Measure Theory
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Lecture 3 - Measurable functions and approximation by simple functions

So, in the last lecture we saw the concept of measurability. We will recall that and go onto look at some properties of measurable functions. The important result we will look at today is that any measurable function, any positive measurable function can be approximated by simple functions which we will define later on. So, let us recall measurability.

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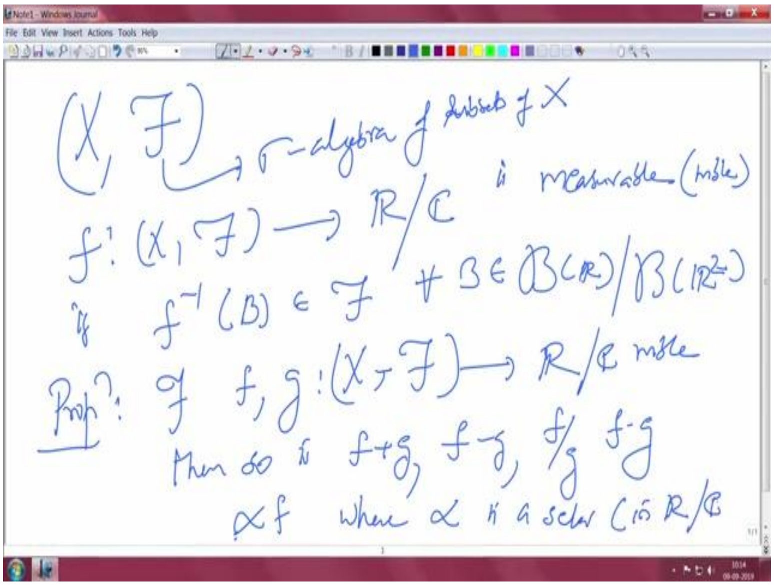

So, recall that we have a set X, we have a sigma algebra, so, this is a sigma algebra of subsets of X and the function f from X to R or C complex plane is measurable. So, we will denote the abbreviation mble measurable if f inverse of any Borel set belongs to our sigma algebra for every B in B of R or B of C. So, we saw this yesterday in the last lecture. Let us look at some easy properties of this. One is if, let us write it as a preposition so that I can prove some of these things for you. If f and g are measurable functions, measurable then so is f plus g, f minus g, f by g, f times g and alpha times f, where alpha is a scalar. So this can be either a real number or a complex number.

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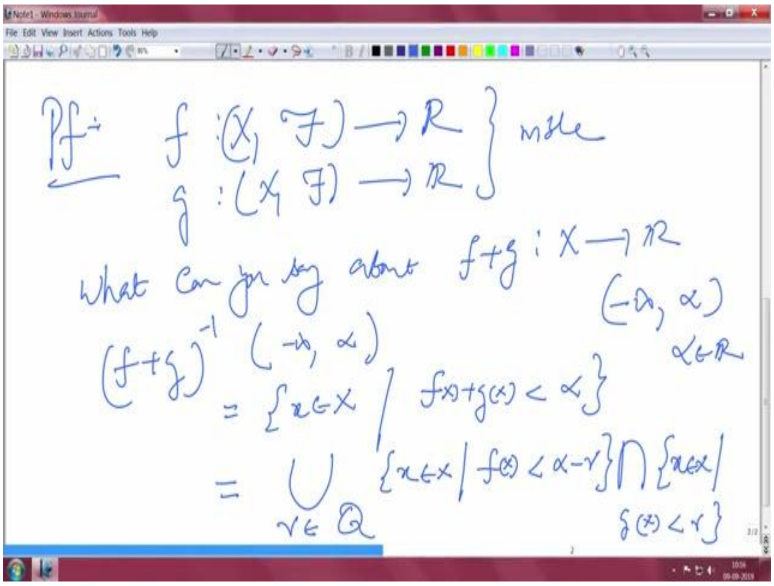

So, we will prove this, proof of this is rather easy, so let us look at just one of them. Suppose I know that f is measurable. So this goes from hope X f to let us say R, same proof will work for complex numbers, and I have g from X to the real line, both are measurable. I want to prove that f plus g. What can you say about f plus g? So f plus g is a function from X to the real line, we want to check if it is measurable. Well, what do we do, we take inverse image of certain Borel sets and see if we land in f. So recall that we do not have to look at all Borel sets in the real line, we need to look at the only certain sets.

So we will look at sets of the form, let us say minus infinity to alpha, where alpha is a real number. So you look at f plus g and look at the inverse image of the set minus infinity to alpha where alpha is a real number. Well, what is this? This is all those points in X such that f plus g at x, so, that is f x plus g x belongs to the set which is same as saying this is less than alpha, correct? Well, I can write this as union over R in rationals x such that f x is less than alpha minus r intersection x such that g of x is less than r.

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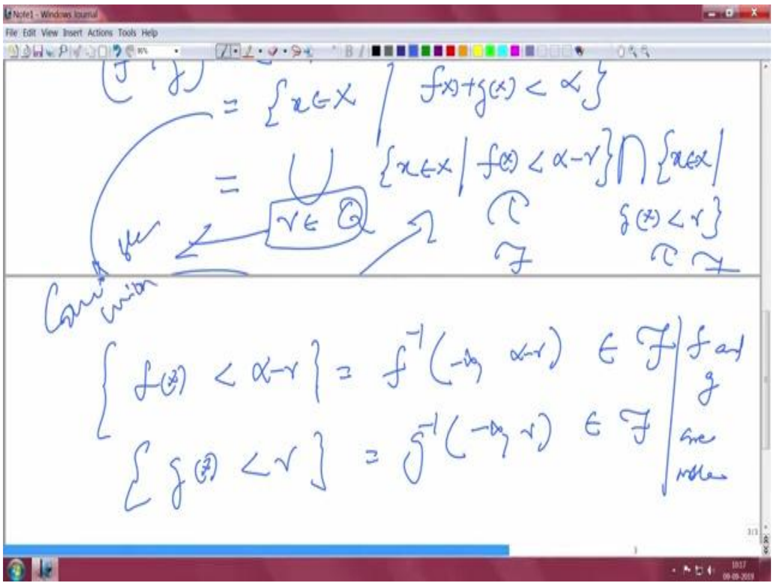

