Measure Theory
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Lecture30
Cantor Set

Okay, so we will continue studying the properties of Lebesgue measure and some Lebesgue sets. In the last lecture we saw how it behaves with respect to the linear transformations. And we have seen some invariance properties earlier. Like it is translation, the Lebesgue measure is translation invariant. It is invariant under dilations, it is invariant under reflections and so on.

We will see that they give us some change of variables formula which you are familiar with, when you studied remand aggression, you have seen such things. So, they sort of continue to hold true here with some justification. So, let me elaborate on that a little bit.

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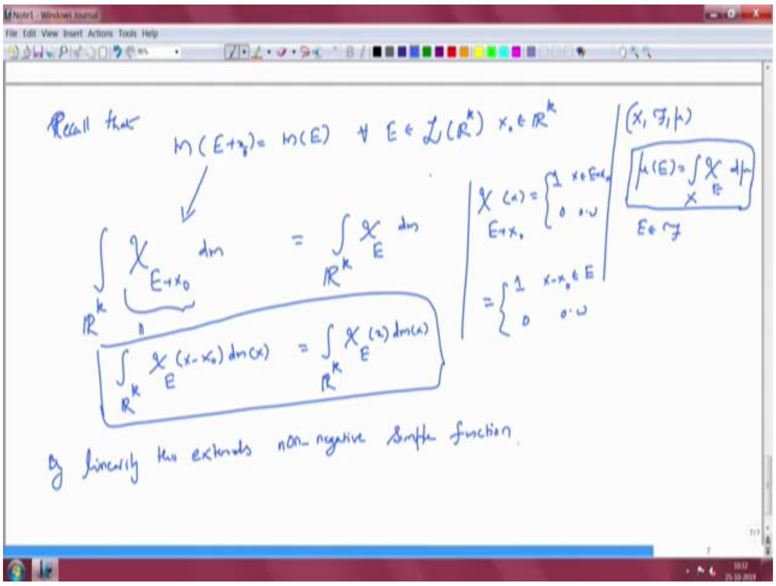

So, we start with invariance properties of Lebesgue measure. So, invariance, recall invariance, recall that measure of E plus x equal to so let us put x naught equal to measure of E. So, this was true for every E in the Lebesgue set on Rk and x naught was a point in Rk. So, we had this from the beginning.

Well, I want to say this gives us some kind of change of variable formula. So, let us see what does it say. It says that, let us write this in terms of integration. Since we know that measure of the set. So, in any triple we started with the measure of a set E is equal to the integral of the indicator of E, whenever E was a measurable set, of course.

And this was the basis for everything, this gave us integration for simple functions and by taking limits for positive functions and then for all functions. So, we can use the same technique. So, well, how do we do that? So, let us look at the left hand side. So, what is the left hand side, left hand side is the indicator of E plus X naught and then you integrate over Rk with respect to measure mu, sorry the Lebesgue measure. So, we have the Lebesgue measure dm.

On the right hand side, it is simply the integration of chi e with respect to Lebesgue measure. But let us look at this function, what does this mean, how is this defined. So, indicator of E plus X naught at the point x is, well, it is 1 if x belongs to E plus X naught and 0 otherwise, which is same as, it is 1 if x minus X naught belongs to E and 0 otherwise. So, the left hand side I can write as integral over Rk chi E of let us say x minus X naught dm x.

So, the x is put to say that, x is the integration is with respect to the variable x. This is equal to integral over Rk chi E. So, let me write chi E of x dm x. So, this is something which you have done in Riemann (())(4:12). So, changing the variable x minus X naught to y and so on. So, this tells me that we have some formula for indicator functions.

So, by linearity, this extends, this extends to non negative simple functions, non negative simple functions. Because, so, if S is a simple function it is a linear combination of indicators and integral is linear.

