Measure Theory

Professor E. K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru

Lecture 28

Invariance properties of Lebesgue measure

Alright, so we will continue with the properties of the measure we have just constructed, it is actually the lebesgue measure. We have seen the equality in the case of k cells, but since it is regular you know the equality will follow for all other sets as well. So, that part I will leave it to you. But let us prove a uniqueness property of the lebesgue measure.

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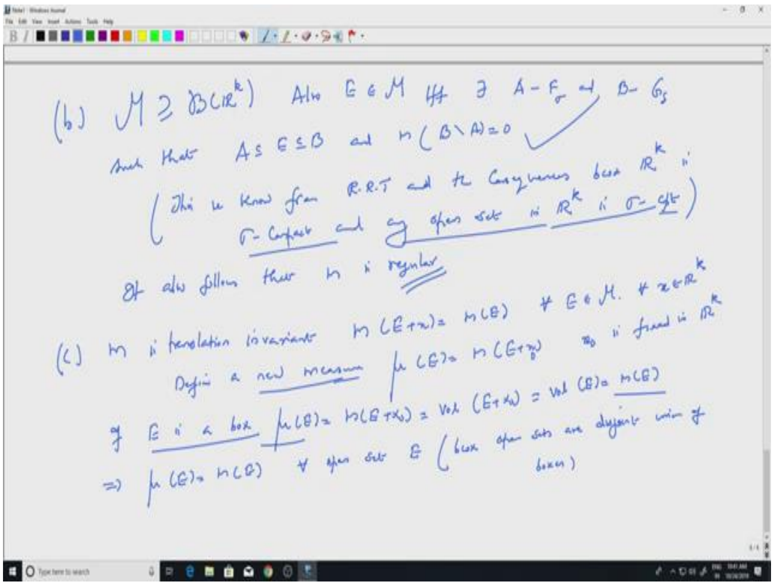

So, that is the property c, which we had written down in the last lecture. So, m is the measure which we constructed is translation invariant, translation invariant. So, we know this from the previous construction but we will prove this for every E in, so this follows immediately from the regularity properties.

So, let me, so what do we do is define a new measure, define a new measure, new measure let us call that mu, mu of E equal to m of E plus x because it is true for every x in r k. So, you fix x in r k. So, let us say x naught. So, x naught is fixed in r k. So, some fixed point I want to say mu is same as the lebesgue measure we constructed m.

So, we have, so if E is a box, if E is a box mu of E by definition is m of E plus x naught but this I know is the volume of E plus x naught because it is a box. But volume is translation invariant, you translate a box, it we still have the same volume. So, it is volume of E, which we know is m of E, so mu of E is m of E, if E is a box.

So, this implies mu of E is same as m of E for every open set E, for every open set E because open sets are disjoint union of boxes and countable additivity, disjoint union of boxes. So, here we are using the fact that mu is a measure but that is trivial to see, because mu is simply m of E plus x naught. And so, countable additivity immediately follow from them. So, we have mu of E equal to m of E for every open set.

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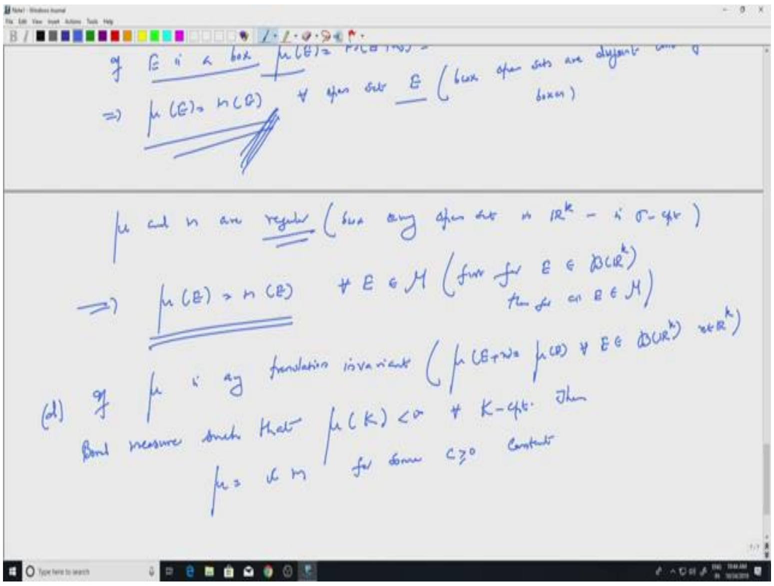