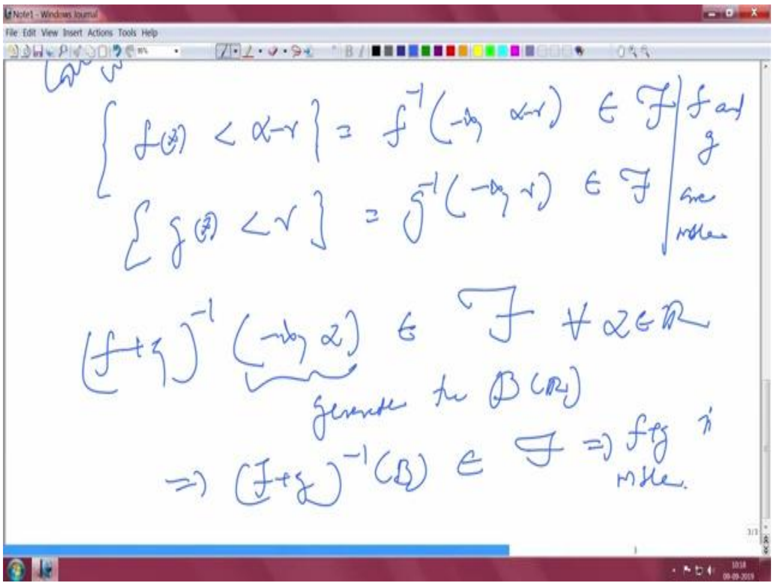

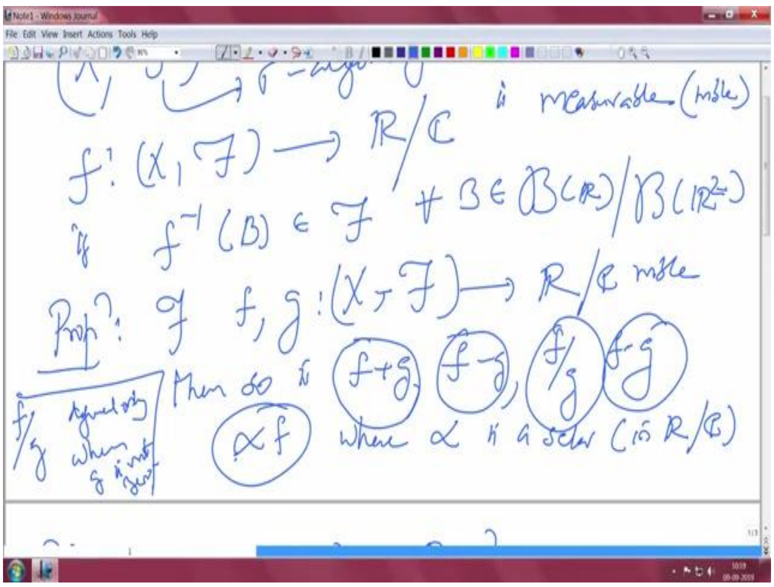

So, if I do this the set where f of x is less than alpha minus are, what is this? This is simply f inverse of minus infinity to alpha minus r. So, this belongs to script f. Similarly, the set where g of x is less than r, this is equal to g inverse of minus infinity to r and also belongs to script f because f and g are measurable. So, now, we go back to the previous duration here, this set is written as a union, union of measureable sets because this belongs to script f, this set also belongs to script f and so the intersection belongs to script f because script f is a sigma algebra and when I am taking union over countable union, union over rational, so this is a countable union, so that will also belong to script f.

So, what we have proved just now is that if I look at f plus g and I look at the inverse image of minus infinity to alpha, then I belong to script f. This is true for every alpha in the real line, but then sets of this form generate the Borel sigma algebra that we have seen. So, this implies that f plus g inverse of any Borel set, so this is the good sets principle we did in the last lecture. If I know that inverse image of collection of sets which generate the Borel sigma algebra is in script f, then it is true for all sets in the Borel sigma algebra.

So, this will belong to script f which implies that f plus g is measurable. So now if you go back to the statement of the procession, all this, so we just proved this for f plus g. Well, same thing can be proved for f minus g or you start with alpha g alpha f, take alpha to be minus 1, you will get this U C f plus g and same proof works for f times g and f by g and so on. You have to be careful about f by g, f by g is well defined only when g is nonzero. So, this is defined only when g is not 0. If g is 0 then we can modify the definition but we will come to that later. So, the collection of measurable functions is a vector space, so that is what we have just proved.