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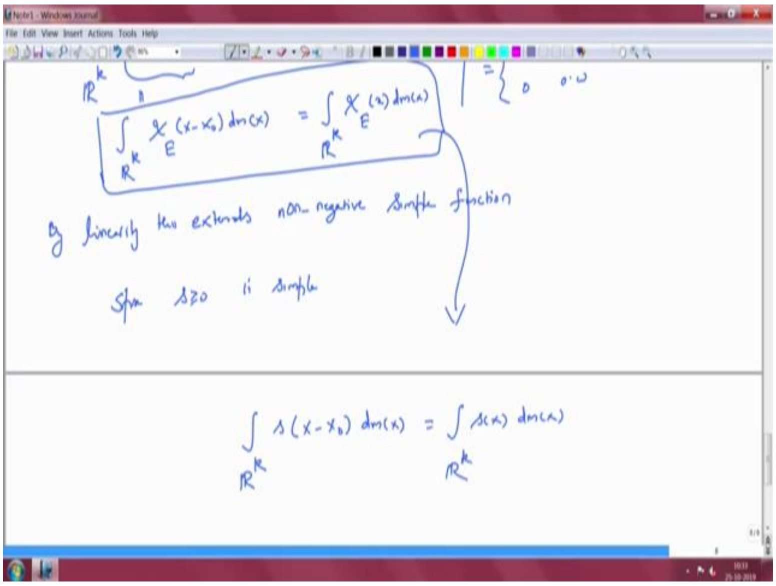

So, we will have, so suppose S is a, suppose S a simple function okay. So, then from here we get integral of Rk of S of x minus X naught dm x. This is equal to integral over R S of x dm x. So, the I am putting the x to indicate that the indignation is with respect to x.

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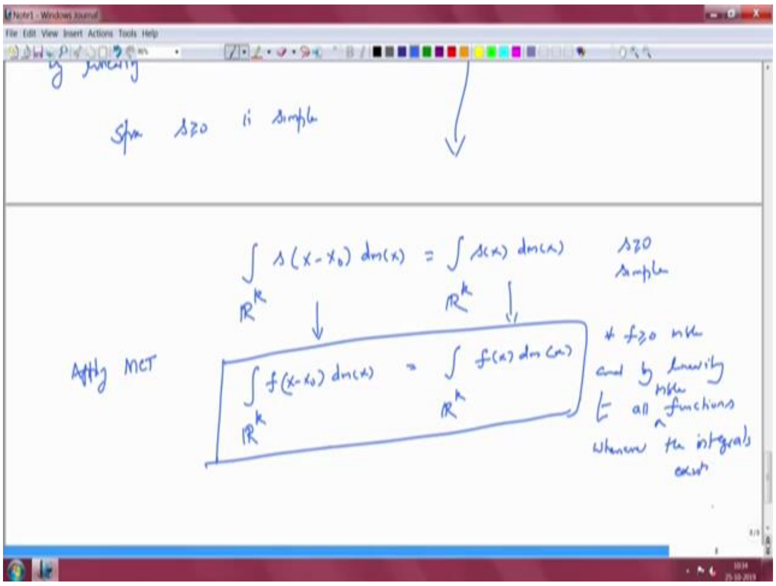

So, now this is true for every simple function positive. So, now I can apply monotone convergence theorem, apply monotone convergence theorem. If I take a positive measurable function, there exists a sequence of simple functions which increases to that. So, that will tell me that on both sides, I have convergence to f of x minus X naught dm x equal to integral over R to the k f of x dm x. So, this is precisely the translation invariant property of the Lebesgue measure.

So, this is true for all positive functions, for every f positive measurable and by linearity, and by linearity to all functions whenever the integrals exist, of course, to all measurable functions, whenever the integrals exist. So, you have seen this for Riemann (())(6:54) and it continues to be true for Lebesgue integration. So, we used translation invariant property. But let us go back to the general thing we just proved. So, maybe I will take a new page here, okay.

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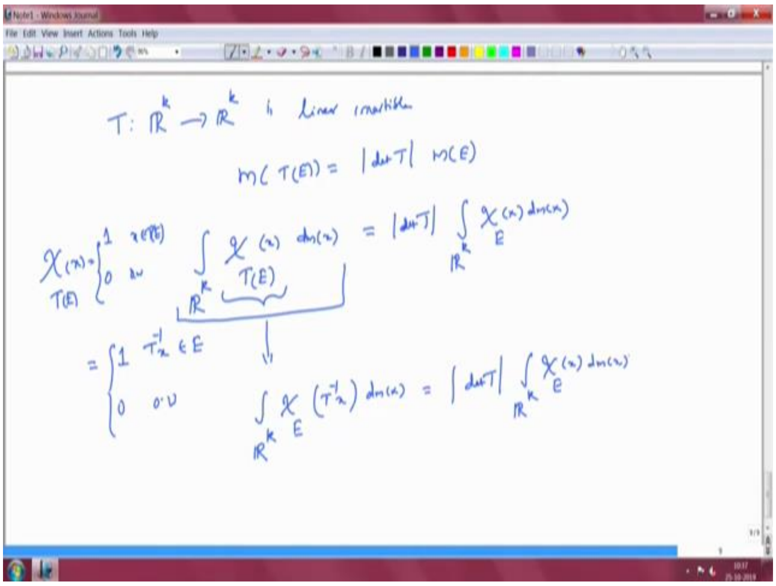

