Measure Theory
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Lecture 38
Product Sigma Algebra

Okay, so in the last lecture, we saw LP spaces, we saw some properties of LP spaces depending on the space X if it had a locally compact Hausdorff space and topology and so on, then we saw that continuous functions with compact support from a dense subspace of LP and things like that. Now, we will go forward, our aim to, our aim in the coming lectures will be to look at product spaces and define what is known as the product measure.

So if I have two spaces, two measure spaces, let us say X and Y with measures on them, we look at X cross Y, the Cartesian product, and then try to define the product measure on that space. So this will also be applicable to, let us say, the real line or Rn in general, you can look at Rn and Rm, we have Lebesgue measures on both of those spaces. And the product measure will be defined. So we will relate all this to whatever we already know in Rn for general n.

But as usual, we will stick to the general settings, abstract settings where we have spaces x and y and we look at X cross Y. And then we put a sigma algebra on x cross why, which comes from subsets of x and subset of y put together. And then we will define a product measure. So the definition of the product measure takes some time, it is a actually a theorem. And then we will see how to (integra) how it affects integration on the product spaces. Okay, let us start.

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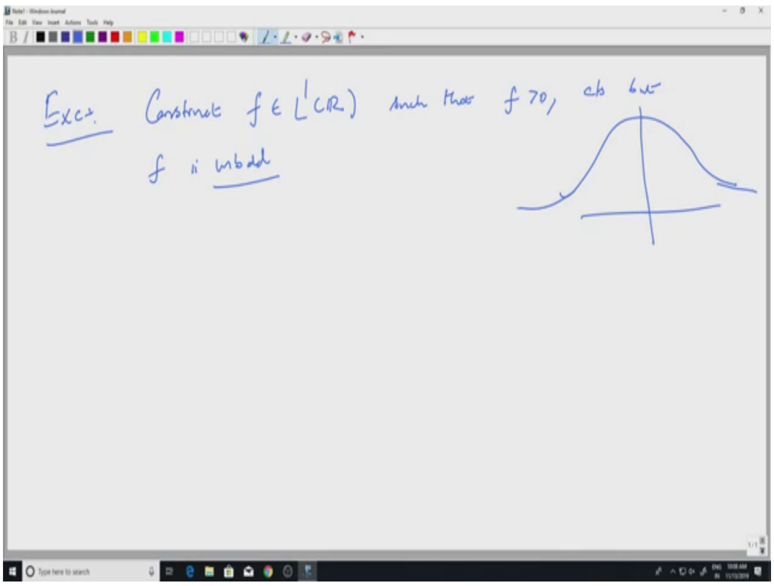

Before we start product measures, let us let me give you a simple exercise. So, we will start with start the day with exercises. Construct an f in L1 of R on the real line such that f is positive, well you can actually take it to be strictly positive. Continuous, but f is unbounded, f is unbounded.

So this is to make sure that you understand the spaces a little bit clearly so, when we say something is in L1, its integral is finite. So, you would want something, which dies down at infinity, but that need not happen always unless you put extra conditions on f.

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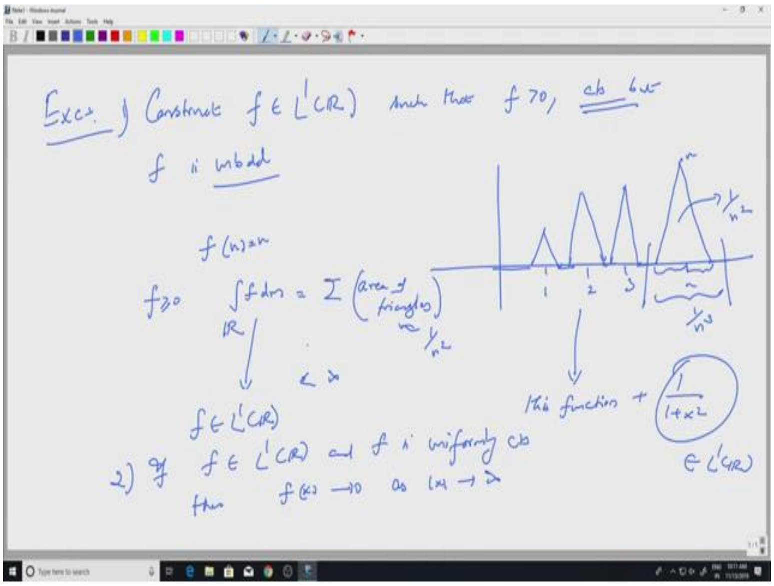

So, let me tell you how this is this is done, give you hints on how to construct f. So what you do is you look at the points, 1, 2, 3, etcetera. So let us say n. So at n, I want f to be n. So let us say this is the height n. So I am trying to construct a continuous function with the property that f of n is n so that it is unbounded, but then around n I locate a triangle like this.

That is graph of my function that is continuous. And I want the area of the triangle to be 1 by n square which means that this length will be of the order 1 by n cube, the area is half base times the height, height is n and base is, if it is 1 by n cube, then I will have 1 by n square as the area.

So around each of them I have these triangles, which are at a height n at n and so on. Remaining places you make it 0 so that it is a continuous function and f is positive so, f is well f is non negative. And integral of f with respect to the Lebesgue measure is the sum of the areas, area of triangles. Because the area under the curve was the Riemann integral of the function and if we look at some interval like this, it is a continuous function. And so Riemann integrable, Riemann integral is equal to the Lebesgue integral and Riemann integral is the area under the curve which is the area of the triangle.