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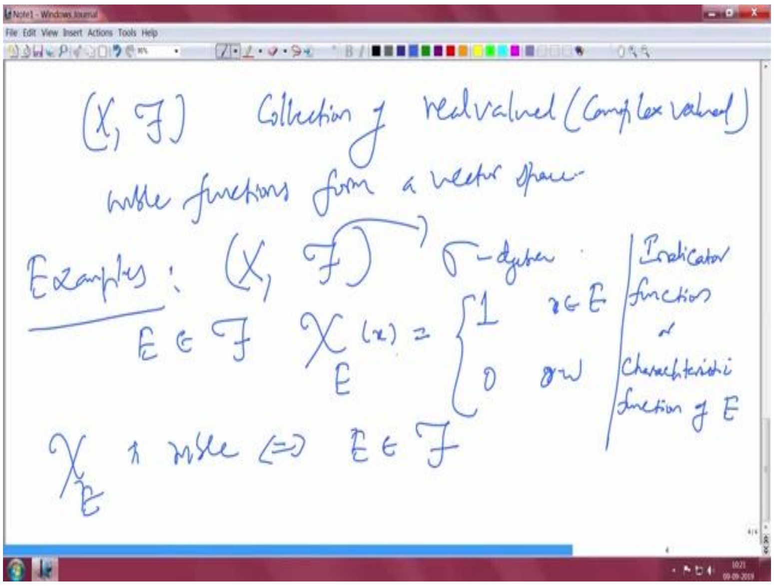

If I take X f and you look at the collection of collection of real valued or complex valued measurable functions form a vector space and vector space is not important, but we will keep using all these properties of measurable functions. So, let us look at some examples. So, I have a space X and I have a space F, I have the sigma algebra F. Take a set in the Sigma algebra and define Chi E, Chi E is called the integrated function or characteristic function of E. So, this is 1 if x is in E; 0 otherwise, this is called the integrated function or characteristic function of E. That is the definition, it takes 1 when x is in E; otherwise 0, this is measurable. So Chi E is measurable if and only if E belongs to the set E belongs to R sigma algebra, if E is not there it is not measurable. So, let us see why.

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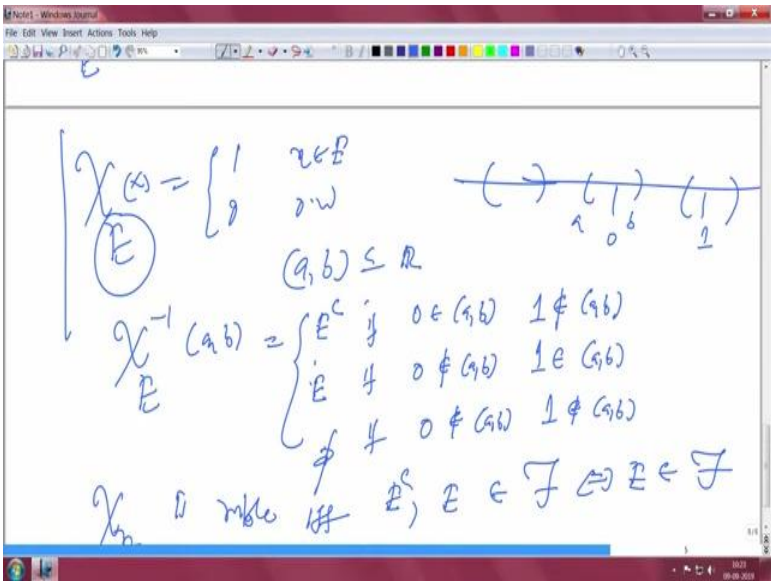

So, look at Chi E so Chi E takes the values only 0 and 1 so, it takes 1 when x is in E, 0 otherwise. So, in the real line, it takes the values only 0 and 1. So, if I take any interval, so let us take some interval a, b inside R and I look at the inverse image, so Chi E inverse of a, b. Well, what will be this b? If the interval is like this then it has 0 and suppose it does not contain 1 then the inverse image of this would be all those points x which are mapped here, which means which is 0. So, that is when it is not in E.

So, this would be E compliment if 0 belongs to a, b and 1 does not belong to a, b. What if 0 does not belong to a, b and 1 belongs to a, b? So, then the interval is like this. So, that will be all those points which are mapped to this interval which is 1, so that is same as E, only the points in E will go to 1. What if 0 does not belong to a , b and 1 also does not belong to a, b? Well, then the interval is something like this and there is no value in this interval which is taken by Chi E. So its inverse image is empty set. So, these are the only possibilities. So, this tells me that this is measurable if and only if these sets are measurable, these sets are in F. So Chi E is measurable if and only if E compliment E are in script f which is same as saying E is in script f.

So let us go back, this set, this function is measurable if and only if the set E is measurable. So that is one of the simplest examples of measurable functions.

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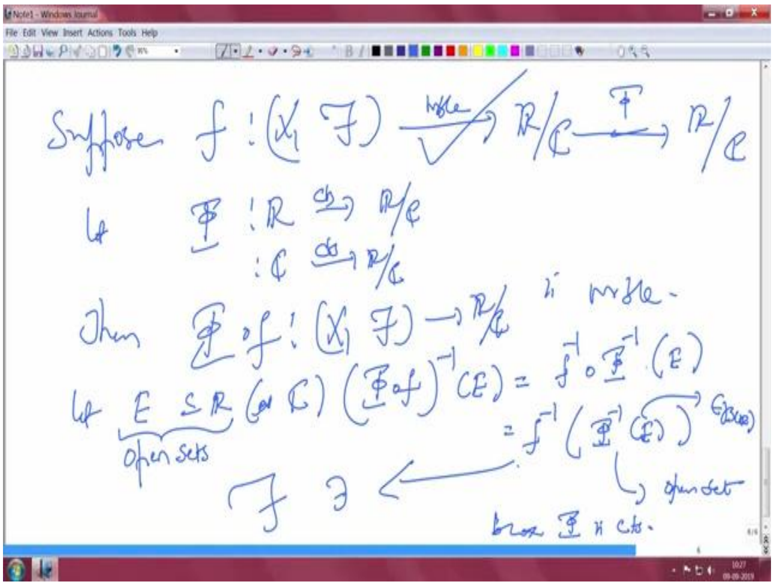

Now suppose, so let us look at some other class of functions. Suppose I have f from x f to R or C measurable. Let capital phi be a function from R to R or C or it could also be from C, R or C measurable, continuous. So these are continuous functions, let us say phi is continuous, then if I compose these 2 functions, so I am going from X to R, or C by F, and from here to R or C by phi and I am looking at the composition. So what is the composition? Composition is phi composed with f. This is a map from X f to R or C. This is measurable.