So, if I, if T from Rk to Rk is linear and invertible, then we know that measure of TE is equal to modulus of determinant of t times measure of E, this is what we just proved. So, let us write that in the integral form. So, this is integral of Rk chi of TE. So, I will put the x just to indicate that integration is with respect to the variable x, because we will be changing the variable now, determinant T integral over Rk chi E of x dm x.

But let us look at the left hand side, this function what does that mean. Chi TE of x equal to 1 if x belongs to TE, 0 otherwise, which is same as, so I want to say that it is 1 if T inverse of x belongs to E and 0 otherwise. So, this is a better way of writing. So, the left hand side integral is simply integral over Rk. Chi of E now, but T inverse x dm x. This is equal to modulus of determinant of t times integral over Rk, chi E of x dm x. Good. So, now we can do the steps we did earlier.

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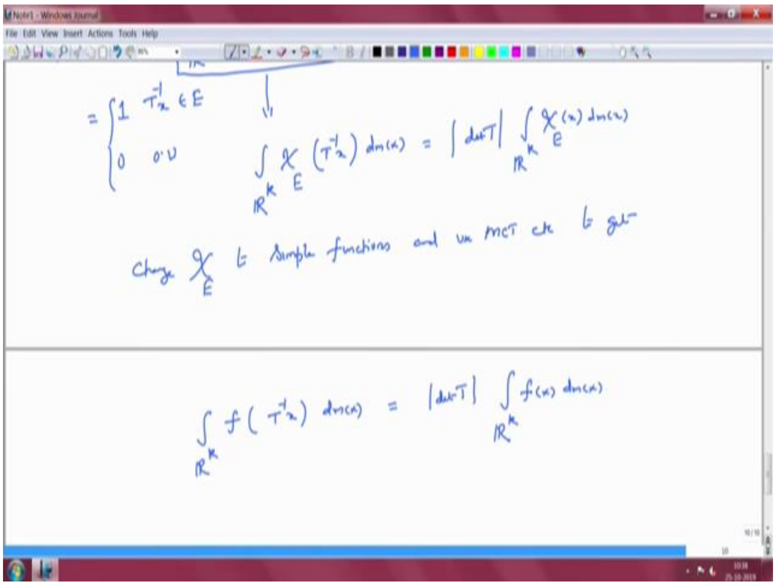

Change Chi E to simple functions. So, change Chi E to simple functions and use monotone convergence theorem etc to get to get integral over Rk. So, I will write the general result now, f of chi becomes S a simple function and S becomes a measurable function f. T inverse x dm x equal to modulus of determinant of T integral over Rk chi E x, sorry f x dm x. Okay, so maybe write it in a slightly better form, so call T inverse something else so call T inverse equal to the linear transformation A.

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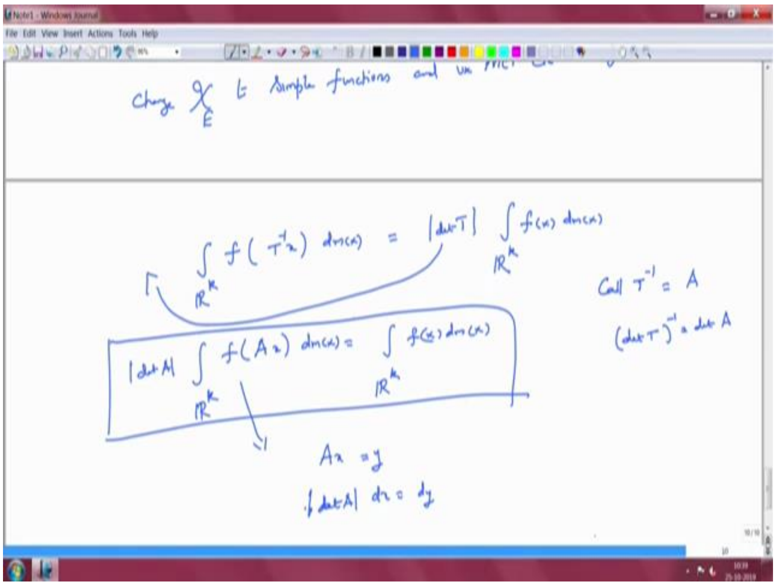