And so, that is this is like 1 by n square. So it converges, so, this will imply that f is in L1 but it is unbounded, but it is not strictly positive. So, what you can do is you can add something which is strictly positive. So, look at this function, this function plus something like 1 by 1 plus x square or something like that. So, this function is also in L1, but it goes to 0 at infinity.

So, this function plus 1 by 1 plus x square will be an unbounded continuous function strictly positive integral is (fine) finite. So, continuity alone will not tell you that it goes to 0 at infinity, but if f is uniformly continuous so, if f is in L1 of R and f is uniformly continuous, then it has to go to 0 at infinity, then f of x will go to 0 as mod x goes to infinity at infinity it has to die down.

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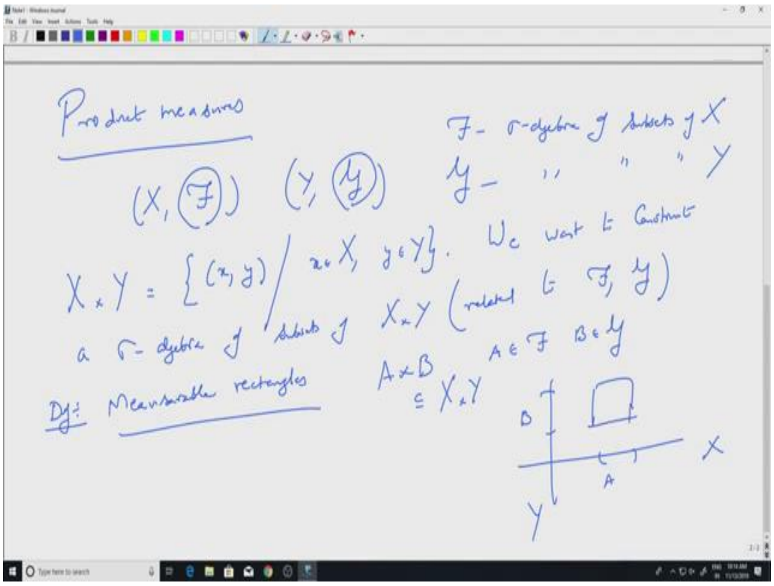

Okay, so with that exercise we can start the product measures. So product measures. So we have two spaces. So first of all we need to define the sigma algebras and so on, so we will start with that. So I have X F so that is a space, X with the sigma algebra F, and I have Y and let us say G, script G. So script F is a sigma algebra of subsets of X. And script G is a sigma algebra of subsets of Y that is how we see it. We want to define, so, we will look at X cross Y. So, X cross Y is the Cartesian product of x and y. So, this is simply all tuples, two tuples ordered pairs x and y such that x belongs to capital X, y belongs to capital Y.

So, this is the usual set theoretic Cartesian product. We want to put, so, we want to construct we want to construct a sigma algebra, construct a sigma algebra on of subsets of X cross Y okay of subsets of X cross Y, but the sigma algebra should be related to F and G. So, related to related to script F and script G.

So, we wanted to find the product of these two. So we start with some definitions, so a definition measurable rectangle. So you have seen this in the case of Rn, it is essentially the same definition, measurable rectangles. So these are sets of the form A cross B. So any set of the from A cross B is a rectangle well. It is called a measurable rectangle if A comes from script F and B comes from script G. So you take a measurable set here, a measurable set here and look at the Cartesian product as a subset of X cross Y.

So A cross B is contained in X cross Y. Such a thing is called a measurable rectangle. So the X will be here, the Y will be here and let us say this is A. It need not be an interval as I have written, I have drawn because these spaces are X and Y are abstract spaces. So we are looking at something like this. If you are looking at the real line, the usual rectangle would be a measurable rectangle. So that is the definition of a measurable rectangle.

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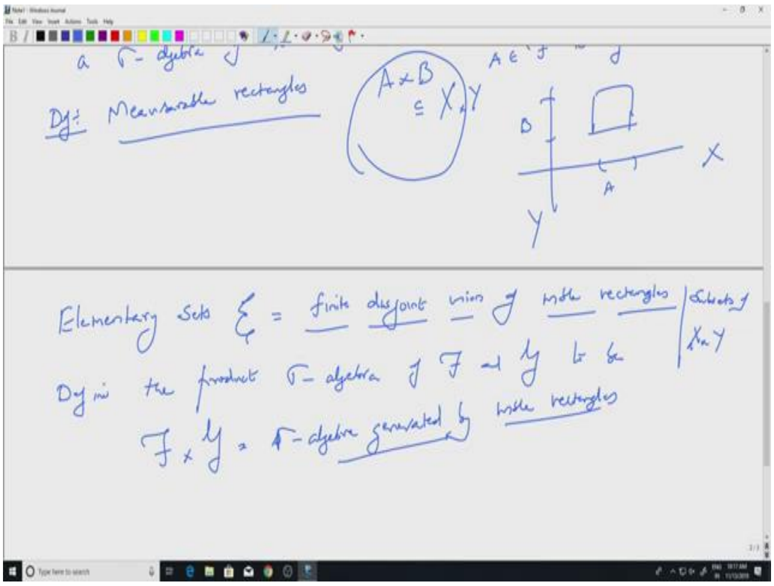