So we have f measurable, phi a continuous function and we are looking at the composition, we want to say that is also measurable. Well, how will we do that? So let us take some set which is inside R or the complex plane. And we look at inverse image, phi composed with f inverse of E. So we do not need to take any Borel set remember, because we need to take only open sets. So this is nothing but because the inverse the way it is defined, it would be F inverse composed with phi inverse of E, which is equal to f inverse of phi inverse of E and this is an open set. Why? Because phi is continuous. So this is an open set and so this will belong to the Borel sigma algebra of R and so f inverse of that would belong to script f and so this is measurable. So we will continue with some more such things.

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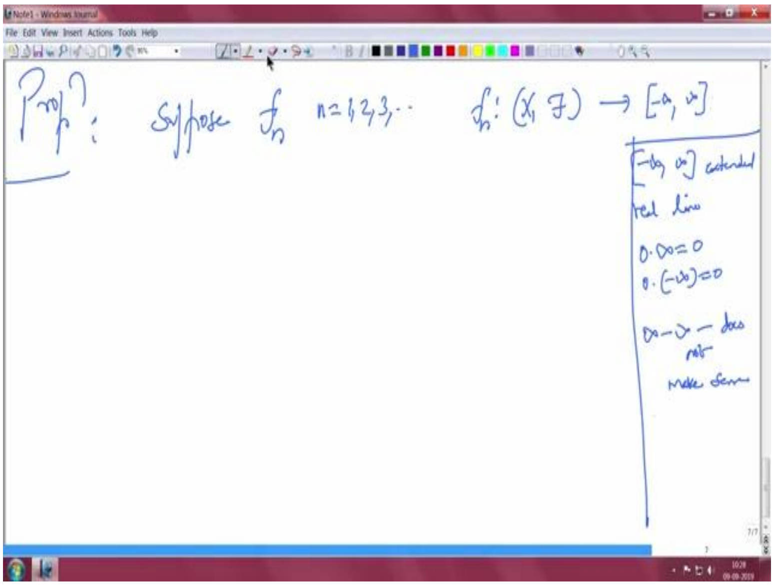

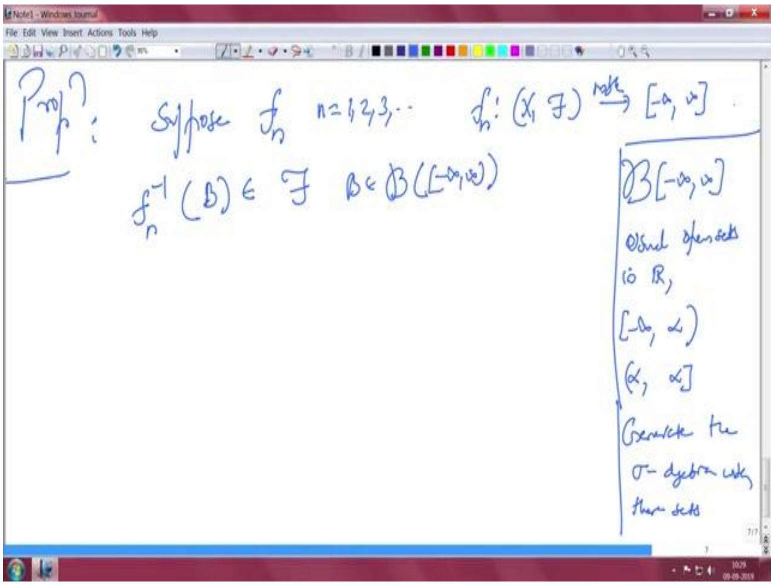

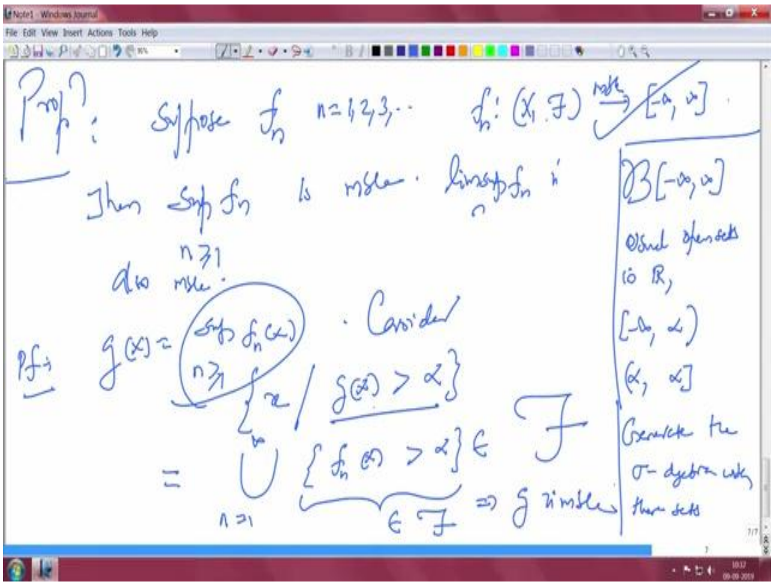

So let us write a proposition, suppose fn, so this is for n 1, 2, 3, etcetera infinitely many, fn's are measurable. So let us say it takes values in closed minus infinity, so let me say a few things about closed minus infinity plus infinity before we go ahead. So, this is the extended real line, here the usual arithmetic operations happen. If I take 0 into infinity, this is assumed to be 0, 0 into minus infinity is also 0, but terms like infinity minus infinity does not make sense. What is the sigma algebra here?

