Measure Theory
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Lecture 41
Fubini’s Theorem I

So, in the last lecture we saw product measures. So if you have two sigma finite measure spaces, you know how to define a measure on the product space. So, on the product space we have a product sigma algebra and the product measure. So, we, the theorem we proved in the last lecture was that you can define a quantity for each set in the production of algebra which is what we call the product measure.

So, we actually did not prove the countable additivity, so I will start with that today, but our aim in the next two sessions will be to prove what is known as Fubini’s theorem, which is one of the most important theorems after we saw all the basic theorems like dominated convergence theorem, monotone convergence theorem, and Fatou's lemma. So, Fubini’s theorem will allow us to interchange integrals, which actually helps in a lot of cases and that is what will be done in the next two sessions.

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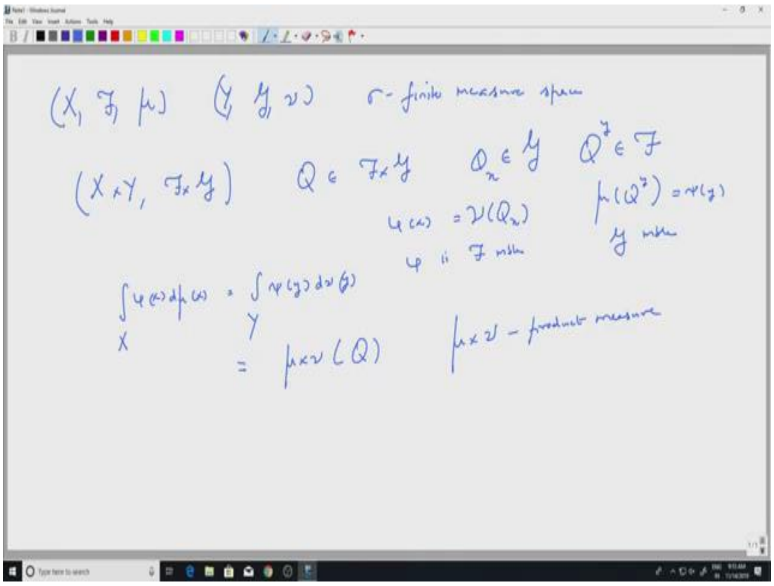

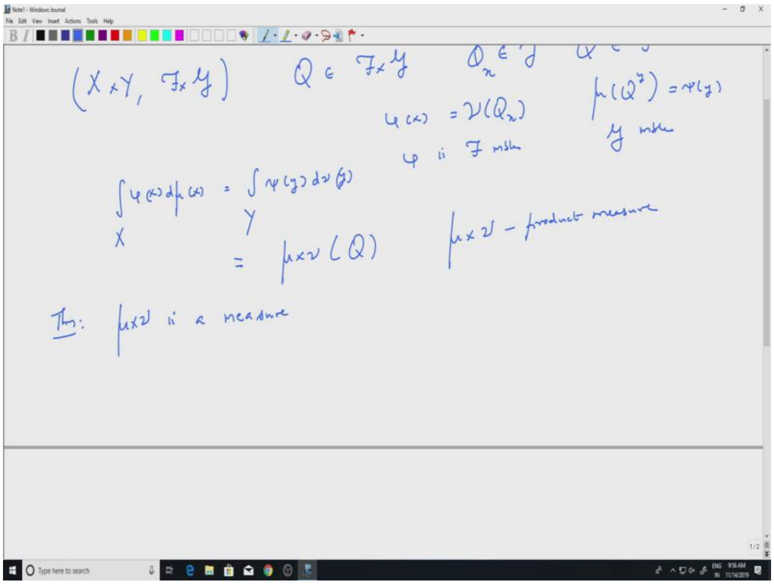

So, let us start. So, recall that we have two spaces now, we have X, F, mu Y, G, and nu. So, these are sigma finite measure spaces that was used very, very crucially, if you remember sigma finite measure spaces. We decompose both X and Y into sets of finite measure and then did analysis on that each piece which helped us in putting together everything to get a measure on the product space.

