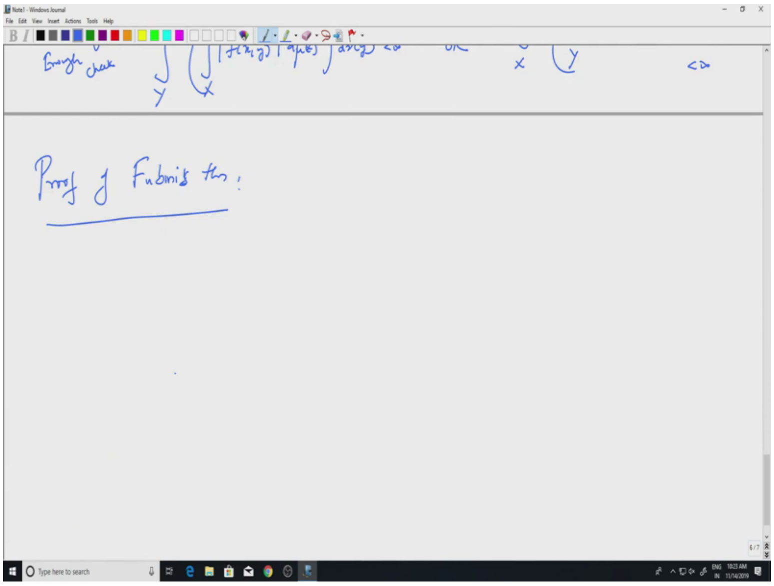
Measure Theory
Professor E.K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 42
Fubini's Theorem II

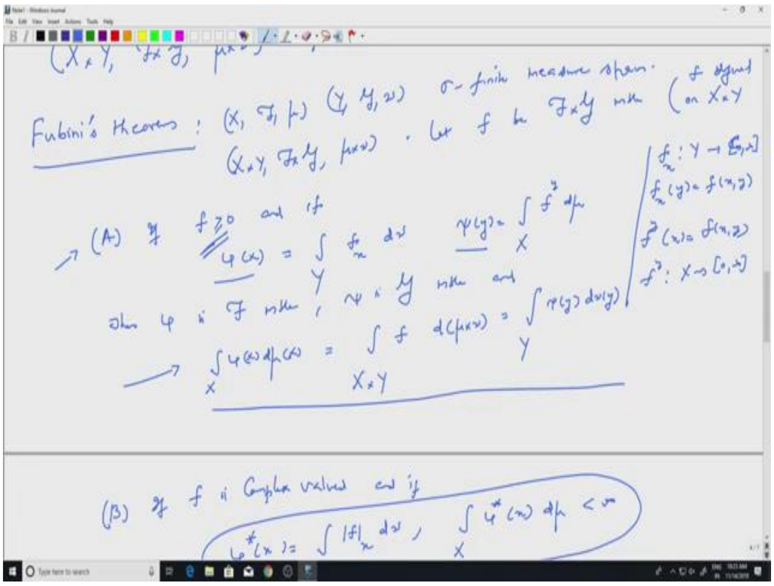
So, in this session we will prove Fubini's theorem, which allows us to interchange integrals for a class of functions in particular for positive functions. So, if you look back at the statement of the Fubini's theorem, we are not for positive functions we are not saying that the integral should be finite, whenever you have a positive function on a product space, it is possible to integrate in whatever order you want, so that is the contend for Fubini's theorem, but if the function is in the L1 space of the product measure, then of course the integrals will be finite. So, we will start with the proof now.

(Refer Slide Time: 01:10)