So, I have Borel sigma algebra of, so I have a Borel sigma algebra for minus infinity infinity closed. Well, what do we do? We take usual open sets in the real line along with sets of the form minus infinity to alpha open, alpha open to infinity closed. Then you generate the same. So, these are the new open sets, you generate the Sigma algebra using this, generate the Sigma algebra with these sets. So, then you have sigma algebra here and measurability here would mean f n inverse of any set belongs to script f as long as that set belongs to the Borel sigma algebra of minus infinity infinity, the usual definition. So, I have a sigma algebra here, I have sigma algebra here, I take the inverse image, I should be here.

So, I assume that there are measurable functions countably many, then supremum f n, n greater than or equal to 1 is measurable. Similarly, limb soup of f n is also measurable, so let us proof this, proof of this is, this is very easy. So let us call that g, g of x equal to supremum over n greater than equal to 1 f n of x. So this is the supremum of f n is a function which is g of x, which at each point x is simply the supremum of the values of F and X. These are taking values here.

Now, consider the set x such that g of x is greater than alpha. Well, what will this be? This is simply, so g of x is greater than alpha meaning the supremum is greater than alpha, supremum is greater than alpha would mean that one of them is greater than alpha. So, that is same as saying this is the union, all sets where f n is greater than alpha. So, let us see why this is true. So, if I take an x on the left hand side, then I know g of x is greater than alpha and so the supremum is greater than alpha.

So, there will be one f n for which it is greater than alpha. So, it could be some f k, it will be some f k plus 1 etc, so that will be here. Similarly, if the point is here, that means for some n f n x is greater than alpha and so obviously supremum will be greater than alpha. Well, but what happens to these sets? F n's are measurable. So these sets will belong to script F. And script f is a sigma algebra, I am taking a countable union and so this would be in script f, so this implies that g is measurable.

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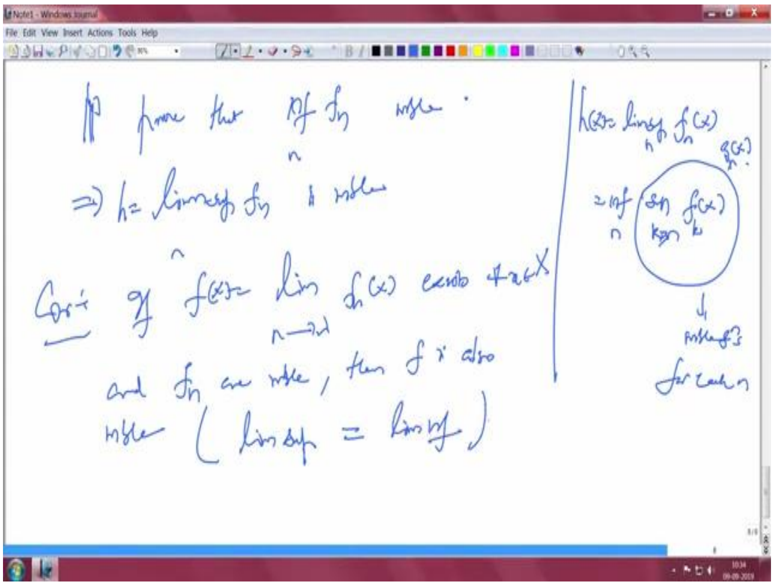

Similarly, prove that infimum of f n are also measurable, same proof. So this would implies that lim sup of f n, well, how is this defined? So let us call this h, so how it is defined, h of x would be lim sup of f n of x. What is this? This is infimum over n supremum over k, f k x that is the definition of lim sup. These are measurable functions, these are k greater than or equal to n. So, these are measurable functions for each n, call that g n, if you like, then I am taking infimum over it.

So, infimum of measurable, supremum of measurable functions is measurable, so g n’s are measurable, then I take the infimum that is also measurable, so this is measurable. Corollary, if f x equal to limit n going to infinity, f n x exists for every x in x and f n are measurable then, f is also measurable. That is because lim sup will be equal to lim n because of limit exists. So if you take limit of measurable functions, you will get the measurable function.

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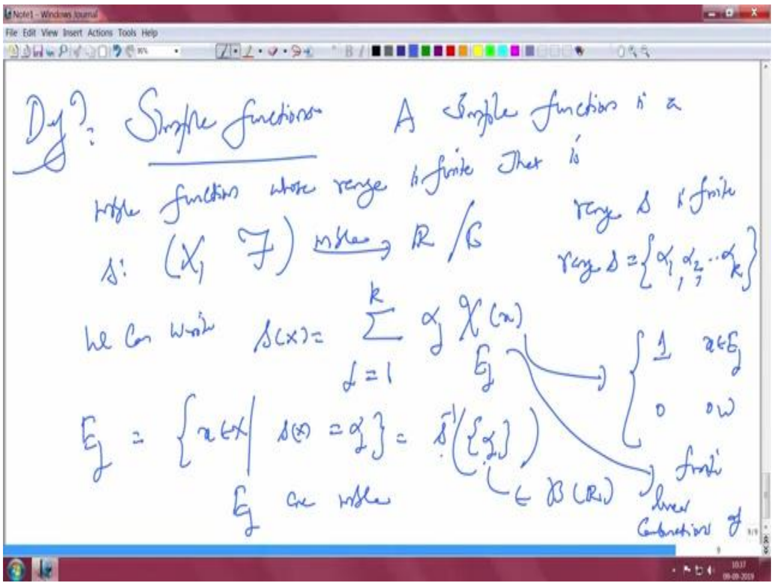

Definition so, a simple function is, it is the definition, simple function is a measurable function whose range is finite. That is let us say I have a space X and a sigma algebra F, so I look at real valued or complex valued of course, range of S is finite, so range of s is finite, so range of S will be finitely many elements from the real or complex. So, let us say alpha 1, alpha 2, etc. alpha k, so I can, so this is supposed to be measurable, we can write s of x equal to summation j equal to 1 to k alpha j Chi E j of x.