So, we have X cross Y, we have F cross G and on the space, so, the theorem we proved was if you take any set Q in F cross G, then we define two functions associated to it. One is, so, we have Q sub x, which is a section of Q, the x section, but this will belong to the sigma algebra G and Q super y, which is another section, but y section which will belong to script F.

And since this belongs to G, nu of Q sub x make sense and similarly, mu of Q sub Q super y makes sense and we call this phi of x and this was psi of y and these were measurable phi was, so phi is script F measurable. So, phi is a function on capital X remember that, and psi is a function on capital Y, and this is G measurable. And so, one can integrate them with respect to the corresponding measures. So, what we proved was, if I have such a situation, then integral over X phi x d mu x is same as integral over Y psi y d nu y this is what we did. So, these two are equal and this quantity is what we called the product measure.

So, cross nu of Q so this is the quantity we associated with Q and this is what we want to call product measure. So, mu cross nu is the product measure. Of course, we need to say that it is actually a measure, so that means it is countably additive, so let us get rid of that first and then we will go on to Fubini's theorem. So, consider this as a theorem if you like, the quantity we have defined mu cross nu is a measure, which means it is a countably additive set function. Well, the proof of this is sort of one line, because we have now done everything in detail.

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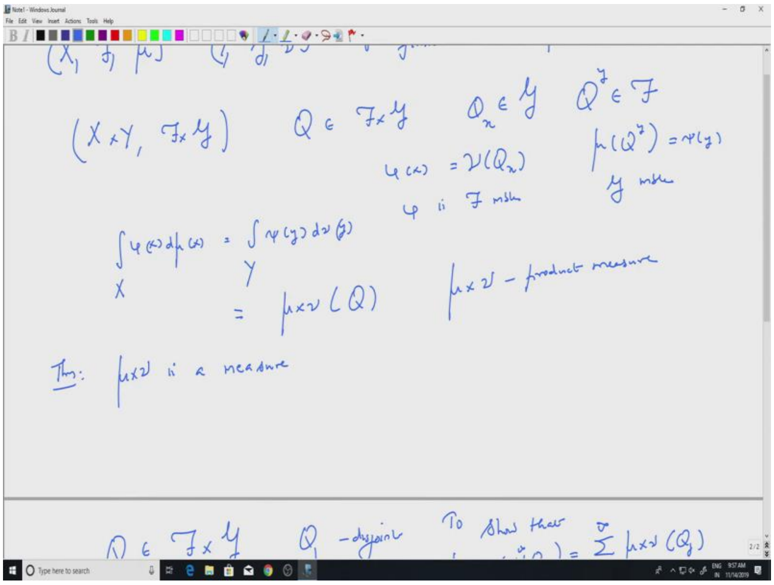

