Measure Theory
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Lecture 27
Construction of Lebesgue measure

So, our aim today is to construct the Lebesgue measure using the Riesz representation theory. So, recall that the Riesz representation theorem tells you that a positive linear functional on the space of continuous functions with compact support on a locally compact has of space x, the place is called ccx.

If I have a positive linear functional on ccx, it is given by an integration against the positive measure. Not just that, so the theorem comes with host of other results that you get a sigma algebra which is bigger than the Borel sigma algebra on x. And there are some regularity properties, which are very similar to what we have seen for Lebesgue measure.

So, what we will do is, we will use continuous functions with compact support on R k to define a linear functional. This is actually the Riemann integral of the function and that will give us a measure which is the Lebesgue measure. And certain regularity properties immediately will follow because R k has a property that every open set in R k is a sigma compact set on such spaces. If you have a measure which is reasonable in the sense that it has finite measure on compact sets, then it is automatically regular we have already done that. So, some of those properties we will follow immediately. So, let us start.

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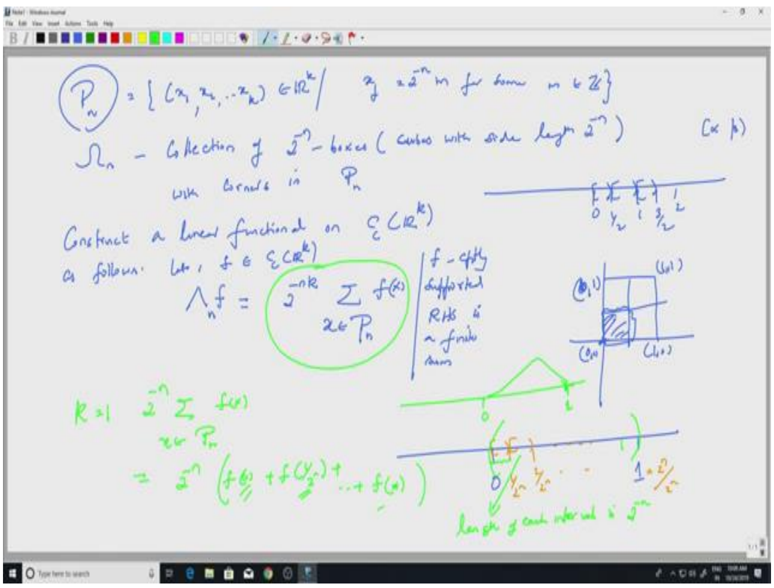

So, I recall some notation. So, remember the set P n, P n was the set of all those points in R k, x 1, x 2, et cetera x k in R k where each coordinate was to the minus n times in integer. So, each x j was 2 to the minus n times m for some m in integers. And we had omega n to be the collection of, the collection of 2 to the minus n boxes, so remember these were cubes with side length 2 to the minus n, remember that was a product of intervals of the type alpha beta. It was open on the right hand side remember that, 2 to the minus n boxes with corners in Pn.

So, in the real line case, we have 1 here, 2 here and so on, you divide this into half and so on and we are looking at intervals of the chi. Similarly, for R 2 and R 3 and so on, so we had 0, 0, 1, 0, 1, 1 and 1, 0 or 0, 1 and then we divide this into 4 equal parts and this area, which is open on the, this fourth portion is not there. So, such things are called cubes to with corner in coordinates in Pn.

So, we use this to construct a linear functional, so construct a linear functional on cc of R k as follows. Well, how do we do this? This we do this at nth level and then take limits. So, let lambda n of f, so f, I am taking f in cc of R k. So, it is a continuous function with compact support, lambda n f is simply 2 to the minus n k.

So, 2 to the minus k is the volume of 2 to the minus k box times the sum of x in P n, so we use a different P, so P n, so P n is this collection of numbers or coordinates and you sum up f of x. This makes sense. So, remember f is compactly supported, compactly supported, so RHS is a finite sum, RHS is a finite sum. But what exactly is this? Well this is actually a Riemann sum. So, let us look at one particular case in the real line, so that this is much clearer.