So, remember Chi E j, chi E j was 1 if x belong to E j; 0 otherwise, what are E j's? E j would be the set where s takes the value alpha j. So this is simply S inverse of the Singleton alpha j. So E j are measurable. E j are measurable because s is a measurable function. This is the Borel set and so, when I take the inverse image, I will get a measurable. So, it is a finite linear combination of indicator functions. So, finite linear combination of characteristic functions that is what a symbol function means. So, the main result we want to prove today is that any positive measurable function is a limit of simple function.

So, you can think of symbol functions as step functions on the real line or an interval. They are the building blocks and we prove that any measurable function is actually a limit of symbol functions which will actually help us in defining integration of positive functions and you will see that the certain theorems which allow us to interchange the integral and the limit follows from this definition very easily. And that is a big advantage we will see as we go along.

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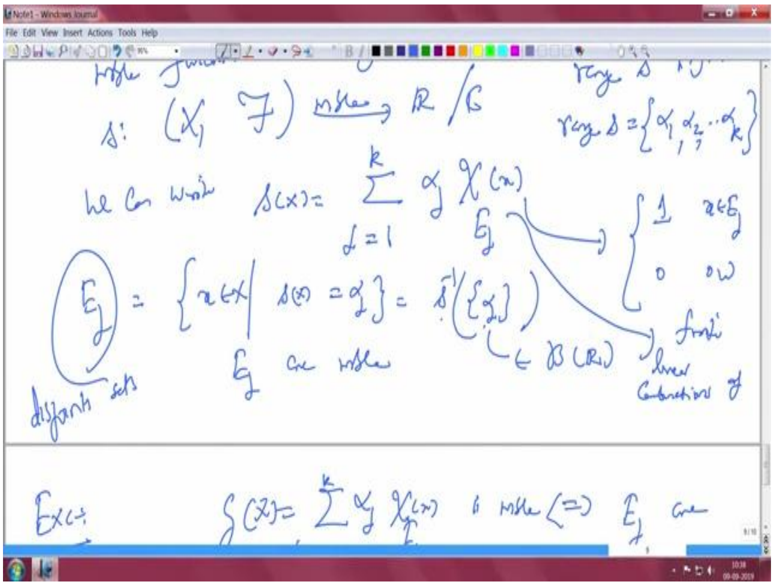

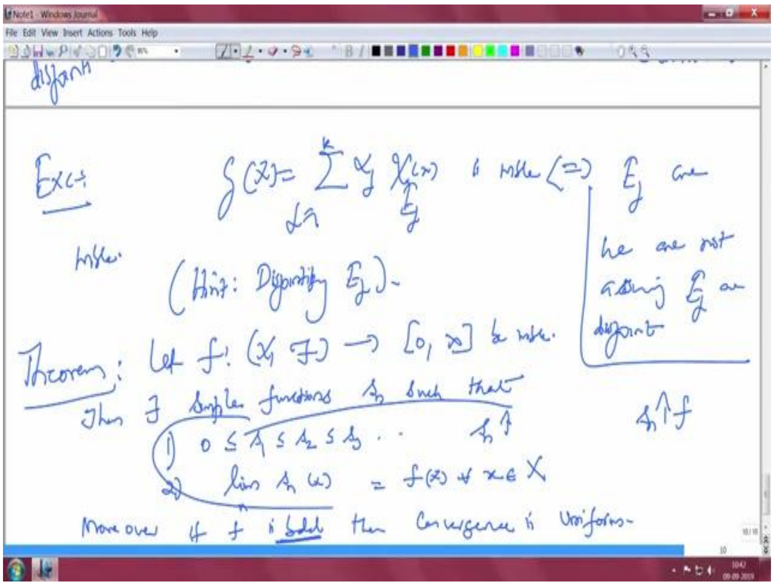

So let us start with an exercise. If g of x equal to summation alpha j chi E j, j equal to 1 to K is measurable if and only if E j are measurable. So, let us recall the what we did earlier. Here the E j's were, these are disjoint sets because alpha j are different distinct elements. So, when I look at the inverse image, they are disjoint sets. In the exercise, we are not assuming E j's are disjoint, we are not assuming E j are disjoint but you can disjointify them.

So, hint is to disjointify E j. So here is the promised theorem, this is very important first non-trivial theorem in this course. So let f from x f, to 0 infinity, so positive miserable be miserable, then there exist simple functions S n such that 1, 0 less than or to s 1 less than to s 2, then to s 3 etc. So, that is S n are increasing to limit of S n at x equal to f of x for every x. So any positive measurable function is an increasing limit of simple functions. Moreover, if f is bounded so the measurable function you started with is bounded then the convergence is uniform.

So if the function is bounded, that will be used at some part, but the main part is this part that you have a sequence of simple functions converging in an increasing manner to F, right, any positive measurable function is an increasing limit of symbol function. So, remember simple functions you think of as slightly more generalized than the step functions. Recall that when you did Riemann integration you had step functions and you approximated the integral of the function via step functions.