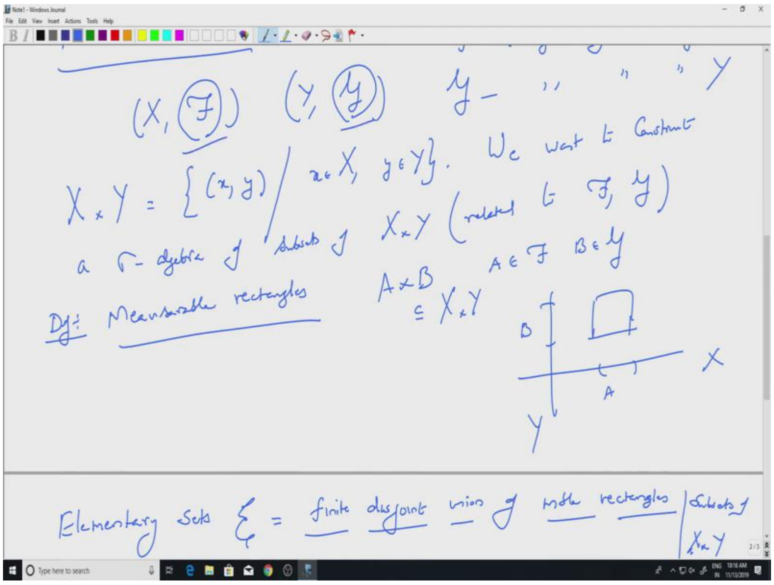

Now define elementary sets, elementary sets. So this is denoted by script E. So maybe I should write it slightly clearly, script E. Script E is the collection of finite disjoint union of measurable rectangles. So, it is finite disjoint union of measurable rectangles. So you take a finitely many rectangles disjoint and look at their union. So, these are all subsets, so elementary sets are subsets of X cross Y.

So, this consists of subsets of X cross Y. So that is one step towards constructing the sigma algebra. Now, I want to define the product sigma algebra. So, remember we have F and we have G, I want to define F cross G to be. So F, define the product sigma algebra, product sigma algebra of script F and G to be. So, this will be denoted by F cross G. So, that is just a symbol. F cross G is the sigma algebra generated by measurable rectangles. So, let me recall so, remember the measurable rectangle is defined here to be A cross B where A is in script F, B is in G.

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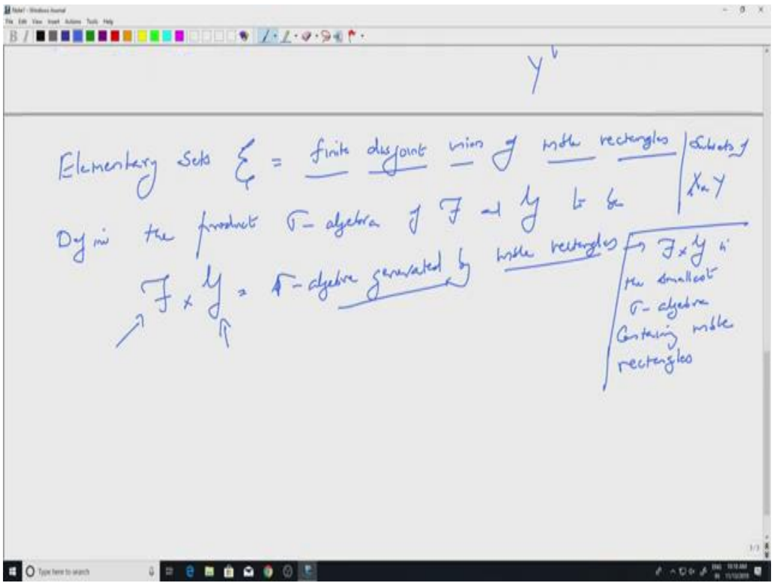