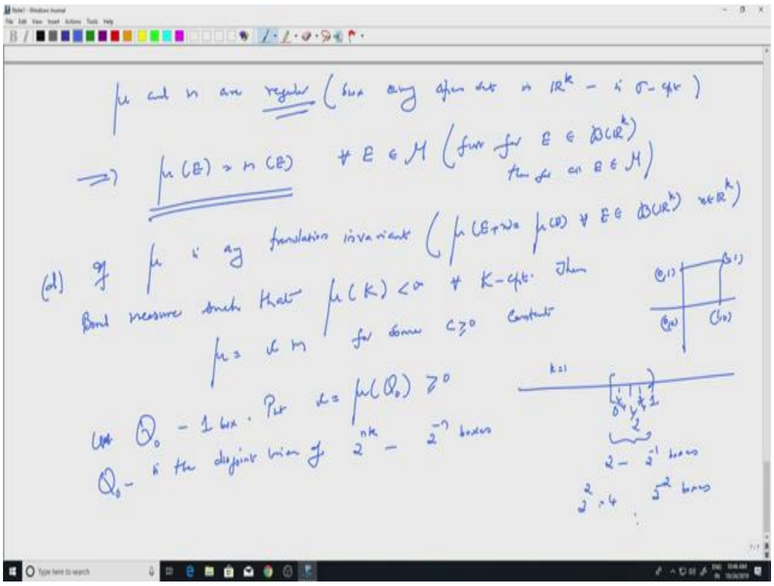

Now, mu and m both are regular because r k is every open set in R k because every open set in R k is sigma compact. And the consequence of Riesz representation theorem both mu and m are reasonable measures in the sense that for compact sets, you get finite measure. So, once you know equality for open sets and regularity this would imply that mu E equal to m E for every borel for every E in m. In fact first for, first for E in borel sigma algebra and then for all set, then for all E in M because every E in M is if just the borel set and a set of measure 0, so because of that, so this is something which we had done earlier.

Remember we proved 2 measures (())(4:52) and proving regularity was essentially this argument that if to two measures agree on open sets then they agree everywhere with regularity first. So, let us go to the next property d. So, this was if mu is any translation invariant, translation invariant, so what does that mean? Mu of E plus x equal to mu of E for every borel set E and x points in R k. So, any translation invariant borel measure, borel measure such that mu of k, so it is a it has to be a reasonable measure, mu of k is finite for every k compact.

Then it is the lebesgue measure then mu equal to some constant c times the lebesgue measure for some constant c, some positive constant c constant. So, we have only Lebesgue measure, which is a nice translation in this. So, it is a uniqueness property of the Lebesgue measures that any measure which is translation invariant is actually a multiple of the Lebesgue measures.

So, let us, how do you prove this? Take a Q naught be a one box, so one box would be, so let us take 0, 1 something like this. In the R in R 2 it will be the unit box here 0, 0, 1, 0, 1, 1, and 0, 1. Put the constant c to be mu Q naught, so this is a number. So, the positive number or a non-negative number, but Q naught is the disjoint union of, disjoint union of 2 to the n k, 2 to the n k, 2 to the minus n boxes.

So, what do I mean by that? So, if I look at k equal to 1, so this is k equal to 1. In the next level I will have 2 boxes. So, I have 2 to the minus 1 boxes. If I again bifurcate them, I will have 4 of them, 4 that is 2 square, 2 to the minus 2 boxes et cetera, et cetera they are all disjoint. Since they are disjoint they will the measure will add up.