Then what we have here is integral over Rk f of Ax dm x equal to determinant of T, so T inverse is equal to A, so determinant of T whole inverse is equal to determinant of A. So, I can take this to this side and I will get modulus of determinant of A equal to integral over Rk f x dm x. So, this is the usual change of variable which you are familiar with. So, what do we do, whenever we have something like this we put Ax equal to y.

And the change of variable should tell you that determinant, modulus determinant of A dx is dy. So, you can change everything to y variable and you will get whatever is on the right hand side. So, let us move to a slightly different kind of result.

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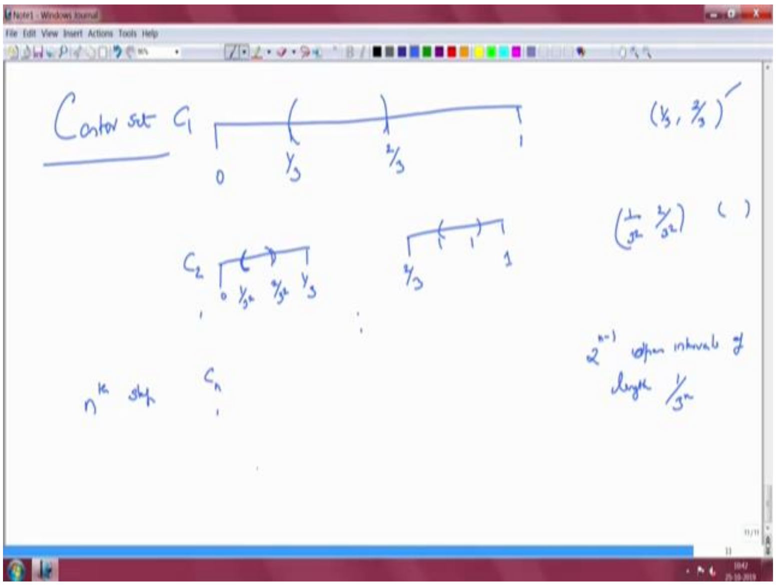

So, this is about sets which are small. So, we will, let us talk about Cantor set. Well, what is Cantor set? So, let us take 0, 1 divide this into three equal parts and then you throw out the middle one third interval, so you leave this. So, you have thrown out the interval, 1 by 3 to 2 by 3. Now, you continue this process. So, you have the 0, one third, you have two third and you have 1. You make them again you take the middle one third and middle one third here.

So, what does it mean to say middle one third, you divide each interval into three equal parts and throw out the middle one third. Okay. So, this would be 0, 0 by 3 square if you like, 1 by 3 square, 2 by 3 square and 3 by 3 square is 1 by 3. So, you are throwing out this. So, 1 by 3 square to 2 by 3 square. Similarly, another interval, so you can find out what these are. So, you have thrown out two intervals of length 1 by 3 square. Here you have thrown out one interval of length 1 by 3 and so on.

So, n step you will be throwing out, you will be throwing out 2 to the n minus 1 intervals, open intervals of length 1 by 3 to the n, this is what you will be doing. So, at each stage. So, let us call this C1, let us call this C2, etc etc I have Cn and so on.

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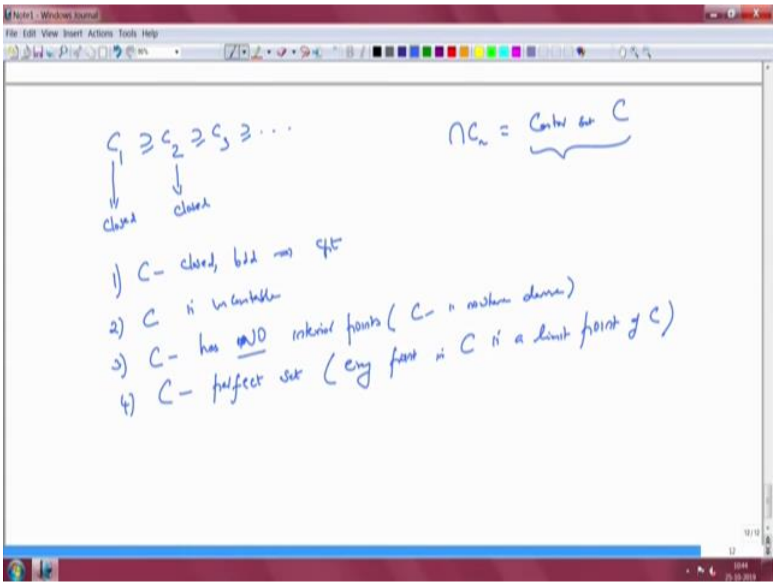