So, I take 0, 1, so let me write the unit interval to be slightly big and then you go to nth level, so then I have 1 by 2 to the n, 2 by 2 to the n, et cetera et cetera, 1 is 2 to the n by 2 to the n something like this. And you are looking at intervals of this form et cetera et cetera. What are we doing? We are, so let us take a take a function which is compactly supported inside 0, 1, so I take 0 and 1 and I have some function here.

So, let us say it is supported inside 0, 1 and we are forming this sum. So, here the k is 1 because the dimension is 1. So, I am simply looking at 2 to the minus n times sum over x in P n f of x. What is this? So, let us (())(6:57), so we are looking at k equal to 1. And we are looking at 2 to the minus n summation x over P n f of x, however we do not have to sum over all the points in P n because f x supported in 0, 1.

So, we will look at only this area this interval. So what is this, this is equal to 2 to the minus n times, well, I have f 0 that is a point in, 0 is a point in P n, then I have f of 1 by 2 to the n plus et cetera, et cetera, f of 2 to the n minus 1, sorry 2 to the n by n, 2 to the n when f of 1 maybe f of 1 is not included the one before that is included.

So, let us not write this point. I will just put a star to indicate that we are choosing the point here. Alright, so, what exactly is this? So, you look at each interval, each interval has length 2 to the n. So here, length of each interval is 2 to the minus n, so you are multiplying by the length of the interval and the value of f inside that interval. So, that is a Riemann sum.

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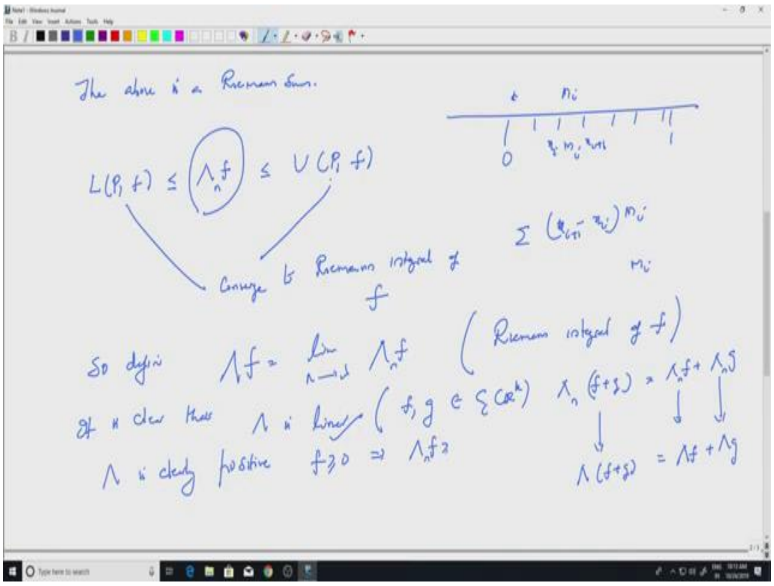

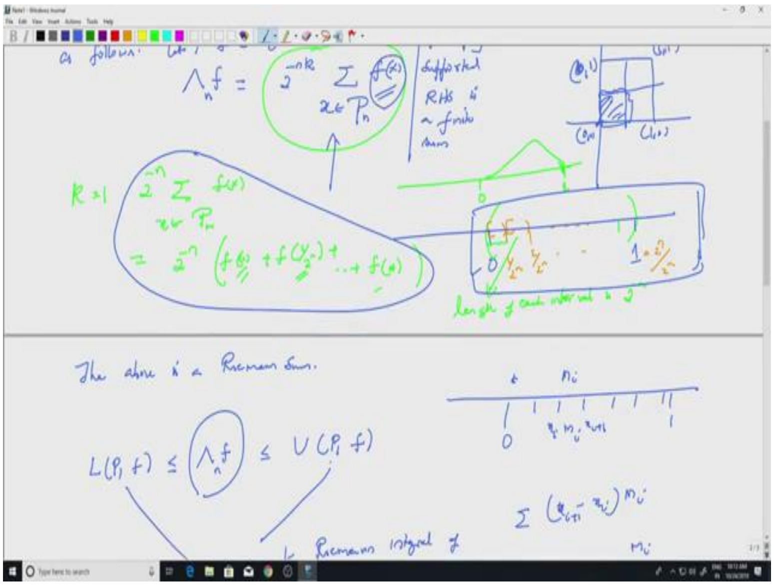

So, the above is a Riemann sum. So, in Riemann integral remember, we divide this into intervals. And then we choose, so ith interval, we choose the supremum and the infimum of the function and then multiply by the length of the interval the value of either the supremum or the infimum.