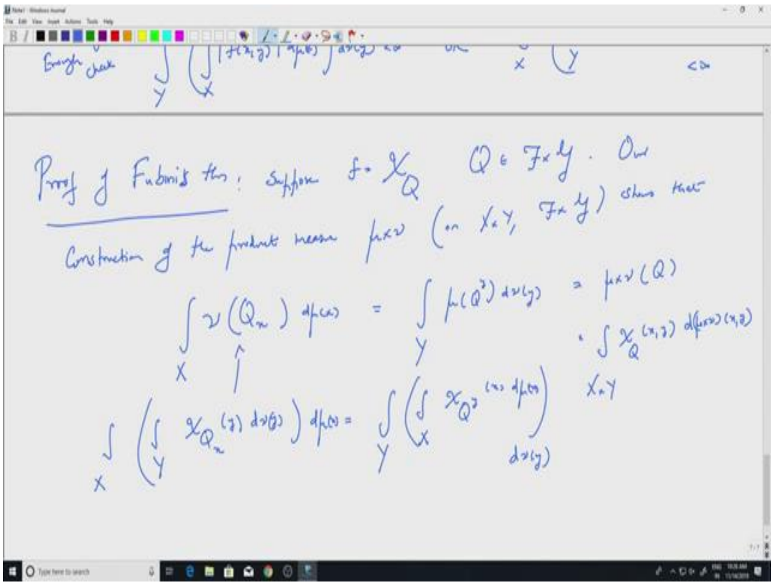
So, proof of Fubini's theorem. So, the proof follows exactly in the expected manner because for sets we have already proved this. So let us let us recall the statement of the theorem.

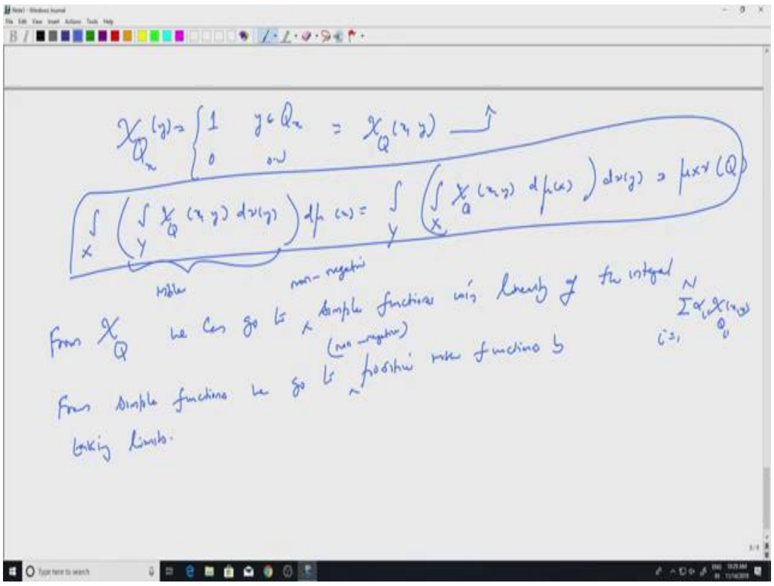
(Refer Slide Time: 1:33)



So, we will look at the statement, A, first. And if you look at the statement A, you see that the function phi and psi which we defined for sets, and we have shown that these integrals are same. So, this is true whenever the function is an indicator function that is precisely the content of the construction of the product measure. So let us start with that.

(Refer Slide Time: 2:00)





Suppose f is equal to indicator of Q characteristic function of Q, where Q is a measurable set, it is an arbitrary measurable set, it need not be a rectangle. Our construction of the measure. So, our construction of the product measure mu cross nu so this was on X cross, so this was on X cross Y comma f cross G, this is our product measure, this shows that.

Well, how did we construct this? This shows that if I look at Q sub x and then you look at nu of that measure that set, you get a function which is measurable and then you integrate this function over X with respect to the measure mu, this is same as integral over Y mu of Q super y d nu y, which is what we call mu cross nu of the set Q, which is integral over X cross Y indicator of Q. Well, I will write X comma Y and you have the measure mu cross nu x comma y as the variables, this is what the construction of the product measure is.

But what is this? So, let us rewrite this, the left hand side is integral over X this is integral over Y indicator of Q sub x at y d nu y, so this would be the measure here and d nu x. Of course, this is equal to integral over Y integral over X chi Q super y at the point x d mu x, so that would be the inner integral, and you have the outer integral with effect to y.