So, I continue this infinitely many steps. So, I have C1 which is bigger than C2 because I am throwing away things from C1. And I throw away from C2, etc. etc. What do I know about them? Each C1 is closed, C 2 is closed. Let us see why are they closed, because we are throwing out the open intervals. So, C1 is this closed interval union this closed interval. Similarly, C2 is a union of 4 closed intervals. Each time we throw out an open middle third. So, what remains is closed.

So, this is a decreasing sequence of closed sets in a complete metrics space, so the intersection cannot be empty as you know. So, the intersection Cn, so this is a close set, okay. So, this is what is called a Cantor set. Cantor set C. So, Cantor set C has various properties which you would have seen in real analysis. So, I will not get into the proofs of any of this. Cantor set C has various properties, one C is closed, of course, because intersection of closed sets, bounded because it is inside 0, 1, so compact. So, there is a compact set.

Two, to C is uncountable, it is an uncountable set, three, C has no interior has no interior. No interior points. In other words, C is nowhere dense, C is nowhere dense and C is a perfect set, C is a perfect set. What does that mean? That means every point in C is a, every point in C is a limit point of C, is a limit point of C. So, these are interesting properties of the Cantor set, it is one of the most interesting sets you will encounter, you can modify this construction, but right now, let us let us stick to C.

So, let us go back. What did we do? We threw out middle one third at each level. So, at the nth level, we will be throwing out 2 to the n minus 1 intervals, open intervals of length 1 by 3 to the n.

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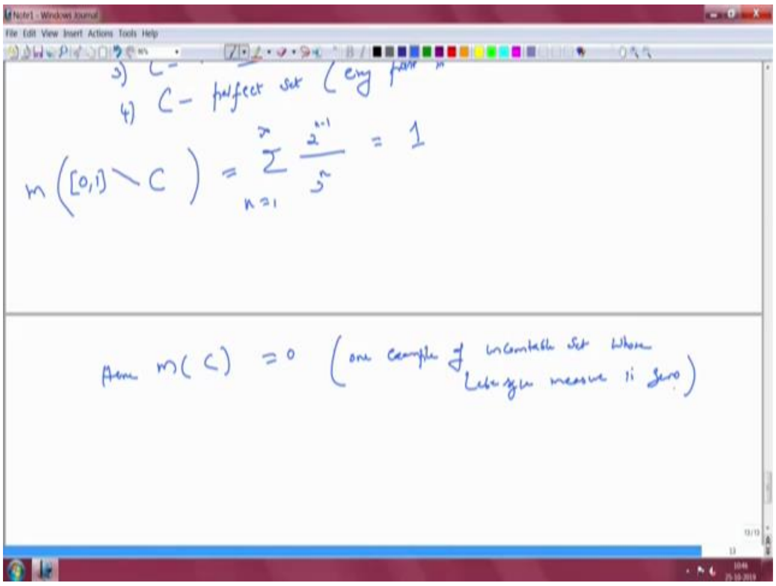

So if you look at the Cantor set C and look at the set 0, 1, everything is inside 0, 1 and look at the compliment. And measure of this, everything makes sense, C is a closed set, compliment is open, so, measure of that will make sense. Well, what is the measure of whatever is thrown out? Well, you will start from adding here, then you add this, then you add this and so on. All these are disjoint open intervals. At nth level, you have to do the n minus 1, disjoint open intervals and each level the open intervals are disjoint from the previous ones.

So, all that you have to do is to add this quantity and see how much you get. So, this is simply the sum of 2 to the n minus 1 by 3 to the n, n going from 1 to infinity. Because of countable additivity and this is 1. So, you have thrown out open intervals whose length adds to 1. What does that mean? Hence the measure of the Cantor set C, even though it is uncountable is 0.