And this sum we have seen here will be something in between. So, the lambda n f is simply bigger than the lower sum and less than or equal to the upper sum in the case of real line and the similar definitions for R k will make sense, what is P? P is partition of the interval we have here.

So, and as length of P goes to 0, we know that this will converge to, so this and this will converge to the Riemann integral of f because f is continuous compactly supported. So Riemann integral x is it will converge. So, similarly the limit of this will exist and that is precise, and that is also the Riemann integral of f, that is the definition of our linear function.

So, define lambda f to be equal to limit n going to infinity, lambda n f. So, this is precisely the Riemann integral of f. We are not going to write it that way, but this is what happens. This is true in R k as well in higher dimension as well, if you define the Riemann integral using the boxes instead of intervals.

So, this is now it is clear, it is clear that lambda f, lambda is linear. Well, why is that? Because if I take f and g in cc of R k then lambda n of f plus g. So, what does lambda n f plus g? This is the sum, this is the sum the sum and if it is f plus g here, then it becomes some of 2 things and so, lambda n of f plus g is lambda n f plus lambda n g.

So, if you take limits, these 2 things will go to lambda of f plus g that is a definition of lambda and lambda f here, lambda g here and so on, if you to multiply by a constant, the constant will come up. So, lambda is linear, lambda is clearly positive, clearly positive, why? Because if f is positive because f is greater or equal to 0 implies each lambda n f is greater or equal to 0. Because you are adding the values of a positive function here these things are positive so, when you add them, you will get positive numbers and so it is a positive linear function.

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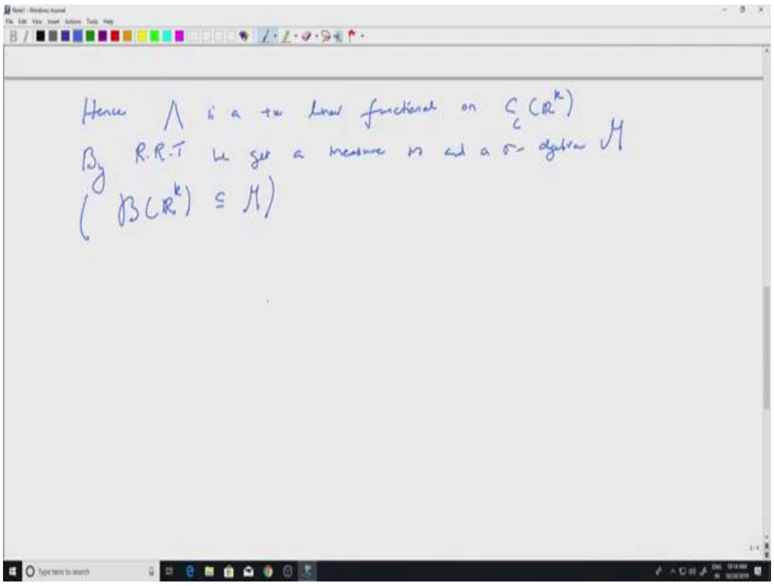

So hence, so, to conclude what we have done is, lambda is a positive linear functional because it takes values in the real line or in the complex plane depending on where f takes values and as a positive linear functional on the space cc of R k. So, remember R k is locally compact (())(12:51) et cetera. So, Riesz representation theorem applies.

Hence by Riesz representation theorem. So RRT is representation theorem. We get a measure which we call m and a sigma algebra sigma algebra script M. So, recall that the Sigma algebra script M is such that the Borel sigma algebra is contained in it that is that comes from the theorem and we have some regularity properties. So, now we prove all the properties which we had stated in the last lectures.