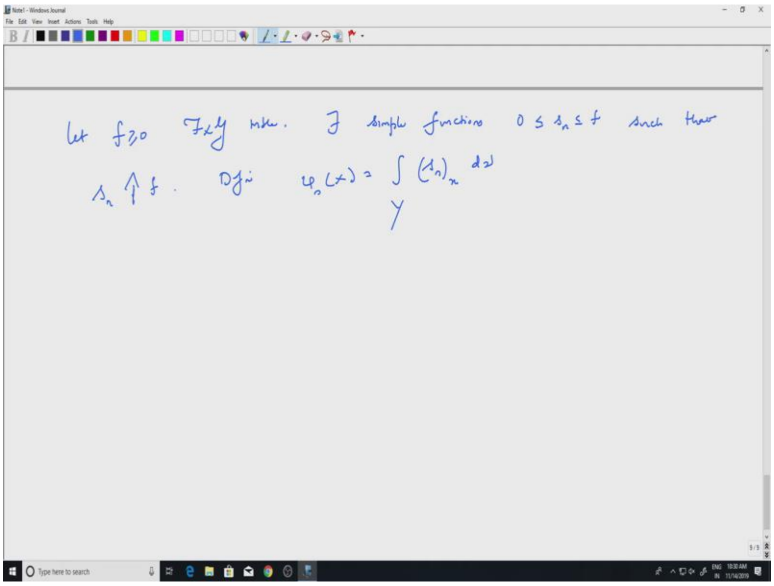
But what are these chi Q sub x y, so chi Q, so let us try to write it down. What is chi Q sub x at y? Well, this will be 1 if y is in Q sub x 0 otherwise. And you see that this is actually equal to chi Q of x comma y, x is fixed, so it is just a section and that is all is needed. So, if I plug that in the previous so you used here in the previous equality. What we have is integral over X integral over Y, chi Q of x comma y d nu of y d mu x, what we have what we are saying is, you do the same thing here, you will get that it is integral over Y integral over X chi Q of x comma y d mu x.

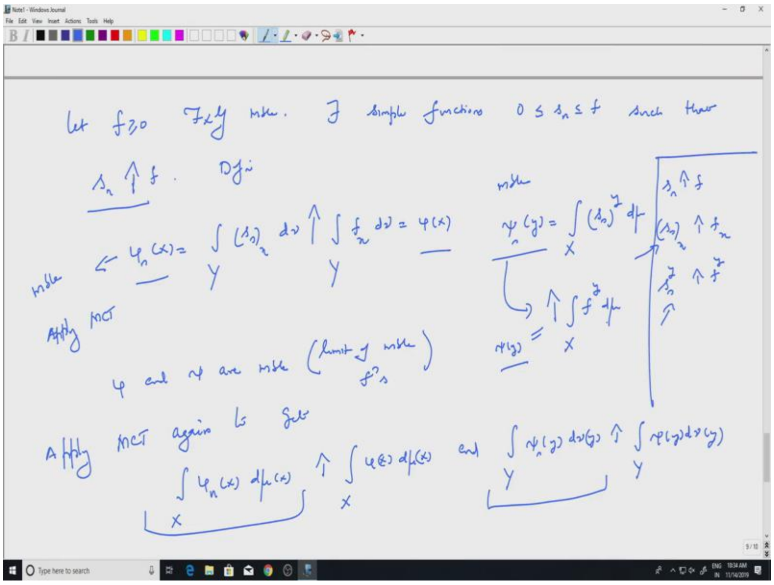
So, now, you are integrating with respect to x first and then integrating with respect to y, which is of course, finally equal to lambda cross sorry there is no lambda that is mu cross nu of Q, and that is the statement of the theorem. So, measurability, so, this is measurable follows, that we have already proved. So, for indicator functions, this is trivial. So, from characteristic functions, so from chi of Q, we can go to simple functions, positive non-negative simple functions using linearity, linearity of the integral.