We are doing something similar, it just that you have more general functions than step functions, but the range is still finite and they are called simple functions and any positive measurable function you can actually approximate by simple functions and that will help us in defining integration. So, let us prove this and then we will go ahead.

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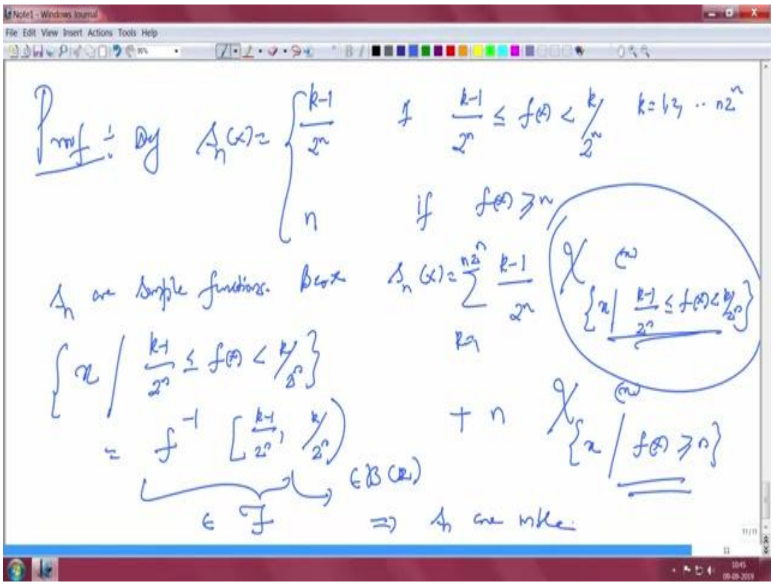

So, proof is rather simple, but one should try to understand this in full detailed, all that we do is define the sequence S n of x to be k minus 1 by 2 to the n if k minus 1 by 2 to the n less than or equal to f of x strictly less than k by 2 to the n. Well, where does k go from? k goes from 1, 2, all the way up to n times 2 to the n. So, when k is n times 2 to the n, you have f of x to be less than n and you take n if f of x is greater than or equal to n. So, that defines my simple function, first of all this is simple so, why is that? S n are simple functions? Why is that? Because I can write S n of x to be k minus 1 by 2 to the n that is the value of S n if f falls here.

So that is same as saying you look at the indicator of the set k minus 1 by 2 to the n result f of x strictly less than k by 2 to the n. So if x falls in this set, then I will have 1 here and 0 in all other places, so I will get k minus 1 by 2 to the n. So, that is summed up from k equal to 1 to n times 2 to the n and when f is greater than and, I have n. So plus n times indicator of the set x such that f of x is greater than or equal to n. So, this is a linear combination of indicator functions.

So to say that it is measurable, all we have to do is to check that these sets are measurable. Well, what is the set x such that k minus 1 by 2 to the n is less than to f of x less than k by 2 to the n? Well, this is nothing but f inverse of closed at k minus 1 by 2 to the n, k by 2 to the n and this is a Borel set, and so this set would be measurable. So this implies that S n's are measurable. So next we prove that S n converges to, so let us try to draw this picture little bit.

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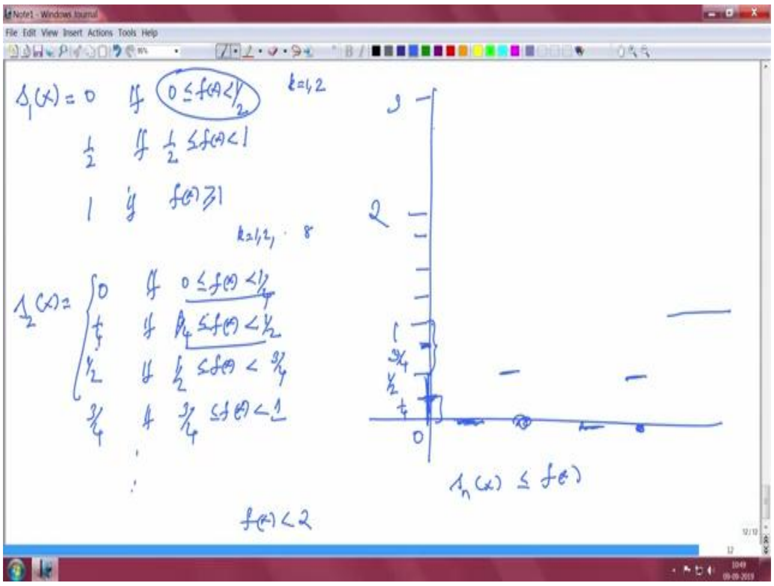

So 0, 1, 2 so let us look at what is S 1 of x. So S 1 of x would be 0, if 0 less than or equal to f of x, strictly less than 1 by 2, so remember k goes from, so k goes from 1 and 2 and it would be half if half is less than or equal to f of x less than 1 and 1 if f of x is greater than or equal to 1. So, what we are doing is we are looking at the range of f and then decomposing it. So we go from 0 to half, if f falls here, then we take 0. If f falls here, then we take half, and if f is greater than 1, we take 1.

So the set where f is between 0 and half, so that would be the inverse image of this set, so that would be some set here, some set here, some set here. There, my simple function is 0. Now, if you look at this area, well, I will get some set here, perhaps some set here and so on. And here I have half and if f is greater than 1, I will get just 1. So at some place, I will have just 1. Well, let us look at what it is S2 x, S2 x so remember k is 2, so k goes from 1, 2 etcetera, 2 into 2 to 2 which is 8. So I will go up to 8, so I will not write everything.