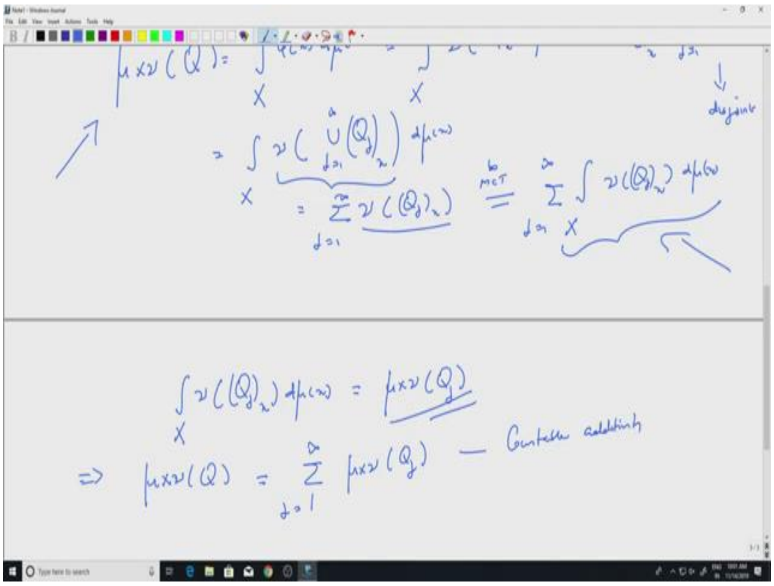

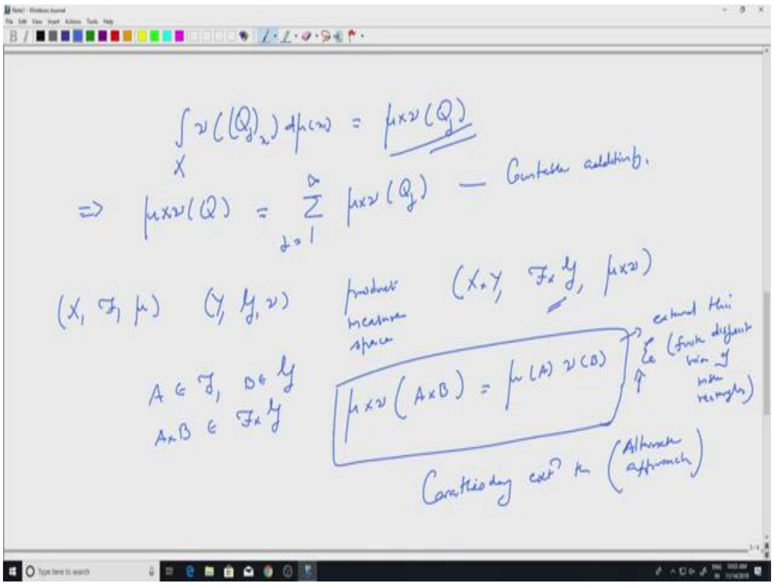

So let us take Qj in script F cross script G in the sigma algebra Qj disjoint. So, we need to show that, so our aim is to show that the quantity we have assigned is countably additive, j equal to 1 to infinity equal to summation mu cross nu of Qj j equal to 1 to infinity. So, let us call that Q equal to the union Qj j equal to 1 to infinity. Of course, this is a measurable set, it is a union of measurable set, so this would be in the product sigma algebra.

Now, let us look at mu cross nu of Q. Well, this is nothing but from our definition, so we have a recall there is a phi and there is a psi and so on, we will let us use phi for the time being. So, phi of x, so you can integrate this with respect to x. So, phi of x d mu of x, so that is the value of mu cross nu at Q. So, what is phi? So, let us recall that phi of x is nothing but you look at Q sub x, so, that is a set in script G and calculate its measure. So, we know it is well defined and it is a measurable function, it is a positive measurable function.

So, this is nothing but integral over X phi of x is nu of Q sub x d mu X. So, what is Q sub x? Q is this union disjoint union so, Q sub x is the union j equal to 1 to infinity of Qj sub x. So, I can write that, so this is equal to integral over X nu of Q sub x is simply j equal to 1 to infinity Qj sub x d mu x. But these are disjoint, this is a disjoint union. And so, Q sub x will also be a disjoint union. So, this is also disjoint and nu is a measure, so this will be sum. So, this would be equal to summation j equal to 1 to infinity nu of Qj sub x.

But now, this is familiar situation you are adding positive functions in x and so integral and summation can be interchanged that is this monotone convergence theorem. So this is equal to by MCT, we have done this several times. So this would be summation j equal to 1 to infinity integral over X nu Qj sub x d mu X. So, we are in good shape. We started with mu cross nu of Q and we got some expression.