So, this is one example of, so one example of uncountable set whose measure is 0, whose Lebesgue measure is 0. Of course, if you take a countable set it has measure 0. So, this is one example where the set is uncountable, but its Lebesgue Measure is 0.

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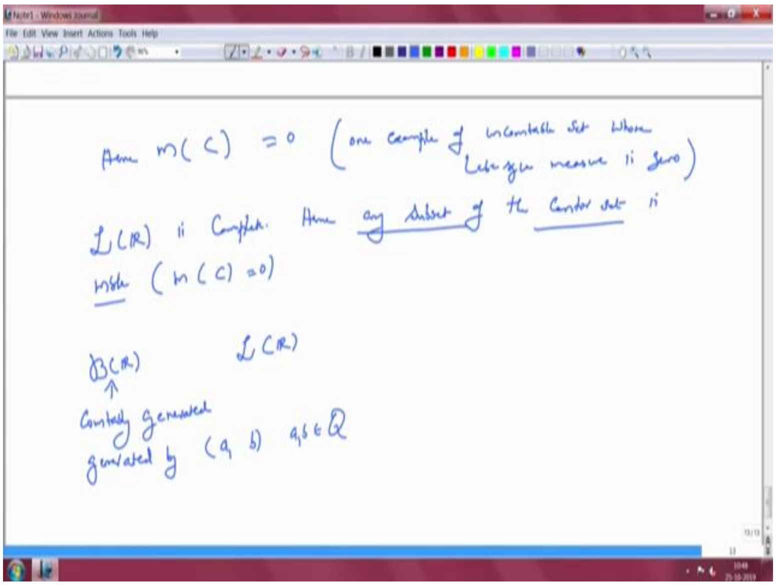

Now, recall that the Lebesgue sigma algebra of the real line with respect to the measure M is complete. Hence, any subset of the Cantor set is measurable because Cantor set has measure 0. So, any subset of the Cantor set is measurable because measure of M is 0, measure measure of C is 0. So, that gives us a lot of sets in the Lebesgue sigma algebra. So, let us, let us compare the Borel sigma algebra of the real line and the Lebesgue sigma algebra of the real line.

Now, this is countably generated, countably generated. What does that mean? So, generated by open intervals a b, where a and b are rationals. So, there are countably many such open intervals and that generates B of R.

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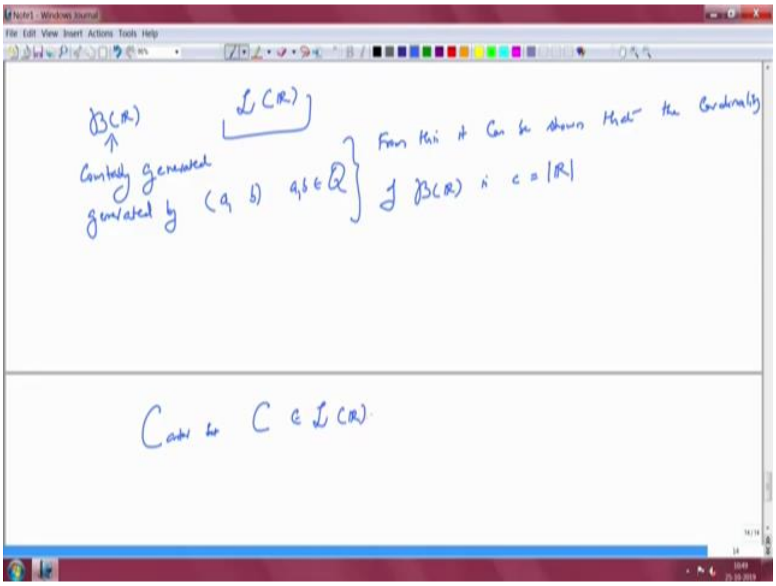

From this it can be shown. So, I will, I am not proving any of this, this is simply an assertion. From this, it can be shown that can be shown that B of R the cardinality of B of R is c. So, the small c is the cardinality of the real line, uncountable, and first uncountable number if you like. But at the same time, if you look at the Lebesgue sigma algebra of R, that has too many sets. Because the Cantor set C capital C belongs to Lebesgue sigma algebra of the real line.