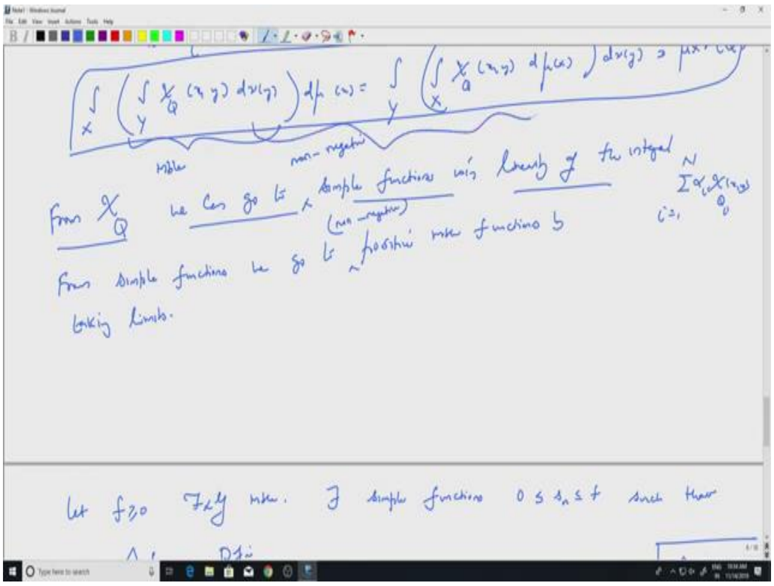
So, what I mean is, if I have this equality for f equal to chi Q, I can replace that with summation alpha chi Qi x comma y i equal to 1 to N. And then because everything is linear, this will become a sum of things and that each term will be equal to the corresponding integral in the other order and so on, so forth. So, I will leave that to you, so this is a trivial step.

Now, from simple functions we go to positive functions positive or non-negative measurable functions by taking limits. So, this is a standard method which we have seen earlier taking limits. So, let us see how to how to do this, so we need to define some more functions.

(Refer Slide Time: 07:46)







So let f be positive and it is f cross G measurable, so it is on the product space. So, there exists simple functions 0 less than or equal to Sn less than or equal to f such that Sn increases to f that is one of the basic results, so we have this. So define, so you can define phi n, well what is phi n? It is a phi function associated with a set. So, phi n of x is integral over Y Sn sub x d nu. And similarly, I have so, let me write it here because then I can show you how the limit is, so, we write it here. Define phi n of x to be integral over Y Sn sub x d nu and we look at psi n of y to be integral over X Sn super y d mu.

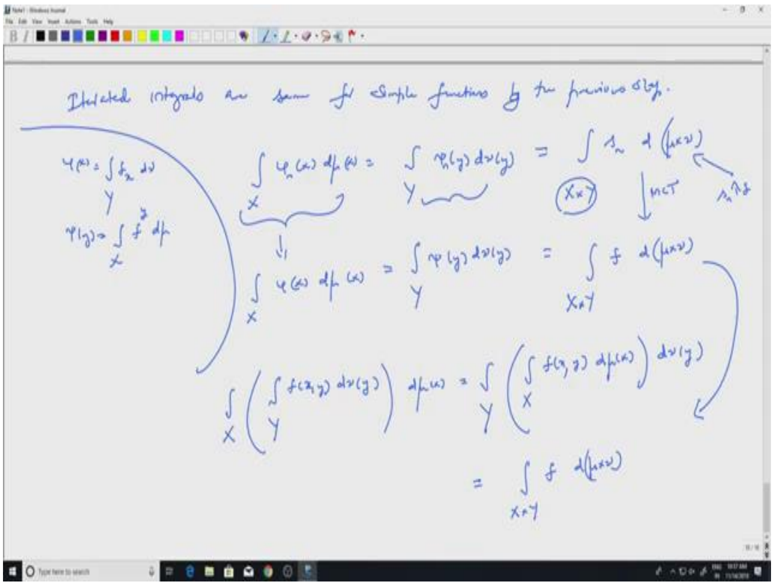
So, these functions, so Sn's increase to f. So, the sections also will increase to the corresponding sections of f. So, Sn increases to, so let me write it here Sn increases to f. So, Sn sub x will also increase to f sub x and Sn super y will also increase to f super y. So, you apply that. Because Sn, so we know phi n's are measurable, well, how do we know phi n's are measurable? Because Sn’s are simple functions, simple functions are linear combination of indicator functions and for each indicator function, we know it is a measurable function, so phi n is simply linear combination of measurable function, so it is measurable, similarly psi n’s are also measurable.