So, what is that expression saying? If you look at the integrant, so if you look at one term there in the summation, so, that is integral over X nu of Qj sub x d mu x. So, what is this? Well, we can trace back and we will get there this is nothing but mu cross nu of Qj that is the definition of mu cross nu of Qj. We know that instead of nu we can also take mu, this functions phi and psi we have defined for Q, we can define for Qj's and we will get this.

So, plug that in you will get that, so all this implies that mu cross nu of Q, so this one I know is this one, which is equal to, so but each of them I have computed to be this. So, this is simply j equal to 1 to infinity mu cross nu of Qj, which is precisely the countable additivity, so countable additive. So, that is what is called the product measure.

So, it is a genuine measure. So, we, if we have two spaces X, F, mu and Y, G, script G, nu, we have the product space. So, product space is X cross Y that is the usual Cartesian product, F cross G, this is the sigma algebra generated by measurable rectangles and the measure mu cross nu, which we have just defined. So, it is a it becomes this is a product measure space.

So, the one thing you should always remember is that if I take a measureable rectangle. So, I take A in script F and B in script G, then A cross B, of course is a subset of X cross Y, will be in F cross G, F cross G is this sigma algebra generated by such things which are called measurable rectangles. So, mu cross nu of A cross B, this is what we computed first if you if you look at the proof, this is simply, mu of A times nu of B. It becomes the product of measure of A and measure of B, this is what a product measure does.

So, one way of another way of doing this would be to start with this definition and then extend it to, extend this to elementary set. So, elementary sets are finite disjoint union of measurable rectangles, so we know it should add up, so measurable rectangles and prove that it is countably additive there and then extend it to the sigma algebra generated by script E that is by monotone class theorem which is F cross G.

So, if you have a countably additive measure on script E, then you can uniquely extend it to script F cross script G that is called the Caratheodory extension theorem. So that is an alternate approach, so alternate approach. So, which we will not be doing this, but one could also start from basic objects like measurable rectangles and then extend the measure to a bigger sigma algebra, so that is one another way of doing it, which we will not be doing.

So, that that finishes the construction of the measure, but now we will look at various properties.

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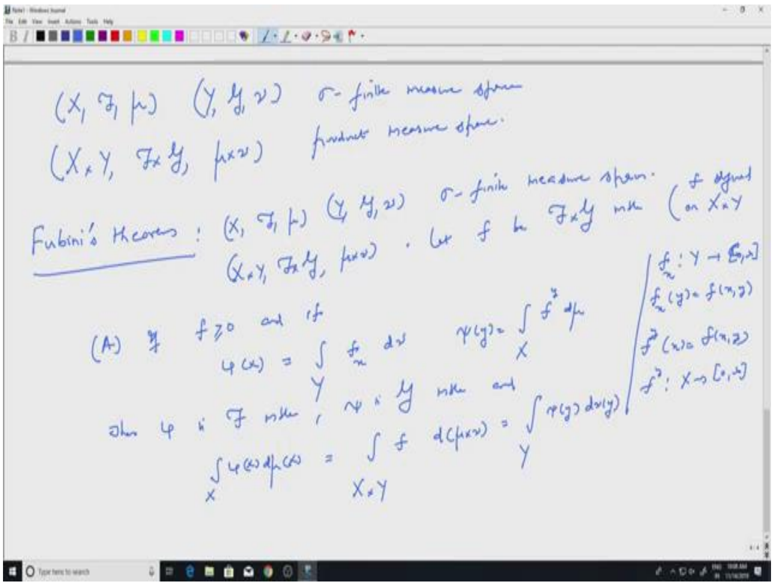

So, as usual we have two of these spaces now, sigma finite, so remember the sigma finiteness always, sigma finite measure spaces. And we have the product measure space, so that is given by F cross G sigma algebra generated by measurable rectangles and the product measure, product measure space.