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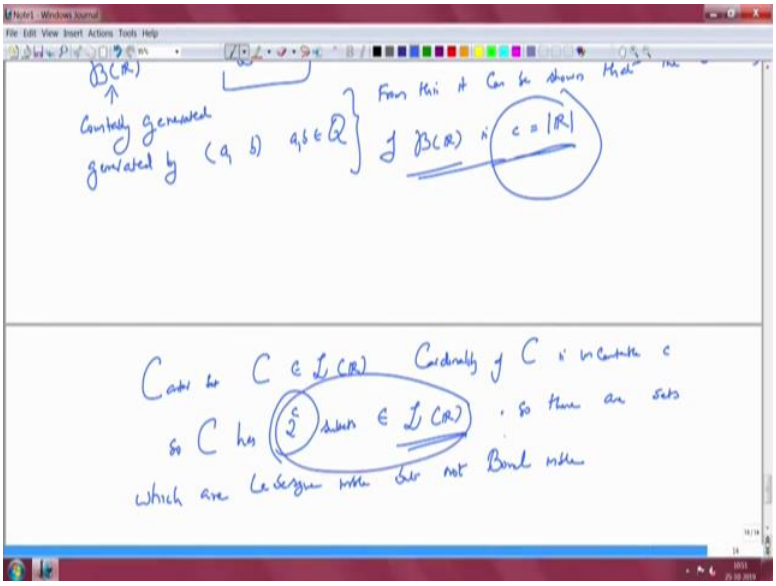

And the cardinality of cardinality of C, cardinality of C is uncountable, uncountable. Because it is an uncountable set, so, that is the small c. So, capital C has 2 to the small c subsets, all of them belonging to the Lebesgue sigma algebra. So, let us recall so here we have B of R having cardinality c, here we have L of R with 2 to the C subsets sitting there anyway.

So, the inclusions, so there are. So, from this cardinality argument, so there are subsets or there are sets which are Lebesgue sets, Lebesgue measurable, but not Borel measurable. From the cardinality argument because here you have only C many sets, but for L of R we have 2 to the C many sets. So, there are too many sets which are not Borel but Lebesgue measurable. Of course, is one way of proving this, we will, so later we will construct using some other, again using Cantor set and the Cantor function we will construct a example.

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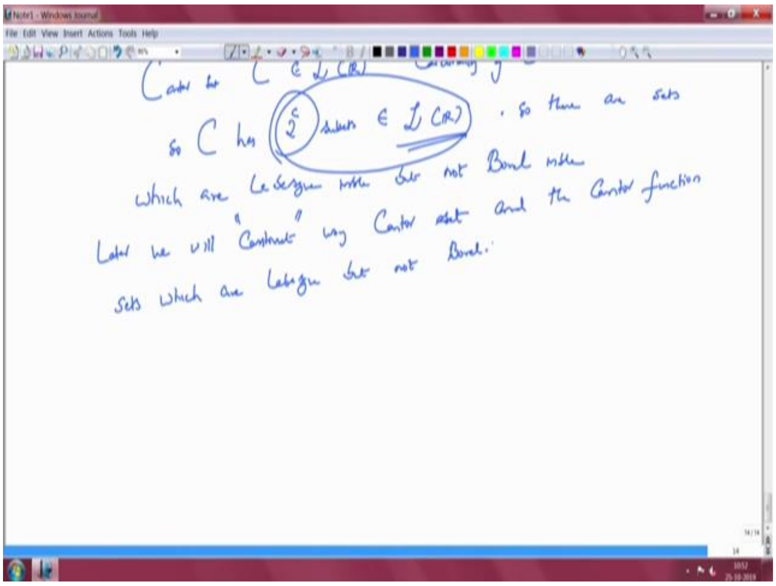

So, I will do this later. So, later we will construct using Cantor set and the Cantor function, so I will explain what that is when construction. Construction is not very explicit it is just a proof of the existence. So, not using cardinality argument. So, construct meaning proof of existence using Cantor set and the Cantor function, sets which are Lebesgue, which are Lebesgue but not Borel.