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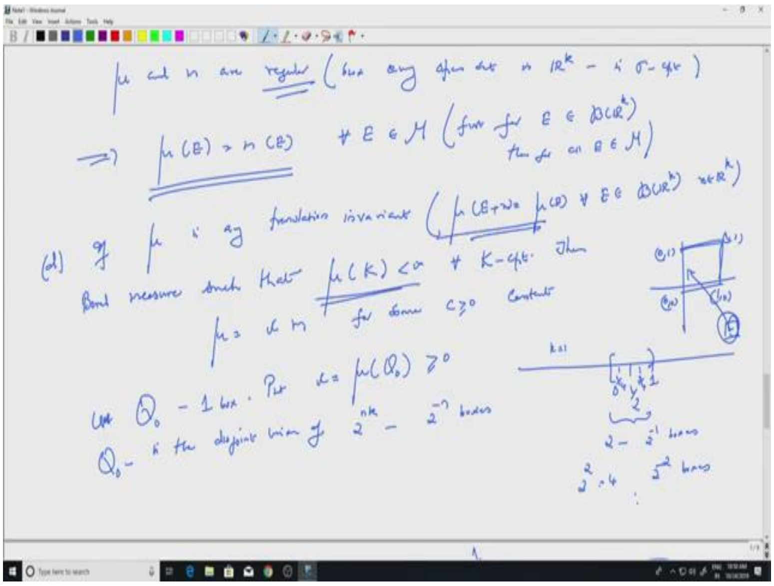