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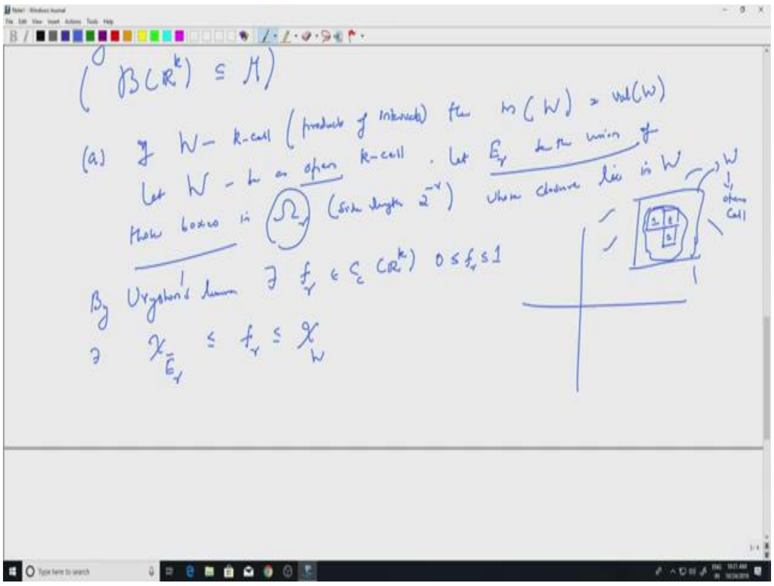

So, we will start from a. So, the first assertion a, so I will write down what it is and then we will, so a the first assertion was that, if W is a k cell, if W is a k cell, so k cell was remember, is the product of intervals rectangle in earlier notation it is productive of interval. Then the measure of the W is the volume of W. So, that is the first thing we want the prove. So, let us proof that. So, we start with an open set. Let W be an open k cell. So, it is a product of open intervals. Let E r, so we have to use the definition of the positive linear functional to conclude this.

So, let E r be the union of those boxes in omega r, so remember the side length here was 2 to the minus r whose closure, whose closure lie in W. So, let E r be the union of those boxes in omega r. So, remember that side length is 2 to the minus r whose closure lie in W. So, we are looking at some W like this and E r is a union of boxes in omega r.

So, omega r gives me a partition of the space with cubes whose corners are in certain prescribed set which we call P r. And this will give me some, so we are looking at the union of boxes in omega r. So, we are looking at some boxes like this. Not the ones which go out but whose closure lies inside.

So, remember the W is an open cell open cell, so the boundary is not included. So, when I take the closure, I do not want to touch the boundary, so things which are strictly inside. So, now by Urysohn's lemma, so by Urysohn's lemma there exist f sub r, which are continuous functions with compact support and they are between 0 and 1.

So, these are all part of Urysohn's lemma statement, you already know that such that chi E bar closure is less than or equal to f r less than or equal to chi W which means that on E r it is 1, outside W it is 0. So, I am looking at something which is 1 on these boxes and maybe on a slightly bigger set but it is 0 outside W. So, these are functions given to us by Urysohn's lemma

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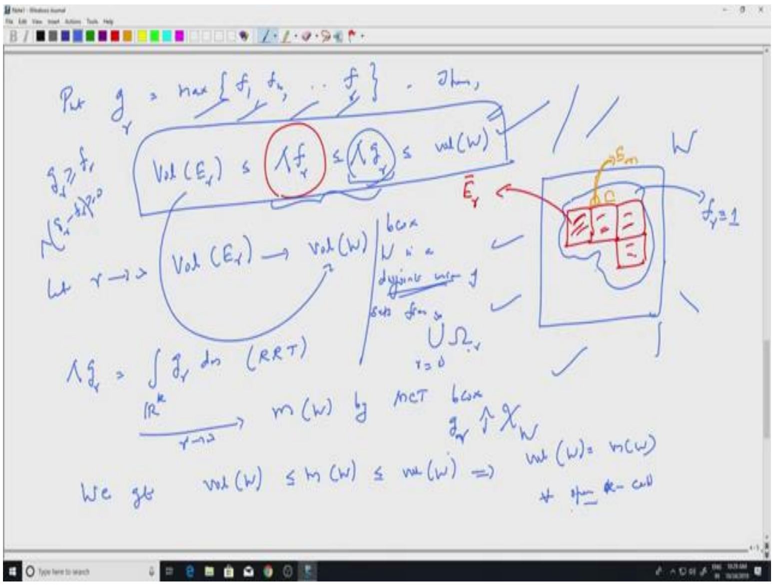