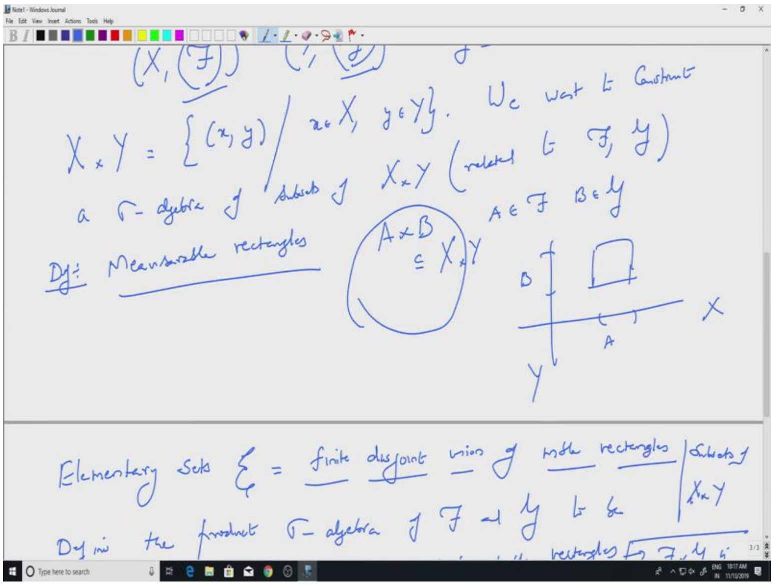

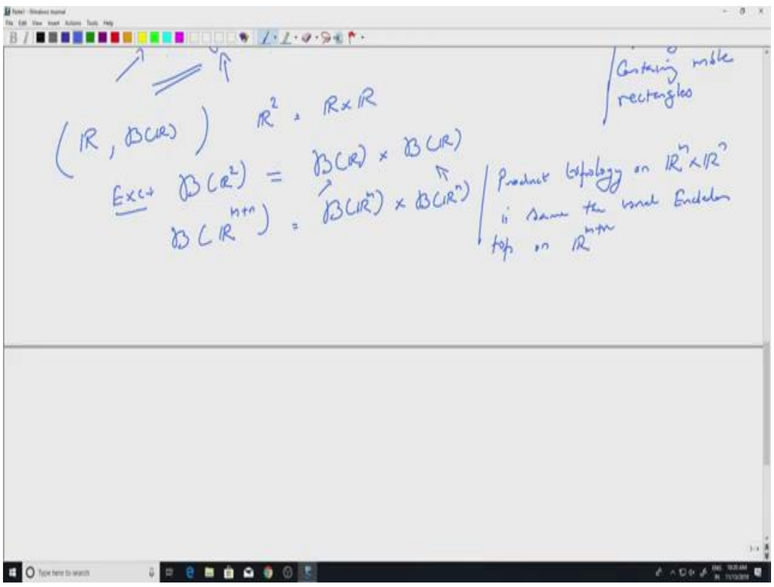

So, you take sets from here sets from here, look at the product set that is a subset of X cross Y, look at the smaller sigma algebra containing all of them. So, this, generated by also means, so, we have seen this before, this is this means that script F cross script G is the smallest sigma algebra containing measurable rectangles. That is what you mean by generated by measurable rectangles.

So, that defines it. So, we have defined several things one is the measurable rectangle, which is simply product set of measurable sets, elementary set is a finite disjoint union of measurable rectangles and the product sigma algebra is the one generated by measurable rectangles.

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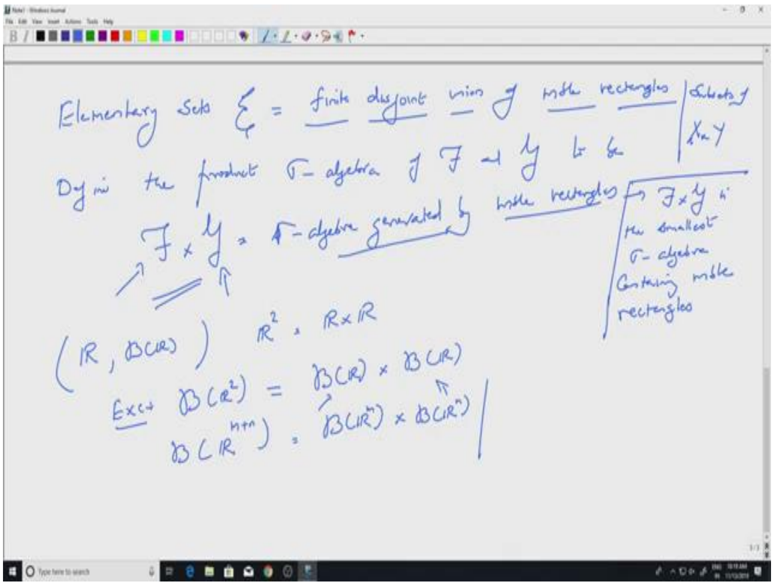

So, let us look at one example and then go ahead. So, let us look at the real line and the Borel sigma algebra or the real line. This is the Sigma algebra generated by the open sets. So, you can look at R2 as a product space that is R cross R. Then, so, exercise. Prove that the Borel sigma algebra of R2. So, this is defined independently of anything, R2 is a topological space. You look at open sets and generate the sigma algebra that is your Borel sigma algebra of R2.

Prove that this is actually equal to the Borel sigma algebra of the real line cross Borel sigma algebra of the real line. So, remember we have two sigma algebras, the product is the sigma algebra generated by measurable rectangles. So, you take a Borel set from here, take another Borel set from here, look at the product, look at all such sets, take the (gener) sigma algebra generated by that.

So, this is not difficult. So, this, or more generally you can prove that Borel sigma algebra of R m plus n. So, this is defined independently of anything, use actually the product sigma algebra of B of Rn. Well, what do we do? Here so, all that you have to do is to use the topological fact that product topology on.

So this this follows immediately from the fact that product topology on Rm cross Rn which is R m plus Rn is same as the usual Euclidean topology on R m plus n. So that will do, if you use that you will get this immediately.

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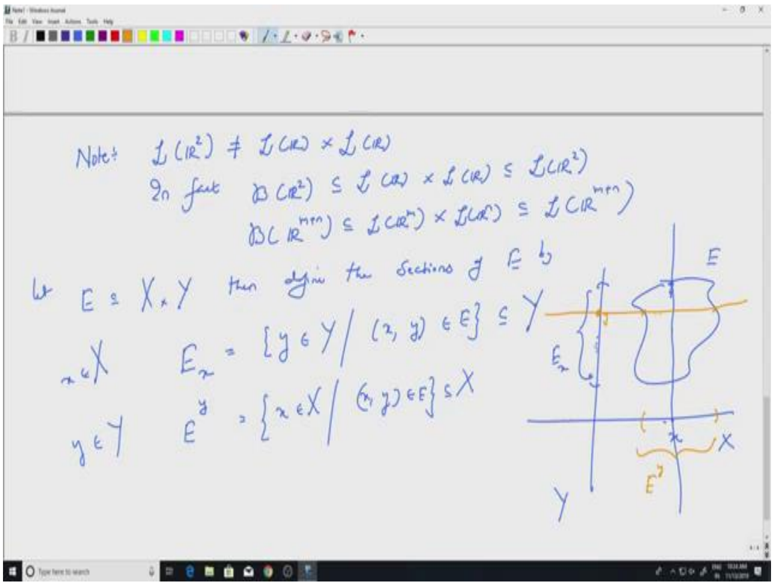