So, now one can state Fubini's theorem, so this is the next most important theorem, Fubini's theorem. So, this gives us conditions under which we can interchange the integrals. So, remember we have X and Y, so when I integrate I can look at iterated integrals and so on. So, you must have done some of this in your B.Sc. and so on. So, think of this as a justification for such computations. So, we start with sigma finite measure spaces X, F, mu and Y, G, nu, sigma finite measure spaces.

And we have the product of course, we have the products product space as well, product measure space. So let F be a function, so let F be script F cross script G measurable, so that is a function on. So, F is a function on F defined on X cross Y, so it is a function of both X and Y in two variables X and Y. So, Fubini's theorem tells you the following, A, if F is non negative and if phi of x, so we have defined this for sets, now we define this for functions is integral over Y f sub x d nu.

So, before I go further, let me recall that f sub x is the x (functi) x section of f, it is a measurable function on y. And similarly psi y, so this is integral over X f super y. So remember f super y is the y section of f it is the function on x d mu. So let us recall that f sub x is a function on y to wherever complex numbers or here it is positive so 0 infinity. How is it defined? f sub x at y equal to f of x comma y. So you can integrate, so this is this will be measurable and you can integrate with respect to the measure nu on y, similarly for the other function, f super y at x is also f of xy, but f super y is a function on x, in this case it is a positive function, so we have this.

So, if f was indicator of a set, characteristic function of a set, this is what we did in the earlier theorem. So, then phi is script F measurable, psi is script G measurable, and the integrals are same that is what we did for sets and that is what is true for functions as well. And integral over X phi x d mu x, this is same as integral over X cross Y f d mu cross nu. So, that is the nu addition here, mu cross nu is a measure and f is a positive measurable function with respect to F cross G. So this integral makes sense, equal to integral over Y psi y d nu y. So, we will discuss the theorem as soon as I write down the whole statement.

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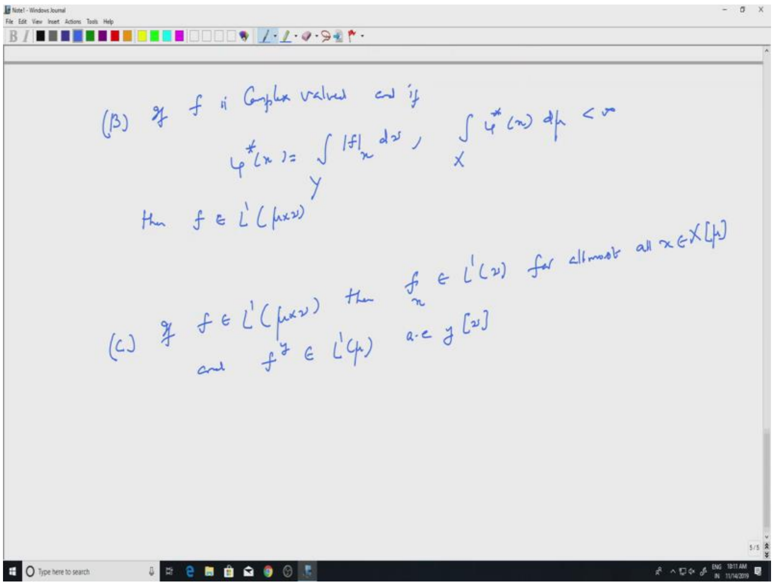