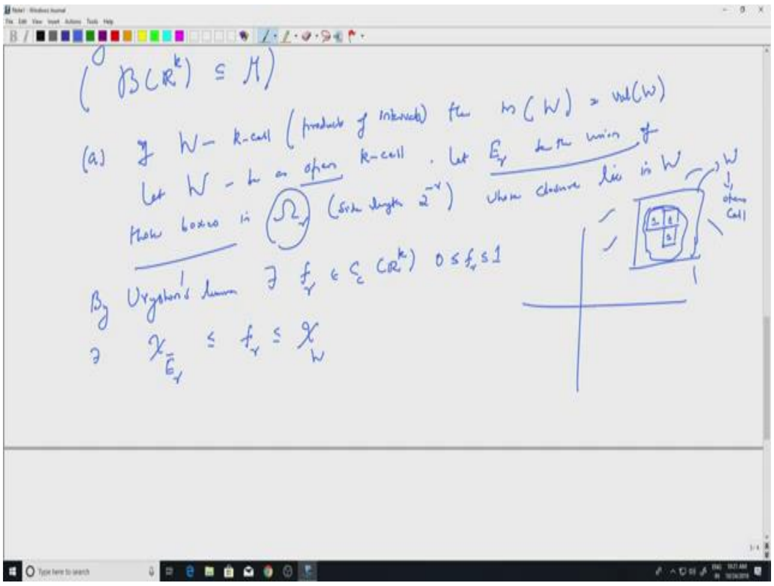

So, put g is a bar equal to maximum of f 1, f 2, et cetera up to f r for each r, I have f sub r I am taking the maximum up to r, f sub r that I called g r. So, then volume of E r, is less than or equal to lambda of f r less than or equal to lambda of g r less than or equal to volume of W. So, let us try to understand what this is? Volume of E r is simply the sum of the volume of the each boxes because they are disjoint. So, you simply add them. Now, so let me draw one more picture to make this clear. So, let us say this is my W. This is my W, mu is another color for the boxes inside.

So, I have some boxes here. They are disjoint because you are looking at product of left hand clothes and right hand open sets, so they are disjoint and that this is what forms if I take the closure, I will get Er closure. And well, what is f r? f r would be something which is 1 on all this set, so it may be 1 on a slightly bigger set as well.

So, f r is something which is 1 here equal to 1 here and f r is of course 0 outside W, it is a compactly supported function. Alright. So, now it is clear that, so what is volume of E r? Well, volume of E r will be the sum of those things, some of the boxes. So, you look at these boxes, you add the volume of each of them. That is same as putting 1 here and calculating lambda f r that is the volume.

But it is 1 f r is 1 on a slightly bigger set. So, there are more things you are picking up. So, this is obviously bigger than volume of E r. And g r is of course, greater than or equal to, g r is greater than or equal to f r. So, g r minus f r is positive. So, lambda of this is also positive and so, we have this inequality and the other one is clear because g r is 1 only on the union of sets coming from E 1 to E r after that it can be less than 1.

So, volume of W will be much bigger than lambda of g r. So, this inequality is easy to see, but that is very crucial for us in proving that measure of W is same as the volume of W. So, let, now let r go to infinity. Let r go to infinity, well what will happen to volume of E sub r? Well, this will have to go to volume of W. Well, why is that? So, let us recall what E r is.