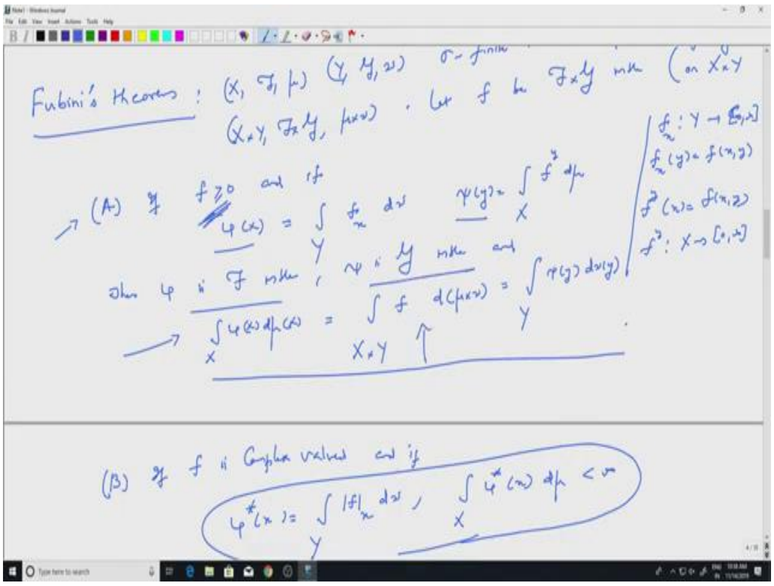
Now, apply monotone convergence theorem. Well, why am I applying monotone convergence theorem? Because I know that there is some increasing limits here, so, and they are all positive, so apply monotone convergence theorem. This will increase, increase to integral over Y f sub x d nu by monotone convergence. Similarly, this will also increase to integral over X f super y d mu.

So, if I call this phi of x, so phi function for f and this to be psi of y, then phi and psi are measurable, because there are increasing limits of phi n’s. So, apply because limit of, so limit of measurable functions, so this was one of the assertions. But you can apply monotone convergence theorem again. Well, where are we applying monotone convergence theorem again? Well, I know that phi n’s increase to phi. So, integral of phi n’s should increase to integral of phi by monotone convergence theorem, psi n’s increased to psi, so that should happen with psi n’s as well.

So, apply MCT again to get integral of X, integral over X phi n x d mu x, this will increase to integral over X phi x d mu x, and integral over Y psi n y d nu will increase to integral over Y psi y d nu y by MCT again. But at the phi n level, we know the equality, the iterated integrals are same for simple functions. So, from chi Q we can go to simple functions using linearity. So, we have the iterated integrals equality for simple functions.

(Refer Slide Time: 12:30)





So, iterated integrals are same for simple functions by previous by the previous step, what does that mean? That means, integral over X phi n x d mu x is same as integral over Y psi n y d nu y. Well, they are iterated integrals are same to the total integral as well, so this much we know X cross Y, this would be Sn d mu cross nu. Phi n and psi n are defined for Sn by taking sections.

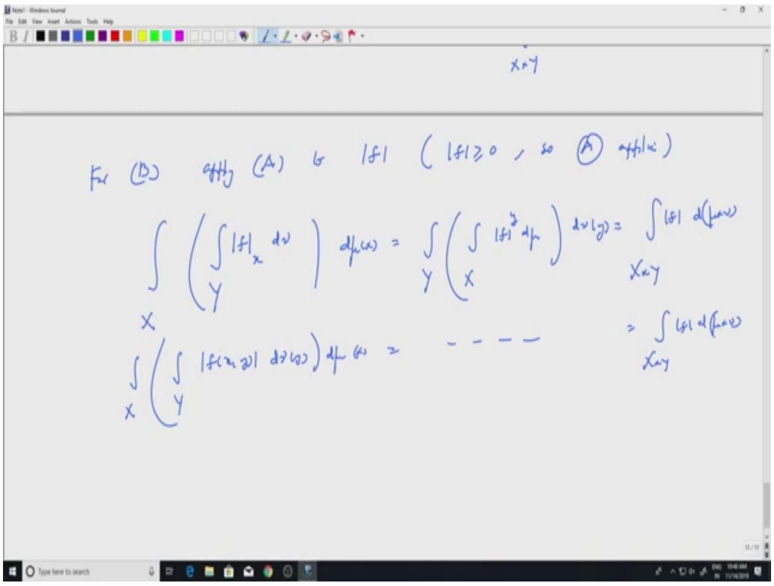
Now, I know that the, this one converges to integral over X phi x, so phi x, so I will write one more step just to be clear, phi x, phi is the function defined for f, which is same as I know this converges by monotone convergence theorem to psi y d nu y. And well, what happens here? I have the space X cross Y and I have a countably additive measure and I know that Sn increases to f, so I can apply monotone convergence theorem again here.

