contained in En now and K compact by inner regularity because that has finite measure.

So I can, I have compact sets so we get compact sets K contained in the big set E such that mu of K is greater than mu of En by let us say 2 or something like that but this goes to infinity. So the K is compact set contained in E will have mu of K going to infinity which is same as regularity mu E is infinity right so the supremum will become infinity. So that is, that is fine. So the c part I will leave as an exercise because this is what we did for Lebesgue measure once we have the statement a then that implies C by taking appropriate unions and intersection.

You remember, if you recall the proof you will see that you run epsilon over 2 to the power minus K or 1 by j or something like that and you take the union or intersection of those sets that is precisely what will give you this. So a implies b, a implies c. So just assuming that X is sigma compact we are able to conclude that mu is a regular measure, the other two are sort of standard things but regularity of mu is important.

So you see that, it immediate, so we can conclude that the Lebesgue measure is also regular if we know that the Lebesgue measure comes from the Riesz representation theorem in this setting. So we will do that at the so we will construct Lebesgue measure again using Riesz representation theorem and so we will get all these results like regularity and so on.

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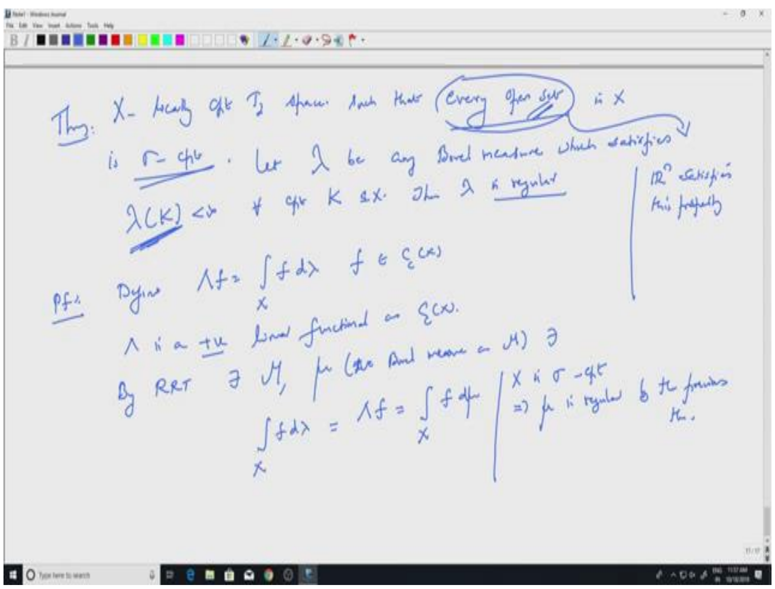

So let me state one more theorem which puts an additional condition on the space so X is again locally compact T2 space, such that every open set in X is sigma compact. So any open set is a countable union of compact sets. Let lambda be any Borel measure which satisfies lambda of K to be finite for every compact K, then lambda is regular. So that is a nice thing to have, so the extra property is that if I have an every open set in X sigma compact then any reasonable, so remember this was our reasonable property which we wanted for measures.

Any reasonable measure is a regular measure so in particular Lebesgue measure is a regular measure. If you know that every open set in Rn is Sigma compact but that is a trivial assertion so Rn satisfies this property. Any open set is the union of close cubes and you can make them compact, so we have seen that. So proof, proof of this usage Riesz representation theorem, so what we do is define so recall this.

This is a reasonable property and that will define a linear functional. So define lambda f equal to integral over X f d lambda. So this makes sense, lambda is a Borel measure, f is a continuous function, it is in C c X. So, it is a linear functional so lambda is a linear functional. Well lambda is a positive linear functional on C c X because lambda is a positive Borel measure.

So by Riesz representation theorem, now this might look a bit confusing but we will, I will explain what this is, by Riesz representation theorem there exist a sigma algebra M and mu, the measure mu positive Borel, positive Borel measure mu on M such that lambda f is given by integration against the measure mu. So measure mu comes from Riesz representation theorem, but we have an additional property here

X is Sigma compact, why, because every open set is sigma compact so X in particular is open set so it is Sigma compact, implies mu is regular by the previous result, by the previous theorem. So will show that, now look at lambda f. Lambda f is defined by X f d lambda. So, now we show that these measures are same. So, we know that mu is regular so we will show that lambda equal to mu. So we know mu is regular from Riesz representation theorem and the consequences of the Riesz representation theorem just like what we did in the previous theorem.

But the measure lambda is given to us and prove that lambda is actually equal to mu, then since mu is regular lambda is regular. So this is also, so this is something which you should notice. So this tells me that the measure is determined by the class of functions f in C c X, if the integrals are same we are saying the measures are same. So, C c X is rich enough to determine the measure. That is what you should understand from this apart from the fact that it is regular. So let us do this quickly.