Now, well, of course, you may ask the same question for Lebesgue sigma algebra, so I will write this as a note, we will prove this later. So, let us say L of R2 is not equal to L of R cross L of R. So if I look at the product sigma algebra of Lebesgue sets on the real line with itself, I will not get the Lebesgue sigma algebra on R2.

So, in fact, so some of this we will see soon. In fact, well, the Borel sigma algebra of R2, of course, this is the product sigma algebra of R and Are, will be contained in the product sigma algebra of L of R with itself, the Lebesgue sets. And that is contained in the Lebesgue sigma algebra on R2. So, we will see just in fact, in general if I look at m plus n, I can have the same containments of sets.

So, R m plus n is contained in L of Rm cross L of Rn which is contained in L of R m plus n. Well this is because these Lebesgue sigma algebras by construction are complete and so, it has all the subsets of sets with measure 0. So, that makes it very big. So, let us try to justify some of this, before that, for that we need what are known as sections. So we will do, we will define this in general.

So, let E be a subset of X cross Y. So remember we have spaces x y and the Sigma algebra script F and script G. Then, define the sections of E, by, so now onwards, the notation will be very, very important. So E sub x, so small x is a point in capital X, E sub x. This is, so x is fixed now, I am looking at the x section of E. This is all y in Y such that x comma y belongs to E. So, notice that this is a subset of Y, the x subsections or subsets of Y. So, let us see what does it mean.

So, let us say I have x here, I have y here, I have some set E right, which is in X cross Y. So, I take some point x here, what is E sub x? So E sub x is, so, you draw the line through x and wherever it intersects E, so, that will give me a set in Y, that is my E sub x, correct? All those points here, which such that x comma y is here. So, that is simply this collection. So, E sub x is a subset of y. Similarly, I define E super y so, that is a Y section. This would be a subset of x right. So, you look at all those points x and X, such that x comma y is in E, this is the subset of x.

So, let me draw this, maybe use a different color. So, let us say this is my y then what do I do? I draw a line through y parallel to X that will intersect E somewhere. So, you look at all those points. So, that gives me some set here and that is my E super y, it is a subset of x. So, I hope that is clear, the sections are clear.

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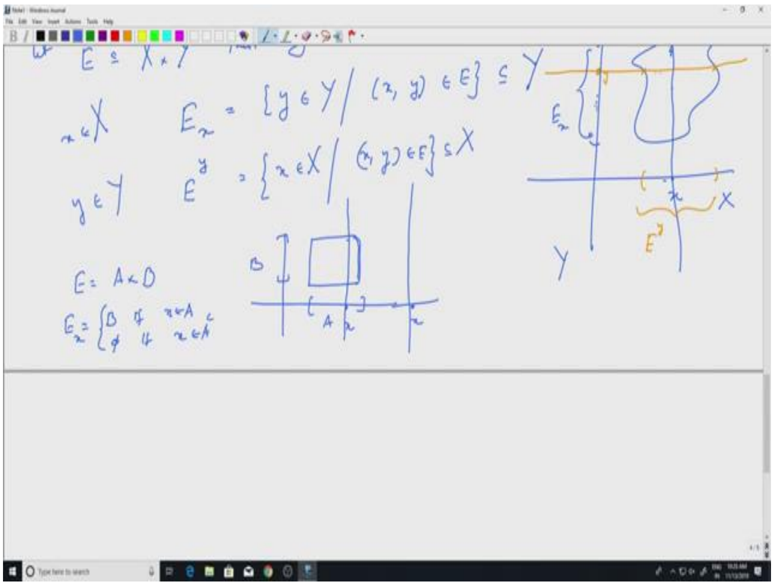

Well, the sections are nice sets, why is that? So, let us start with the. So, let us look at some trivial example so that this is even clearer. So maybe take E to be A cross B. So I will draw this slightly smaller. So let us say this is A, A here. So let some you can think of this as R2 and some interval if you like, does not really matter. And I have B here, some set. So A cross B is the measurable rectangle here. What are the sections? So E sub x, so I fix an x, either here or it can be outside. If it is here, I draw the line and I will get only B.

So E of x is B. If x is in A. So if x is in A, if x is outside A, so if x is here, then this line does not intersect A cross B at all. So then it is empty. So it is empty set if x is in A complement. Similarly for y sections, correct? So we continue. So that is a easy example.