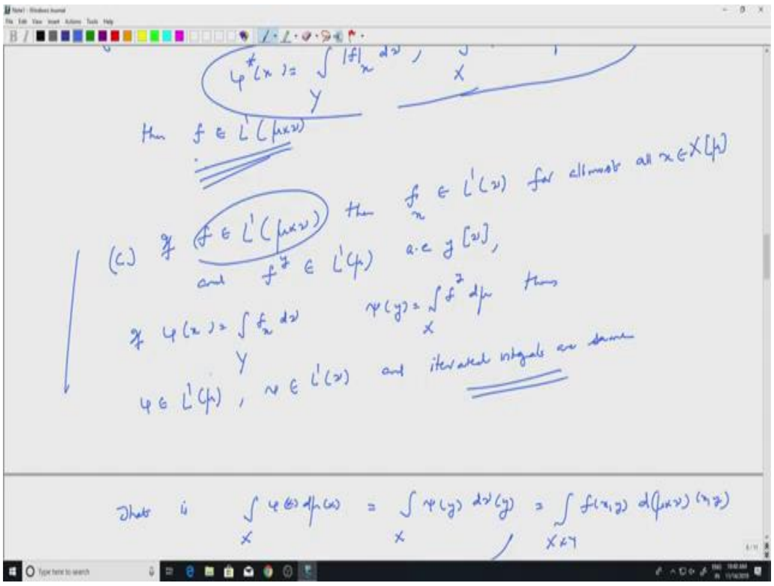
So, MCT saves the day, MCT here says at X cross Y f d mu cross nu. So, now I can put together everything. So, let me write it in full detail what is what was phi? So, let us recall phi here, phi of x was integral over Y f sub x d nu and psi of y was integral over X f super y d mu, so you write that. So, this would be integral over X, phi is an integral over Y, so integral over Y f sub x at y, so that is just f of x comma y.

And you are integrating with respect to nu that is with respect to the Y variable. And then you integrate with respect to that is your phi of x and then you integrate with respect to the X variable. So that is same as the next integral is integral over Y, psi is an integral over X, again, you will have f super y at x so f super y at x is f of x comma y d mu x because you are integrating with respect to X first, this is same as integrating with respect to, so iterated integrals are same, but more importantly, they are all equal to the total integral of f X cross Y f d mu cross nu. So, to compute for a positive function, the total integral, you can do the iterated integrals. So that is the first part of the theorem.

So, let us go back to the statement. So this was a long statement, we proved A. For f positive, we have measurability and the iterated integrals are same equal to the total integrals, so this is what we just established. Let us look at the complex valued case. Complex valued case, well simply follows from the first statement. So, let us let me indicate that so I will leave the details to you because it is sort of straightforward now, so we can start here.

(Refer Slide Time: 16:22)