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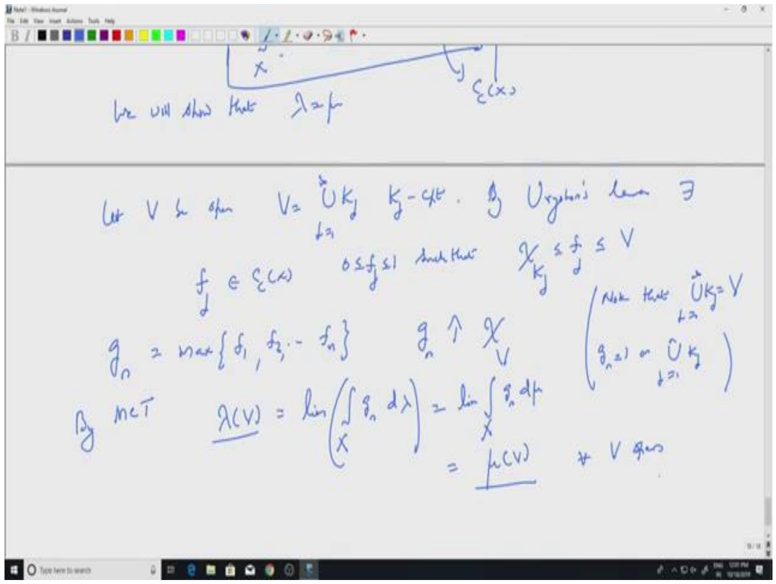

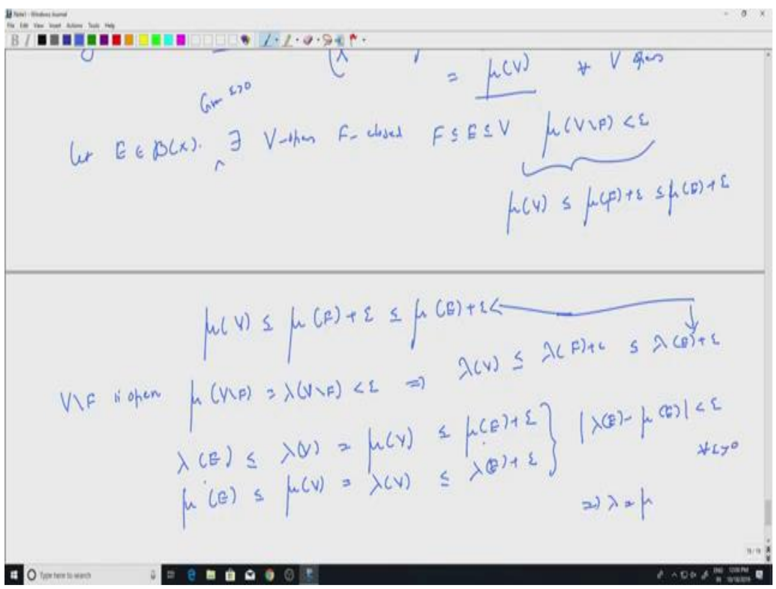

Let V be open then I know that V is union Kj, j equal to 1 to infinity Kj compact because any compact set is a sigma compact set. So by Urysohn's Lemma, so recall Urysohn's Lemma there exist, well we have fj, these are continuous functions and between 0 and 1 such that they are 1 on Kj and they are 0 outside V. Because each Kj contained in V so I have this compact set contained in the open set and I have functions like this by Urysohn's Lemma.

So you take gn to be maximum of the first n ones f1, f2 etcetera, fn. Then gn's are increasing of course, you are taking maximum up to n so gn plus 1 is maximum up to n plus 1 so it is increasing. Well it will increase to indicator function of V because Kj is union of Kj is, so this is because note that union of Kj is equal to V. That is one of the things and gn equal to 1 on union Kj up to n, because you are taking the maximum of 1 over Kj. So you will get 1 there.

So that is the reason. So by Monotone Convergence Theorem lambda V, lambda is a nice measure, so lambda V is the limit of integral over X gn d lambda because gn converge to chi V and so by Monotone Convergence Theorem integral over X gn d lambda will converge to integral over X chi V d lambda which is lambda V. But gn's are in C c X so this integral, inside integral we know is the integra of gn with the measure d mu given by Riesz representation theorem and by this MCT this converges to mu of V.

So what we have proved is lambda V is equal to mu V for every V open. From this we have to go to Borel sets, so let us do that. So let E be a Borel set, so then there exist V open and F closed such that F is contained in E, contained in V and mu of V minus F is less than epsilon. So given epsilon of course, given epsilon positive this is true. Well, what does it say? So this tells me that mu of V is less than or equal to mu of F plus epsilon which is less than equal to mu of E plus epsilon.

So, let me write that again. We have mu of V less than or equal to mu of F plus epsilon less than equal to mu of E plus epsilon. Now so that is one such now, V minus F is open because F is closed V is open, so mu of V minus F will be equal to lambda of V minus F because they agree on open sets. So this will be less than epsilon because mu of V minus F is less than epsilon and so the above inequality holds. So you will have lambda V less than or equal to lambda F plus epsilon less than or equal to lambda E plus epsilon.

So, let us put together these two, this and this put together, you get lambda E less than or equal to lambda V less than or equal to, equal to mu V because V is open and lambda is equal to mu on open sets, less than equal to mu E plus epsilon. So we started with lambda E and we are ending with mu E. Similarly, mu of E I know is less than equal to mu of V but this is equal to lambda of V because on open sets they agree which is less than or equal to lambda E plus epsilon.

So, now if you put together these two you see the distance between lambda E and mu E is less than epsilon. So this is true for every epsilon positive implies lambda equal to mu. So if you have equality on open sets you can go to other sets using appropriate regularity properties. That is what we have proved. So this is a general proof, you can think of it as a general proof. So we will stop here. What we have done so far is to state down state the Riesz representation theorem, we did not prove it.

It is a long proof, I have referred you to the book by Rudin's Real and Complex Analysis. You can see the proof there. We have looked at consequences of the Riesz representation theorem. So, one of them is that on spaces like Rn any reasonable measure is going to be a regular measure. So in particular the Lebesgue measure is regular. We already know that it is outer regular on all sets and inner regular on sets with finite measure but with this now we have proved that it is actually regular on all sets. Anyway it will be completed with the construction of the Lebesgue measure in the next few lectures. So, we will start with the construction in the next lecture.