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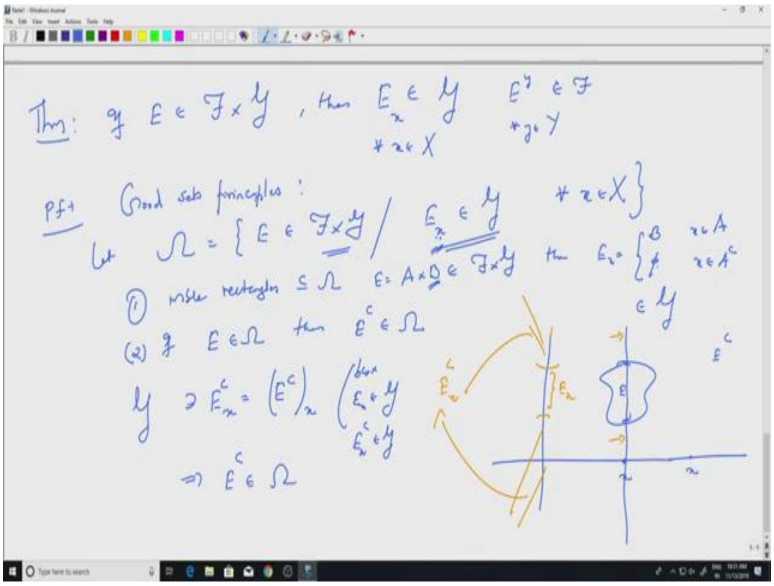

Well, the sets in X cross Y can be complicated it may not be measurable rectangle always. But the sections are going to be nice. So, that is the first theorem. So if E belongs to script F cross script G, so that means we have a space x, we have a space y and we have F cross G, then the sections are also measurable, E sub x, so remember E sub x is a subset of y. So it is going to be measurable in G. It is going to be set in G and E super y. So, this is true for every small x in capital X, E super y is a subset of x. So this would be in script F for every y in Y.

So let us prove this. Proof uses good sets principles. So, let us look at good sets principles. Well, what is good sets principles? Well, we will collect all the good sets. So, let omega be the collection of all those sets E in script F cross script G such that E sub x belongs to G. Remember E sub x is a subset of y. So E sub x. Yeah, for every x in X, for each x section, we land in script G, you look at all those sets, we want to show that good sets are all the sets, all the sets in F cross G are good sets.

Remember, F cross G is the product sigma algebra, it is the sigma algebra generated by measurable rectangles. So it contains more than just the measurable rectangles, unions complements, intersection and so on. But let us see what omega is, omega is the collection of good sets. So let us see what all it contains.

So, first of all measurable rectangles are in omega. Why is that? Because if I look at A cross B, which is in script F cross script G, then the section so if I take E sub x, remember? We just did this, this is B if x is in A empty otherwise. X is in A complement. So E sub x is either B or phi. But B is in script G. So this would be in script G all the time. So this is satisfied.

So measurable rectangles are in omega 2. Well, now you can see where it is going. If E belongs to omega, then E complement also belongs to omega. Well, so now I am not assuming E to be a measurable rectangle, E is an arbitrary set in omega, which means E is a set in F cross G such that the sections are in G, x sections are in G. I want to say E compliment also has the same property.

Well, why is that? So what do we do? Look at the section of E complement. So I want to look at this section of E complement. So let us draw some picture. So let us say this is E, E complement is whatever is outside, So whatever is outside is E complement. So I take a set's point x. Well, maybe I take a point here that is that will show it better.

You draw a line through x and see where it intersects, that gives me E sub x. But what is E complement sub x? Well, you draw the line through x and see where it intersects E complement. So it intersects E complements at these places. So I will get something from here to upstairs and something from here to downstairs, this portion which is the complement of whatever is here. Which is, but this is simply E sub x.

So this portion will consist of Ex complement, so which is same as E complement sub x. So this is simply E complement. So you look at E compliment first as a subset of F cross X cross Y, and look at the section.

But then this tells me that E complement has the property that the x sections R in G, because this is in G because E (comple) E sub x is in G. So, that means, because E sub x is in G, E sub x complement will be in G, which is same as E complement sub x. So, E compliment has the property that all sub x x sub sections are in G. So this implies E complement itself is in omega. So omega is closed under complementation. And you can do same thing with a countable union.

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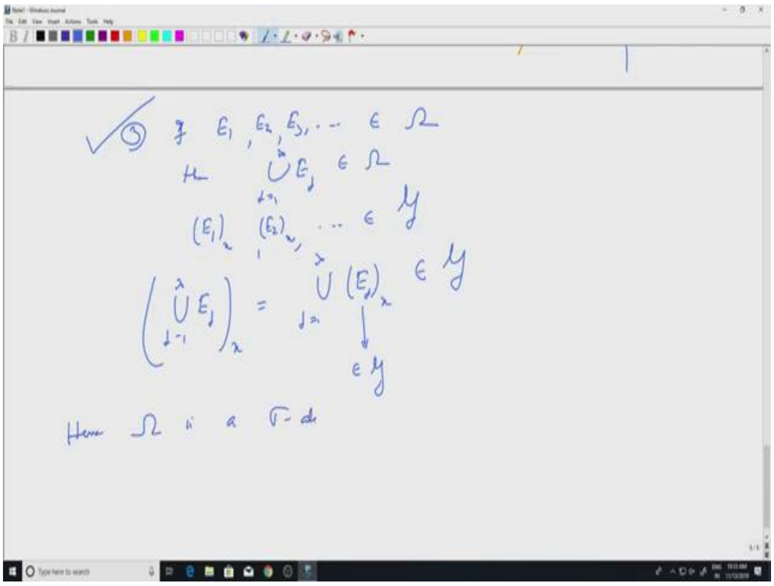