E r is the union of all the 2 to the minus r boxes inside W, remember that W is a disjoint union of boxes from union of omega. So, this is because, so because W is a disjoint because any open set is a disjoint union of sets from union omega r or equal to say 1 to infinity, or 0 to infinity.

So, E r as r goes to infinity, so this may be E r and when you go to E r plus 1 you will pick up more smaller ones and so on, so forth. So, I can call this E r plus 1 and so on and so forth. So, you will fill up other places by taking r's bigger and bigger, because the side length becomes smaller and smaller and W is finally the union of those things.

So, this follows, so on the left hand side, so if you keep this in mind, on the left hand side this I know converges to volume of W. And lambda g r, so what happens to lambda g r? Well lambda g r is integral of g r dm by this is by the Riesz representation theorem. We have the measure m and g r is a compactly supported continuous function.

So, it is measurable and it, and lambda is given by that measure m. So, this is the representation theorem and this converges to as r goes to infinity we will go to measure of W by monotone convergence theorem. Why is that? Because g r increases to the indicator function of W.

Well let us see why, what is g r? g r is maximum of f 1, f 2, f r and so on. So, gr, so remember f 1 would be 1 on 1 boxes, f 2 will be 1 on half boxes and f 3 will be 1 or 2 to the minus 2 boxes and so on and so forth. So, when I take g r, g r is the maximum of those things. So, it will be 1 on a large number of boxes completely inside W. And W is disjoint union of sets from this, so as r becomes bigger and bigger, g r becomes 1 on a larger and larger set, which finally becomes W finally converges to W and so g r will converge to the indicator of W.

So, by monotone convergence theorem, lambda g r will converge to m W. So, from here we get, so, let me right here itself. So, we get volume of W is less than or equal to measure of W because the middle term goes to measure of less than equal to volume of W because the other, this side there is no dependence on r. So, this implies that volume of W is same as measure of W. So, for every open cell, every open k cell. But this is true for all cells, why?

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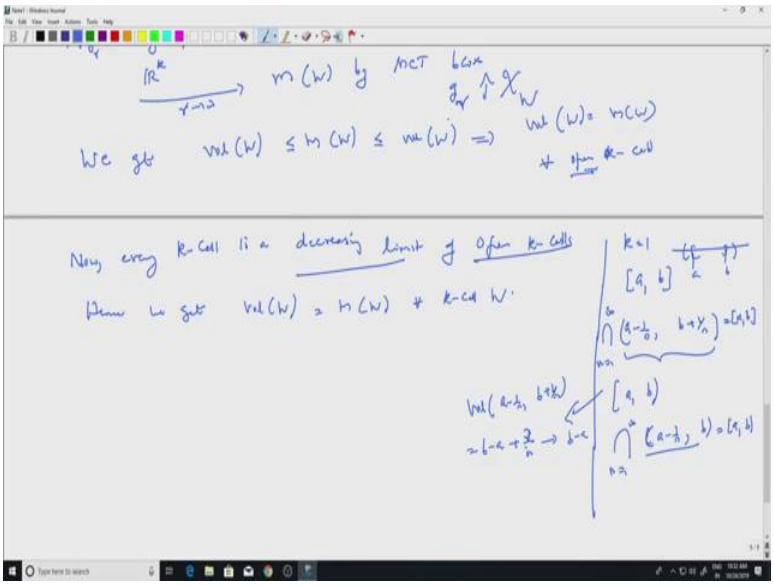

Because any now, every k cell is a decreasing limit of open k cell, the decreasing limit of open k cells. So, what does that mean? For example, if I take for k equal to 1 dimension 1, if I take cell like this, let us say closed interval and this is of course, the intersection of, this is the intersection of, you look at intervals of this form.

So, you can look at a minus 1 by n, b plus 1 by n, you take the open interval and then take the intersection, this is equal to the close interval a, b. But these are open k cells they are open intervals. Similarly, if I have one side open, well then what should I do? I will look at intersection n equal to 1 to infinity. Open at a minus 1 by n and open at b.