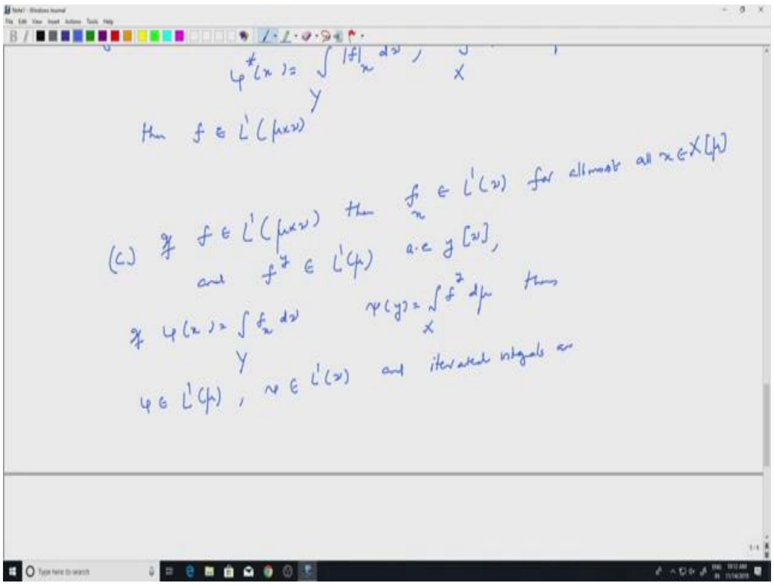

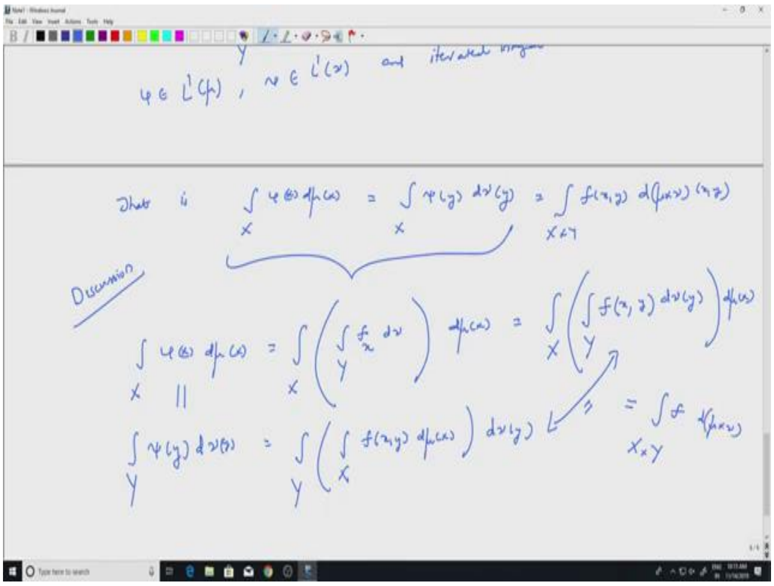

If so, here f was positive, now we will look at f complex valued, if f is complex valued and if phi star of x equal to integral over Y mod f sub x. So, remember mod f is a measurable function sub x is the section. So, nu d nu mod f sub x is a function on y and so now, that is a positive function, so you are simply integrating a positive function, comma integral over X phi star x, so, those are positive measurable function integrate. So, if this is finite, so, that is one integral which we are looking at then f belongs to L1 of mu cross nu that is one more (stat) one more statement in the theorem.

If f is in L1 of mu cross nu, then the sections f sub x is in L1 of nu, f sub x is a function on y, so I can talk about whether it is in L1 of nu or not, for almost all x in X. So, the almost every L will be with respect to the measure mu on X. And similarly, and f super y will be in L1 of mu, remember f super y is a function on capital X. So, it will be in L1 of mu, so almost everywhere y, but that would be with respect to the measure nu on y, correct, for almost all y, f super y is a function on x will be in L1 of mu.

And the iterated integrals are same. So, comma, the function phi, what is phi? So, phi of x is integral over Y f sub x d nu and psi y is integral over X f super y d mu. Phi will be in L1 of mu, phi is a measurable function on X, I am saying it will be in L1 and psi will be in L1 of nu and so, I can write if then psi will be in L1 of mu and iterated integrals are same.

What are the iterated integrals? So, this is a longer statement. So that is integral over X phi x d mu of x is equal to integral over X psi y d nu, d nu of y. Of course, they are all same as X cross Y f of x comma y d mu cross nu x comma y. So, I am just writing the variables x comma y. So, these are the iterated integrals, why are they called the iterated integrals? So, let us, let us look at that before we go into the proof of these theorems. Let us look at the first integral, so first integral is, so this is just a discussion, this is not the proof.