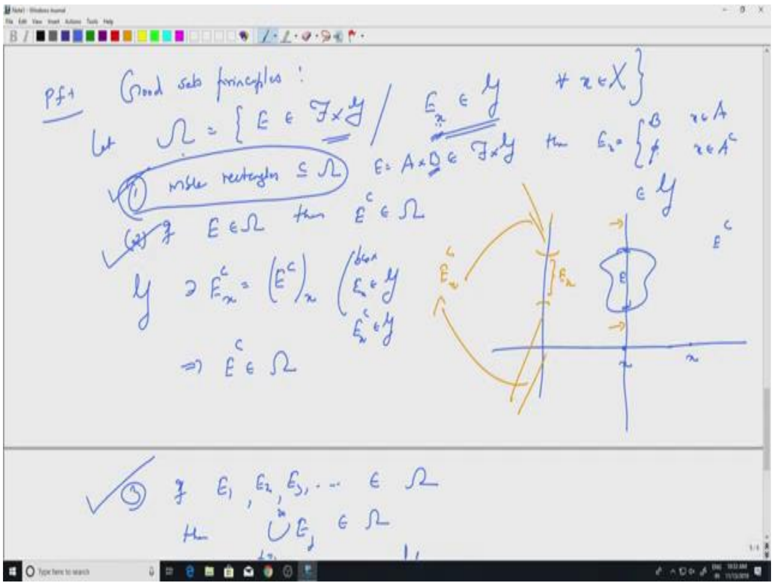

So 3, if E1, E2, E3, etcetera belong to omega then union Ej, j equals 1 to infinity also belongs to omega, why? Well, what I know is E1 sub x, E2 sub x, etcetera, etcetera, they all belong to script G. I want to show that union belongs to omega. So, you look at union Ej, j equal to 1 to infinity, and you will look at the x section. Well, just like what we did earlier, we will get that this is actually you look at the x section of each of them, j equal to 1 to infinity and take the union but each of them belongs to script G.

So, the union also belongs to script G because it is a sigma algebra So now, we have proved three properties, what are the properties? One is measurable rectangles are in omega, omega is closed under complementation, omega is closed under countable union, which means omega is a sigma algebra. Hence, omega is a sigma algebra.

Well, the whole space is there, the empty set is there, it is something which you can check very trivially. So, it is a sigma algebra, but it is a sigma algebra which contains the measurable rectangles.

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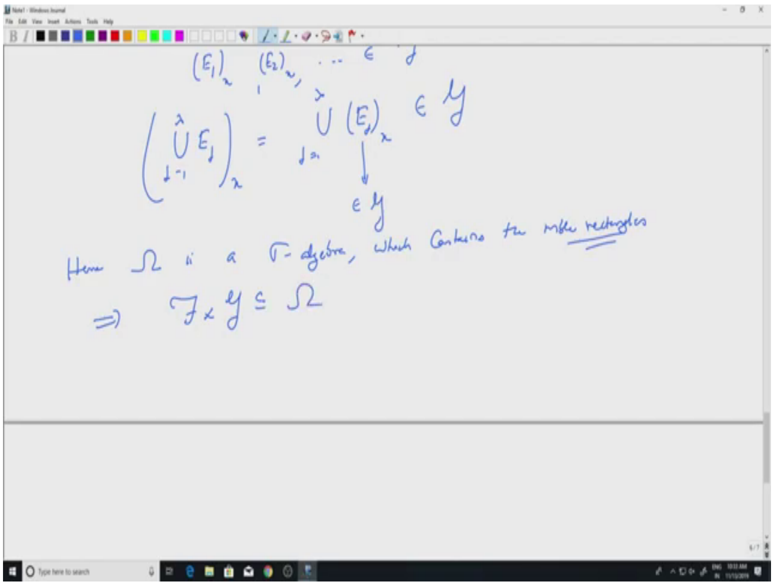