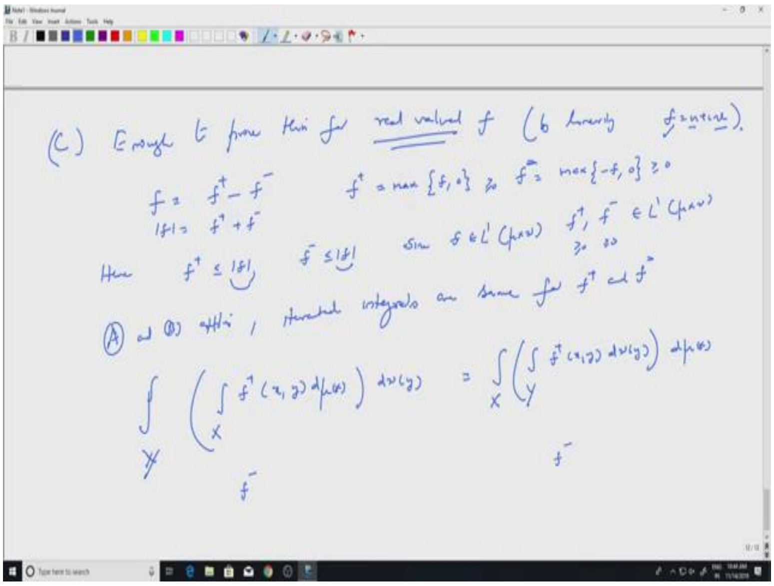


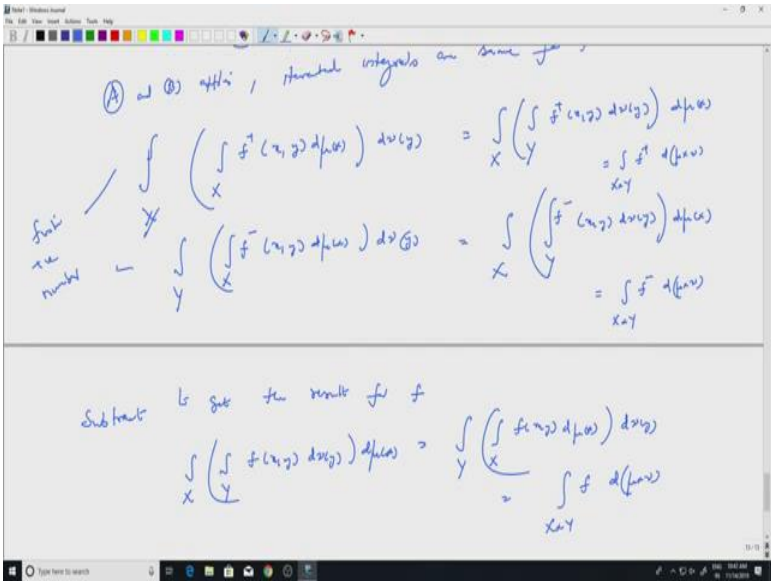
So, for for B, apply A to mod f. So, mod f is positive, so mod f is a positive function, so A applies, so A can be applied. So, if A applies, what does it say? We will get that if you look at the mod f the x sections and you integrate with respect to Y that is the phi function for mod f x. So, I will write this as d nu and then you integrate with respect to X d mu x, this is same as integral over Y integral over X mod f super y d mu and d nu y. This is same as integral over X cross Y mod f d mu cross nu. So, it is enough to write it. So, this is true for mod f, because mod f is positive.

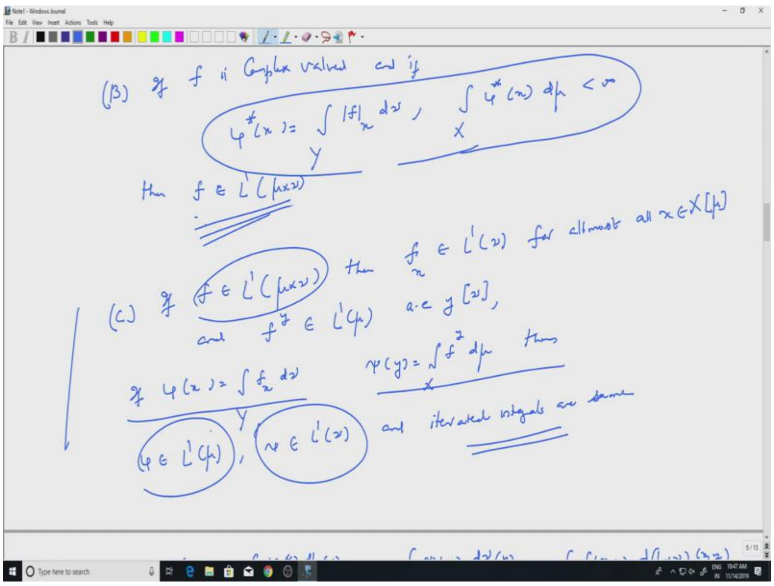
So, let us write it in the proper form. So, this is simply integral over X integral over Y mod of f of x comma y d nu y d mu x equal to etcetera, etcetera, equal to integral over X cross Y mod f d mu cross nu. So, I am not writing the whole detail. So that is all the second statement is, so that is the interesting part of the theorem because you can always take mod f and then apply iterated integrals and see if it is finite. So, if that is finite, if any of the iterated integrals is finite, then f will be in L1. Any of the iterated integrals of mod f is finite, then f will be in L1 of the product space.