So, these are open cells and intersection would be equal to close at a open at. So, if I know the measure of these things since is a decreasing sequence and the measures are finite, I can apply the theorem we know, if I have an decreasing sequence of sets m, E, n will converge to m E if E n converges to E or E n decreases to E provided one of them has finite measure.

So, that you apply. So, hence we get volume of W, so that also of course will converge, volume of W will become, so it is like volume of a n, so let us look at this again, volume of a minus 1 by n to be plus 1 by n. This is nothing but b minus a plus 2 by n and of course that converges to b minus a whatever the interval is, which is the volume of that interval.

So, we get volume of W equal to m W, well m W because m is a measure and if you have a decreasing sequence of sets you know it converges, so volume of W equal to m W for every k cell W. So, that proves one assertion. So, let me write down the second one, this we already know.

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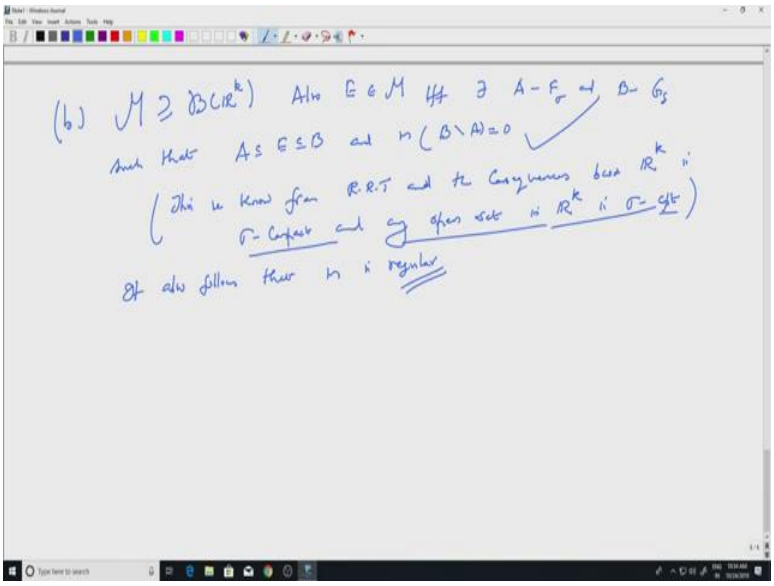

So, b the assertion b was the Sigma algebra M we got contains the Borel sigma algebra so this follows from Riesz representation statement of Riesz representation theorem itself. Also, a set belongs to M if and only if there exist a set A which is f sigma and a set B which is G delta such that A is contained in E contained in B and M of B minus A is 0. So, this we already know.

So, this follows from this we know from the Riesz representation theorem and the consequences because R k is sigma compact, R k is sigma compact and any open set in R k is also sigma compact in R k is in R k is sigma compact. So, if we have the second assertion that any open set in R k is sigma compact, then the consequences of the Riesz representation theorem, which we looked at the final properties will also be true.

So, this follows immediately from whatever we already know more importantly, it also follows that, it also follows that and m is regular. So, Riesz representation theorem gives you outer regularity and inner regularity for sets with finite measure, but that we have been able to change if the spaces sigma (comp) any open set in R k is sigma compact. So, we can use the second consequence of the Riesz representation theorem to say that any reasonable measure is regular.

So, this directly follows from Riesz representation. So, let us stop here, we have just constructed the Lebesgue measure using the Riesz representation theorem by constructing a positive linear functional on cc of R k, what we have just proved tells us that it is actually the Sigma algebra we obtained from Riesz representations theorem is actually the Lebesgue sigma algebra, because it is the sets in the Sigma algebra differ from Borel sets by a set of measure 0. So, it is the completion and it equals the volume of each k cell. So, we have equality there.

Now, we will look at other properties that it is translation invariant and we know that this is the unique, well we will also prove that this is the unique translation invariant measure on R k which gives finite measures to compact sets. And we will see how it in how the linear transformation affects the measure.

So, remember a constant will come out that constant is actually the determinant which we will prove later on, right now we will prove that if you take a linear transformation Lebesgue set will be mapped into Lebesgue set and its measure is a constant times the measure of the original set.