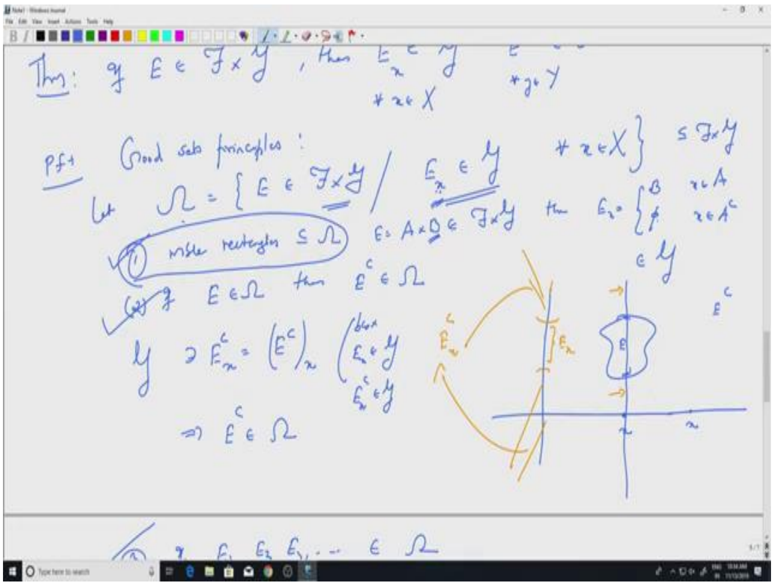

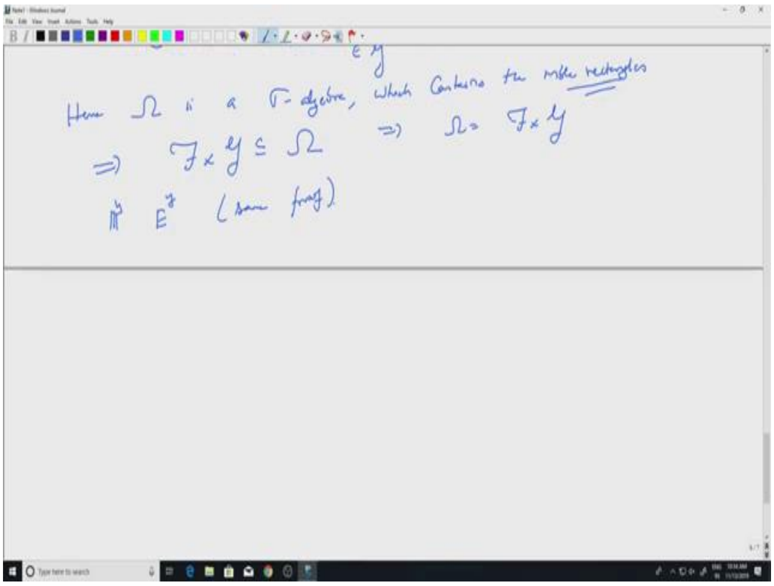

But the smaller sigma algebra containing measurable rectangles is F cross G because that is the product sigma algebra which contains the measurable rectangles. Hence, it will contain the sigma algebra generated by measurable rectangles. So, this implies script F cross script G which is the sigma algebra generated by measurable rectangles, that will also be contained in omega because this is the smallest sigma algebra containing measurable rectangles and so it will be contained in omega.

So, now if you look at the good sets definition, this omega is something which is contained in F cross G. Now, we have proved the other way. So, omega is equal to F cross G which means whatever set I take in F cross G, I have this property that sections are measured. So, this implies omega is actually equal to F cross G. So, all sections are measurable similarly for E super y. Same proof. So, well this immediately tells us something

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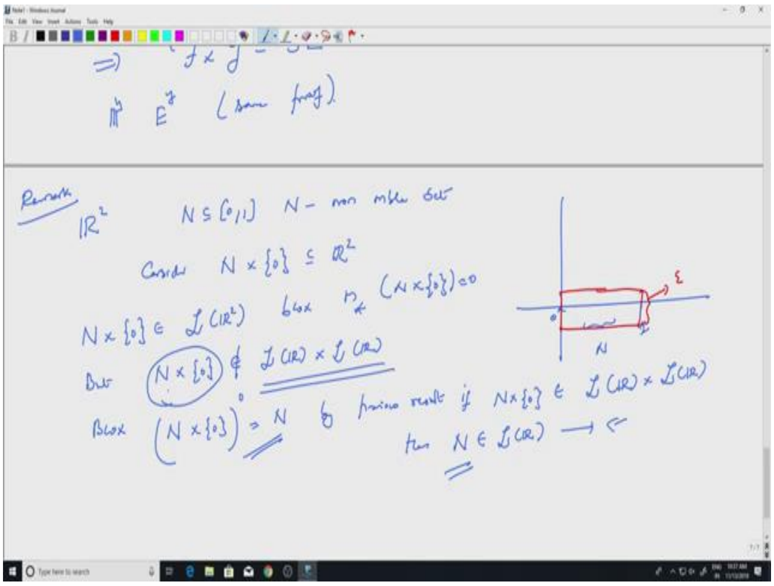

So, let us, we will finish with that. Let us look at R2, so maybe example or remark. Let us look at R2 and let us take A, which is a, or N, let us call it N, which is contained in 01. So, remember we defined, we constructed a non-measurable set, non-Lebesgue set. And consider, N cross 0, so this is contained in R2. So, I have 01 here and N is here, it is a subset here. And I am looking at cross 0.

So, it is on the on the real line as a subset of R2. Now, N cross 0 belongs to Lebesgue sigma algebra of R2. Well, why is that? Because, the outer measure of N cross 0 is 0, so we have done this; is 0, because I can cover this with small rectangles of length epsilon. So, take the width of the rectangle to be epsilon and the length is 1. So, the area of this would be epsilon and epsilon can go to 0.

So, that will tell me that N cross 0, anything inside the real line will have outer measure 0, because the real line itself has outer measure 0 in R2. Remember that we are looking at the two-dimensional Lebesgue measure. So, because of that this is a measurable set, but it is not, but if I look at N cross 0, this cannot be in LR cross LR. Why is that? If it is the product sigma algebra here then if it belongs to the product sigma algebra, the sections will be in each sigma algebras, because N cross 0 if I look at, super 0.

So, that is a 0 section of N cross 0. That is just N, this is just N and by previous result, if N cross 0 was in the product sigma algebra, then the Y sections will be in the second sigma algebra. This because of this, so, but that is not true. This is a contradiction.

So we will stop here. We have just defined the product sigma algebra to be the smallest sigma algebra generated by measurable rectangles and we have seen that sections are measurable. So, that happens only if you are underlined sigma algebra is a product sigma algebra. We will continue with this, we will define the measures using the measures on x and y. We will and the Sigma algebra we have just defined on X cross Y. We will define a measure on X cross Y, that is our next step.