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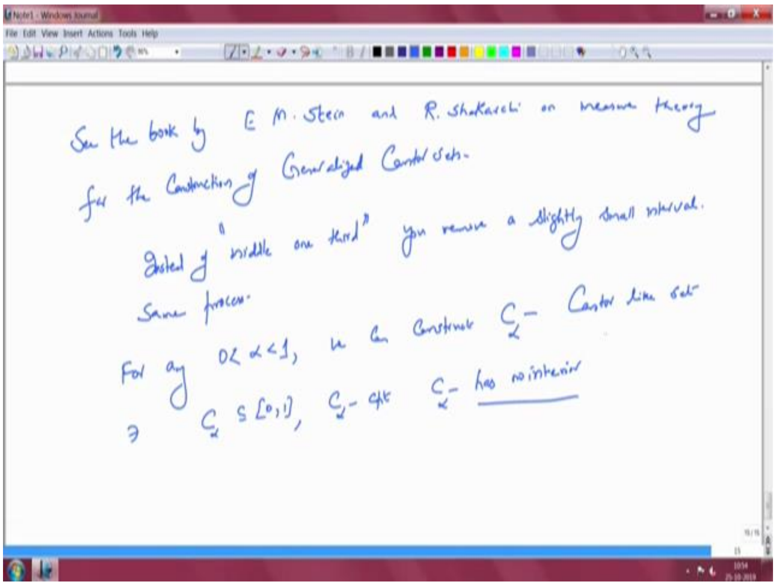

So, let us conclude this session but let me explain one more construction of the Cantor set, Cantor like sets. So, let us go back to the Cantor set. We started from 0, 1. And then we threw out the middle thirds, so you can modify the construction. So, I will give you a reference. So, see, see the book by. See the book by "Stein and Shakarchi". So, "E M Stein" and "Rami

Shakarchi” on measure theory for the construction of light, for the construction of generalized Cantor sets, generalized Cantor sets.

So, what do you do? Instead of, instead of middle one third what do you do is, you remove a slightly smaller interval. So, this has to be done with some care, but you can do this and same process. So, what do you get is for any alpha, for any alpha between 0 and 1, we can construct, let us say C alpha Cantor like sets, Cantor like set such that C alpha is of course sitting inside 0, 1, C alpha is compact, C alpha has no interior, this is interesting, no interior.

And measure of C alpha instead of being 0, you can make it alpha. Okay, so, the point to note is that you can slightly modify the construction of the Cantor set and you will get some interesting sets.

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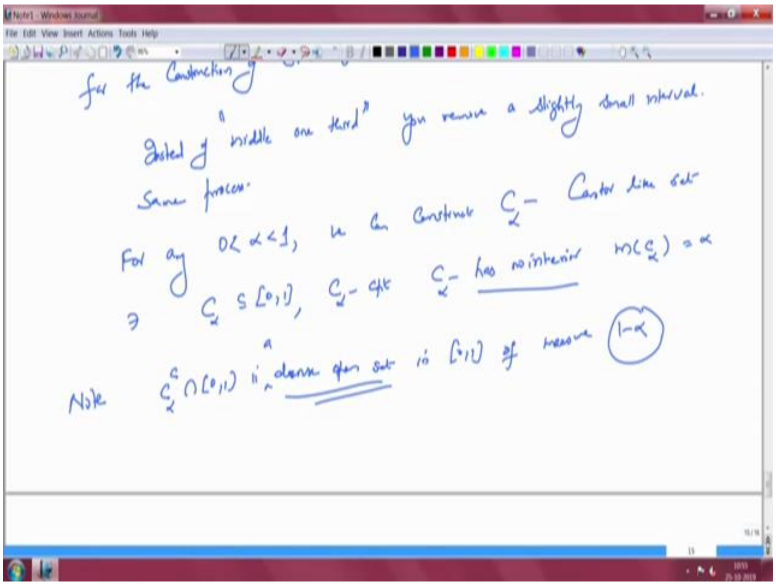

So, notice that the compliment C alpha compliment inside 0, 1 is a dense open set in 0, 1 of measure 1 minus alpha. So, note that. So, you can make this as close to 0 as you want. So, you have very small dense open sets, which you can construct, of course that is not very very surprising.

Okay, we will stop here. In the next lecture, we will continue this circle of ideas and then use the Cantor function. So, Cantor function will be a function first defined on the Cantor set, but then you can extend it to the closed interval 0, 1 as a continuous map. The point of the Cantor function is that it maps the Cantor set into a set which has measure 1. And using that and the fact that homeomorphisms will map Borel sets to Borel sets we will justify the existence of non Borel sets but Lebesgue sets. That is what we will do in the next lecture. Okay. We will stop.