So, let us look at C. In C, you are starting with an f in L1 and you are saying the iterated integrals are same that is what we want to prove. So, this is not difficult. So, I will try to give the details, but maybe only the sketch of the proof because it is not very difficult, the first part is the main part.

(Refer Slide Time: 18:46)







So, C. For this, so first assume or so maybe enough to do this for, so enough to prove this for real valued functions real valued f, f can be complex valued that is by linearity because you can always write f as u plus iv and if the iterated integrals are same for u and v, it will be same for f as well. So, assume f is real value, so, I can write f equal to f plus minus f minus, remember the positive part and the negative part.

So, what is f plus? f plus equal to maximum of f and 0. So, whenever f is 0, f is positive, it is f plus, f is negative f plus is 0 and f minus is maximum of minus f and 0. Wherever f is negative, you take the negative of that that will be your f minus. So both are positive measurable functions and f is equal to f plus f minus. So, also notice that. Well what is mod f? Mod is f plus plus f minus. So, this also tells me that hence, f plus is less than or equal to mod f comma f minus is less than equal to mod f but f is in L1.

So, since, f belongs to L1 of mu cross nu anything. So, mod f will have integral finite, so this will tell me that both f plus and f minus are also in L1 of mu cross nu, also they are positive. So, A applies, so A and B if you want, A and B applies. And iterated integrals are same for f plus and f minus. So, I will write it instead of bringing in phi and things like that, let me write it in one go. So I look at f plus of x comma y with respect to d mu x. So, first integrate with respect to X, and then integrate with respect to Y, so that would be same as the integral over X integral over Y f plus of x comma y d nu y and d mu x.

Similarly, for f minus and both of course, are equal to the integral of total integral over X cross Y, so I am not writing that. So, but maybe let me write this portion, because I am going to subtract d mu x d nu y integral over Y equal to f minus of x comma y d nu y, so first integrate with respect to Y and then integrate with respect to X d mu x. So, these two are same for both f plus and f minus because they are positive functions in L1, so the first two cases apply.

Now all you have to do is to subtract these two. So, there is some measurability part, which I am skipping, because it should be clear to you that what is measurable and so on. So, all you have to do is to subtract these two. So, remember that both the integrals are also same as, so this is equal to integral over X cross Y f plus d mu cross nu and similarly, this is same as integral over X cross Y f minus d mu cross nu that is the first and second part.

Because of this subtract to get the result for the result for f. So, you have to make sure that when you subtract you are not getting an infinity minus infinity term, but these are all in L1, so they are all finite quantities, so these are all finite numbers. So, finite positive numbers, in fact, because they are all positive.

So, I can subtract them and then I will get, so if I subtract I am going to get integral over X integral over Y f plus minus f minus that is my f, f of xy d nu y d mu x equal to integral over Y integral over X f of xy d mu x and d nu y, which is of course equal to the product, the full integral f d mu cross nu. So I am writing the same thing again and again, you can subtract because they are all finite. So, there is one small thing which I have skipped, but that is sort of trivial, so I will leave it to you to think about it that is the.

So, let us go back to the statement. Yeah. So, we have this and this they are measurable and they will be in L1 because their integrals are finite. So we have just proved that if I look at the plus part and the negative part, then the integrals are finite. So, we will stop here. The proof of Fubini's theorem is rather simple, but it requires you to do, it requires you to construct the product measure first and the construction of the product measure essentially tells you Fubini's theorem for indicator functions. And then from indicator functions, you can go to simple functions and positive functions.

Once you do it for positive functions, you will have it for all other L1 functions by appropriate linearity and things like that. So that is what we have done. So, the upshot of all this is if you have a positive function, the iterated integrals are same or if the function is in L1 of the product space iterated integrals are same. So, we will stop here.