So, discussion of the theorem, phi of x d mu x, what is that? Well, what is phi? We know how phi is defined, so that is an integral over Y f sub x d nu, this is my phi. And then I integrate with respect to mu, this is what we had seen. Well, let us write this in a slightly better form, so integral over X integral over Y f sub x d nu, so that is with respect to the variable y. So, instead of writing sub x, so, f sub x at y is f of x comma y d nu y. So, I am integrating first with respect to the y variable and then with respect to the x variable that is the phi x integral.

What about the psi y integral? So, integral over X integral over Y psi y d nu y. So, we do the same thing, so integral over Y integral over X, this would be simply f of f super y, so that is f of x comma y d mu x. So, you integrate with respect to x first and then integrate with respect to y. We are saying these two integrals are same under some conditions, so these two integrals are same. So, iterated integrals are same. So, you whether you integrate with respect to y first and then with respect to x does not matter they are same and they both are equal to the total integral of the function f over the product space X cross Y, this is what the content of Fubini’s theorem is.

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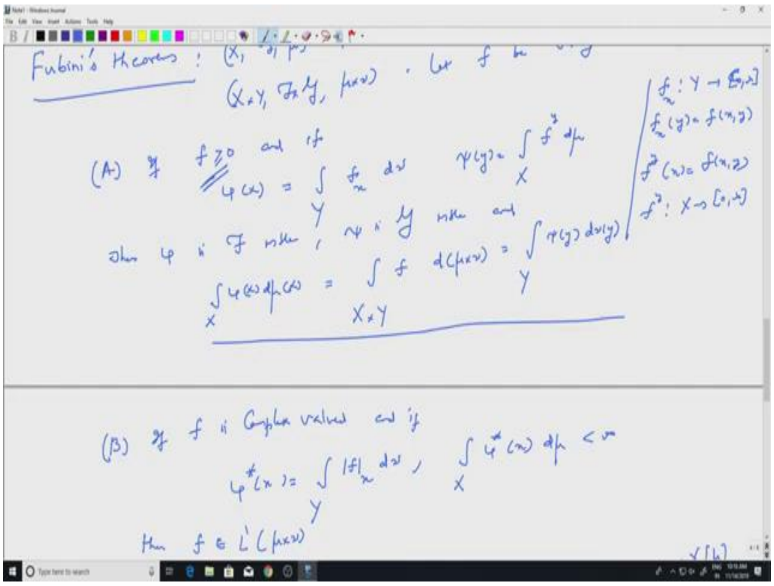