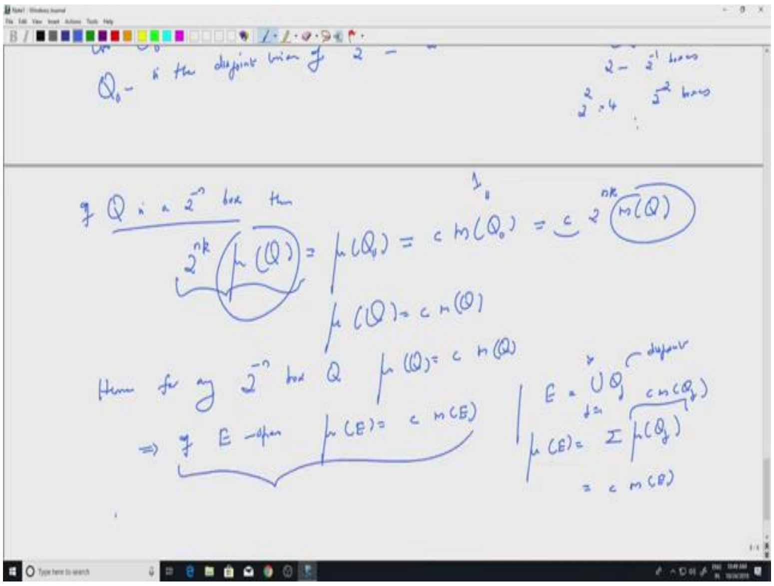

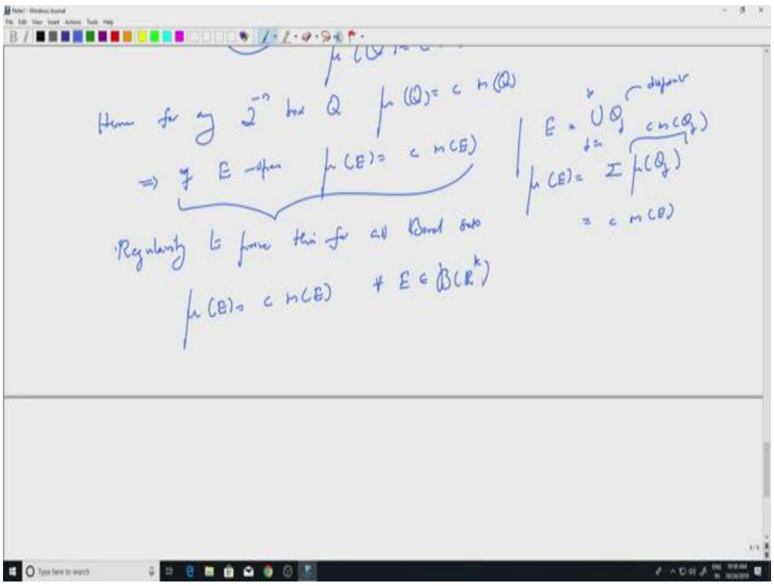

So, if I look at only Q naught and I do this, I will get if Q is a 2 to the minus n box, then 2 to the n k times mu of k equal to mu of Q naught. Remember the Q, Q can be a 2 to the minus n box anywhere you want, you can translate it and bring it inside Q naught because mu is translation invariant.

So, if the box, if the Q, 2 to the minus n boxes here, you can bring it inside by translation and the measure do not change. But the Q naught is 2 to the n k disjoint boxes of this kind. So, the measure will add up. So, this is simply, so this is simply writing Q naught as disjoint, 2 to the n k disjoint 2 to the minus n boxes.

But mu of Q naught is c times, it is c, the constant C times m of Q naught because m of Q naught is 1. This is the one box. Its volume is 1 but m is a measure, so the same computation tells me that this is 2 to the n k times any same box Q. I write Q naught as the 2 to the n disjoint, 2 to the minus n boxes.

So, this tells me that mu of Q is c times m of Q. So, mu of Q equal to c times m of Q, what did we proved? We proved that for any 2 to the minus n box. Hence, for any 2 to the minus n box Q, we have mu of Q equal to c times m of Q. So, this immediately implies that, if E is open then mu of E is equal to c times m of E. Why is that? Because E can be written as, so E can be written as union of boxes, countable disjoint union of boxes.

So, mu of E is the sum of mu of Q j because they are disjoint, but for these are boxes. For boxes, we know how it acts. So, this is c times m of Q j. But m is a measure and so it adds up. So, this is just c of c E. But now, we know what to do because of regularity, so from here, we simply use regularity to produce for all borel sets, for all borel sets.

So, we will have mu of E equal to c times m of E for every borel set and so for every Lebesgue set also, but that is not important here. So, the only translation invariant measure on the real line or r k with this property is the Lebasque measure. So, that is the uniqueness property of the Lebesgue measure. And so, we have one more property how it behaves with the linear transformations.

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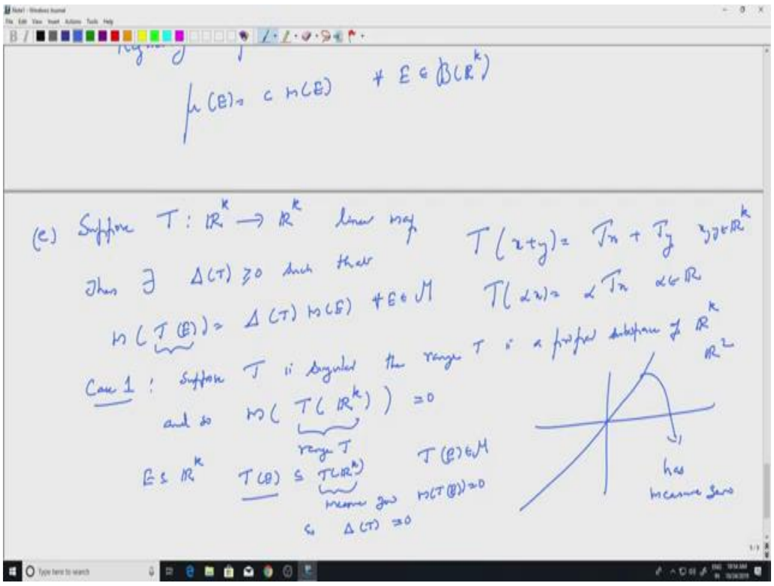

So, let us write down that suppose, I have a linear transformation, suppose T is a linear transformation from r k to r k it is a linear map. So, recall what is the linear map T of x plus y is equal to Tx plus Ty and T of alpha x equal to alpha times Tx where alpha is a real number, x and y are in r k.