So I will have 0 if 0 less than or equal to f of x less than 1 by 4. I will have one-fourth, if 0, sorry, 1 by 4 less than or equal to f of x strictly less than half. I will have half if half less than to f of x less than 3 by 4. So I will write one more, 3 by 4, if 3 by 4 is less than or equal to f of x, strictly less than 1 and so on. So it will go up to 2, f of x less than 2, and f of x is greater than 2, the simple function is 2. So let us see what did we do. Initially, we had 0 to half, half to 1 that became 0, to one-fourth and one-fourth to half, so this interval got divided into 2. Similarly, this interval got divided into 2. And this interval got divided into 4 equal parts, and we have 3 and so on.

So in the first step, if f is between 0 and one-fourth, then the simple function is 0. But if f is between, one-fourth and half, then we take the lower end point one-fourth. So remember in the first case it was 0, between 0 and half the simple function was 0. In the next case it may go up, but it is either 0 or slightly more than 0. So, it is increasing so, if you write down s1, s2 s3 and so on you will see that it is increasing, it always stays below F. And what we are doing is we are decomposing the range of F into smaller and smaller intervals as we go along. That will give us simple functions S n of x which is always less than or equal to f of x, because it is always the lower end points of various intervals which we are looking at.

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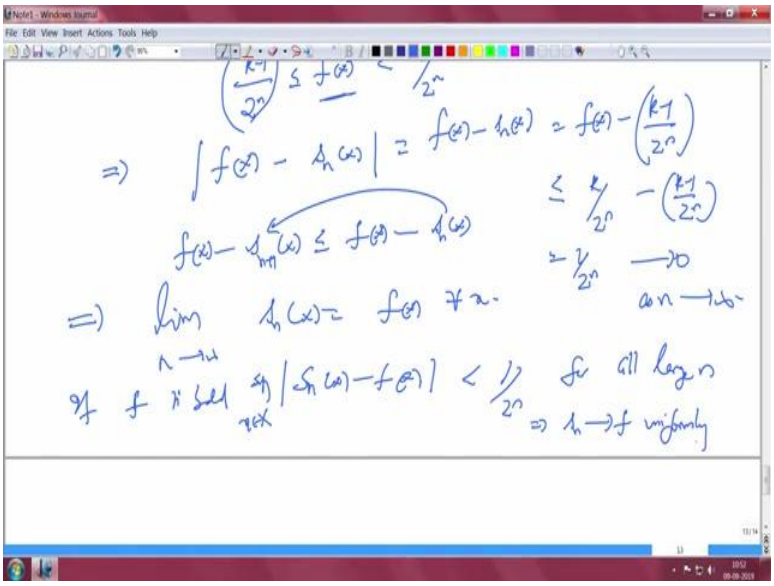

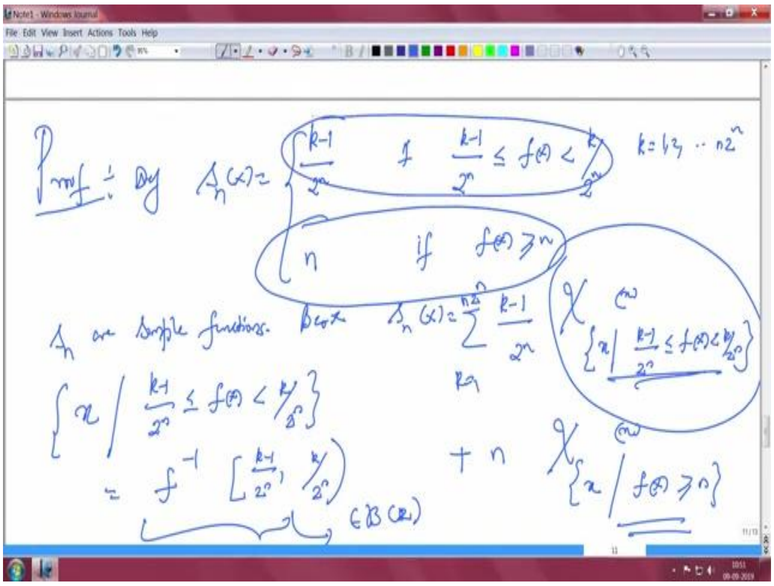

And more importantly, if I take an X, there exists some k and n such that k minus 1 by 2 to the n is less than or equal to f of x less than k by 2 to the n. For some k and n which tells me that if I look at the difference between f of x and that particular S n of x, well, this is of course f of x minus s n x. But f of x is here and the value of S n x so, this is f of x minus the value is k minus 1 by 2 to the n, the lower end point. So, this will be less than or equal to 1 by 2 to the n always because f of x is greater than, less than k by 2 to the n. So, let me write one more step just to be clear. So, this is less than k by 2 to the n minus k minus 1 by 2 to the n equal to 1 by 2 to the n and this goes to 0 as n goes to infinity. Remember s n's are increasing.

So, if I look at fx minus s n x, fx minus s n plus 1x, this is of course, going to be less than to f of x minus s n x, right because s n is smaller than s n plus 1 and so this goes to 0, so this implies that limit of n going to infinity s n x is actually equal to f of x for every x. If f is bounded then that greater than or equal to n part does not arise. So, let us see so, for a large n this will not be there because f is bounded so f will be less than or equal to some fixed n. So, only these are the values of, only these will be the values of the simple function. So, that will tell me that if f is bounded, modulus of s n x minus f of x and I take supremum over x in x, this would be less than 1 by 2 to n for all larger, that implies s n converges to f uniformly.