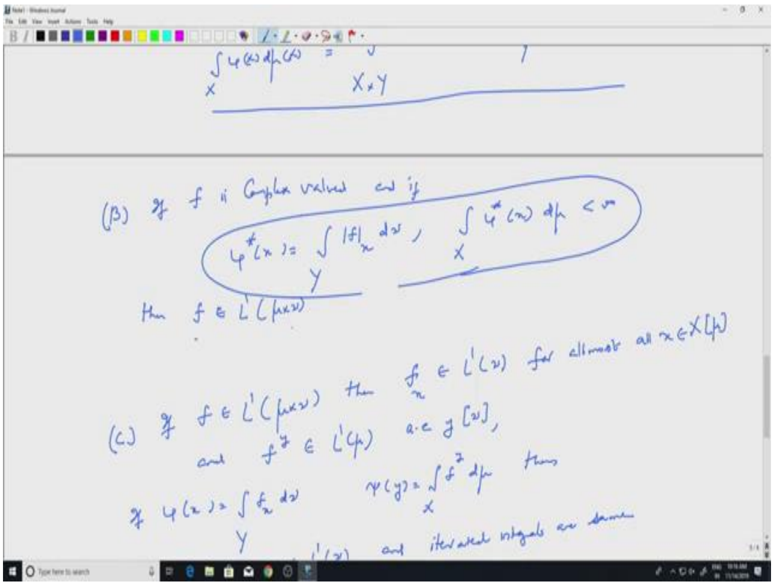

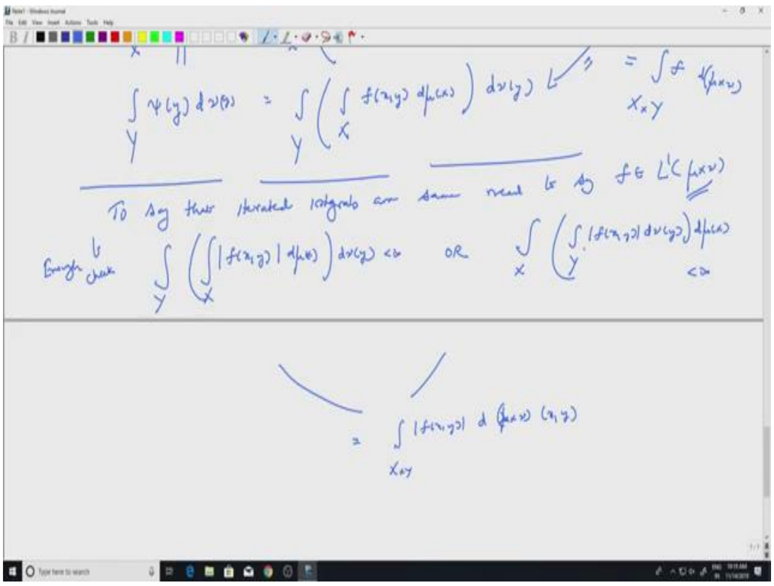

But what does it say? Let us go back to the statements. If f is positive, then you have this equality. So for any positive function, it does not matter which order you integrate, you the iterated integrals will be same. What is the second condition? If f is complex valued and if you have one iterated integral to be finite, you are looking at, you are not looking at the psi function, you are looking at only phi function, you can of course define psi star in a similar manner and you will have the same result. So if one of the iterated integrals is finite, then the function is in L1. And the c says if function is in L1, then the iterated integrals are same.

So, how does one check that iterated, how does one use this theorem? Well the, it is very easy to use a theorem, because if I want to say iterated integral same, I need to show that. So, to say that iterated integrals are same, we need to say f is in Ll of mu cross nu that is what we need to say.

But how will we do that? We will look at mod f, but if you look at mod f, mod f is a positive function and for positive functions, the theorem says that iterated integrals are always same. So for mod f all that you have to do is, so enough to check that you look at mod f, so modulus of f of xy, so let me write it in that form. Mod f of x comma y. Now, this is a positive function, so iterated integrals are the same. So, you look at either this with respect to x and then with respect to y or whether this is finite or integral over X integral over Y mod f of x comma y d nu y d mu x. This is finite, because iterated integrals are same as the integral of mod f over mu cross.

So, all these both of them are equal to integral over X cross Y mod f of x comma y d lambda mu cross nu of x comma y that is what the theorem says. So, it is very easy check, all you have to do is to take mod f, look at iterated integrals and see if one of them is finite. If one of them is finite the other is finite, they both are equal to the L1 norm of f, with respect to the measure mu cross nu. So, this is a good time to stop.

So, what we have just done is stating the Fubini's theorem in full. So, if you have a positive measurable function, you can simply compute the iterated integrals and see that it is actually in L1 in the product space and once it is in the L1 of product space, you can look at the iterated integrals. So, this becomes an extremely useful theorem when you want to interchange integrals. A special case is when X, when both the measures are counting measures, so this is a cell which you would have seen much earlier. If I have a double summation with positive terms, then I can interchange the order of the summation that is, of course, Fubini’s theorem. So we will stop here.