So, that is the statement E then assertion is that there exist some number depending only on T greater than or equal to 0 such that measure of T E equal to constant delta T times measure of E for every E in m, how will you prove this? So, there are two things here, one is I need to know what is the property of T E, so that I can write m of T E. So, m of T E will make sense only if T E is measurable.

So, let us do this in two cases. So, case 1, suppose the singular, suppose T is singular that means determinant of T E is 0. Then well, by from linear algebra we know that range of T is a proper subspace of proper subspace of R k. And so, measure of T of R k. This is the range of T equal to 0 because it is contained in a proper subspace of R k. So, let us recall that in R 2, if I take a subspace proper subspace that will be either 0 or (())(14:49) and this has measure 0. This line has measure 0, we did this. So, similarly, any proper subspace of R k will have measure 0.

So, if I take any E which is an R k, then T of E will be contained in T of R k, which has measure 0, this has measure 0. So, it is a subset of a set of measure 0 by completeness T of E will also be in M and M of T E is 0. So, delta T, so delta T in this case is 0, this is actually, this is going to be the determinant which we will prove.

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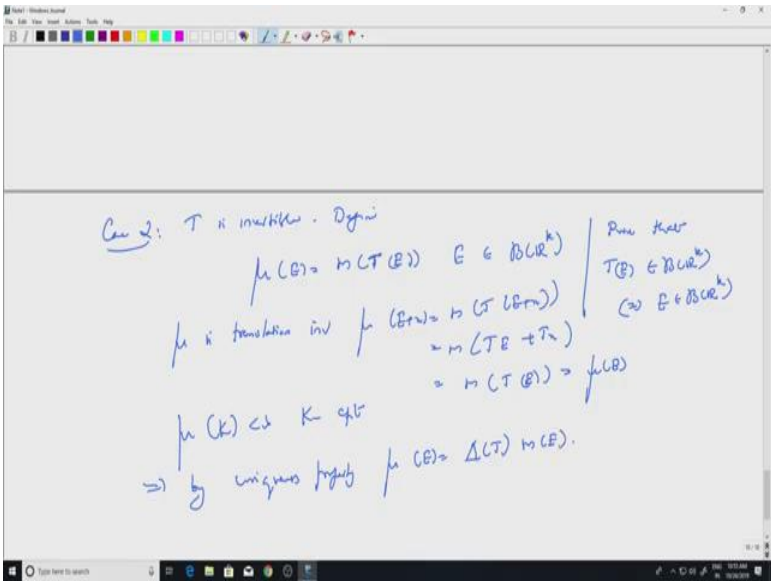

So, we will, let us do the case 2 as well. So, case 2, T is invertible, invertible. So, you define, define a new measure mu of E equal to m of T. So, I will for E in borel sigma algebra of R k. So, here you have to prove that T E is a borel set, if and only E is a borel set. So, this is easy because T is continuous and T inverse is also continuous.

Now, mu is translation invariant because mu of E plus x equal to m of T of E plus x equal to m of T E plus Tx because T is linear, which is m of T E which is mu of E, so it is translation invariant. And mu of k is finite for every k compact that is trivial, because T of k will be compact.

And so, by uniqueness property, uniqueness property, mu of E will have to be equal to some constant which we call delta T times m of E, this is all we wanted to prove. So, we will stop here. So, we just looked at some more properties of the Lebesgue measure, how it behaves with the linear transformation, more importantly the uniqueness of the Lebesgue measure with respect to translation invariants.

So, we will continue this, we will look at more properties of the Lebesgue measure and Lebesgue measurable functions in the coming lectures. More importantly, we will construct or at least explain why there are Lebesgue sets which are not borel. So, we have seen an example of a set which is non-measurable, non Lebesgue measurable, but we will see examples of Lebesgue sets which are not borel sets.