Measure Theory

Professor. E.K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru

Lecture 43

Completeness-of-product-measures

So, in the last lectures, we saw Fubini's theorem, which allows us to interchange the order of the integration. As long as the function is positive or is in the L1 space of the product space, Product Measure. Then we know how to interchange the integral, so there will the iterated integrals will be equal.

So, that was the content of the last two sessions. We will look at the product spaces a little bit more closely, this would be essentially a discussion on completeness properties of the products spaces. So, product spaces need not be complete, product measures need not be complete, we have to complete them and in particular, we will look at how the Lebesgue measures behave.

So, for example, if I look at the Lebesgue measures on the real line and look at R cross R, I can look at the product measure on R cross R, how is it related to the two dimensional Lebesgue measures. So, we will see that it is done via completion of the product sigma algebra. So, that is our aim in the coming lecture. So, let us start.

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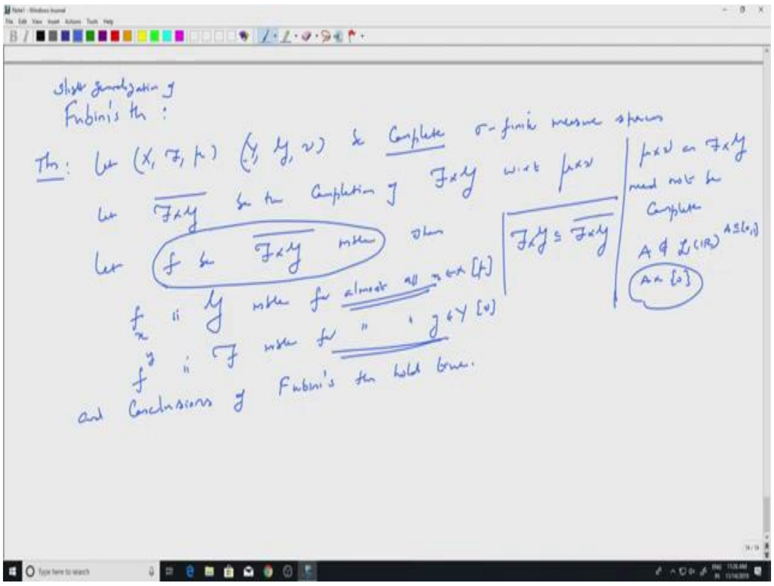

So, we had Fubini's theorem, Fubini's theorem. So I am not going to write this again, but it is possible to. So, slight generalization of this, so slight generalization of Fubini's theorem. So I will sort of explain this I will not prove it because it is essentially the same proof. The issue is that the product space, product measure need not be complete.

So, let me state this as a theorem. So, we start with two complete sigma finite measure spaces Y, G, u be complete so complete and sigma finite measure spaces. Let F cross G bar be the completion of F cross G. So, F cross G is the sigma algebra generated by measurable rectangles, F cross G bar is its completion with respect to the product measure with respect to Mu cross Nu.

Well, this is because the product measure need not be complete Mu cross Nu on F cross G need not be complete we have that, need not be complete. It is enough to look at a non-measurable set A which is not an L of R and look at A cross something 0. You will see that that, so the measure we have constructed was, the set we had constructed was between, inside 01 and you will see that, well actually, you can take appropriate subsets and you will see that this has outer measure 0 in L2, in R2 and so, or it is a subset of us set which has outer measure in R2 and so it should be in the Sigma algebra.

But it is not in the product sigma algebra. So, there are certain things here which I may explain later. So, as of now, one needs to realize that the product of two complete measured spaces need not be complete. So, you complete it by putting subsets of measure 0. Let f be F cross G bar measurable.

So that means it is a bigger sigma algebra. So, keep that in mind, F cross G is a sigma algebra and the completion is just putting more sets there. So, that is a bigger sigma algebra and if it is measurable with respect to the bigger sigma algebra, it need not be measurable with respect to the smaller one. Then well the first assertion is that f sub x. So, sections make sense, is G measurable. So, if we had the product sigma algebra instead of the completion of it, then it is G measurable always.

But once you complete it, there are sets of measures 0 coming in. So, this will be g measurable for almost all x in x with respect to a force the measure Nu. So, it is not true that for every x we have f sub x is measurable. But for almost all x it is and similarly f super y is script F measurable for almost all y in capital Y with respect to the measure Nu and conclusions of Fubini’s theorem, Fubini’s theorem hold true.

So, you can look at iterated integrals and they will be equal provided F is positive or F is in L1 of the product measure. So, we have enlarged class of functions to f cross G bar measurable, not just F cross G. Because they do not differ much. So, there is not much in this theorem. But you need to realize that the certain conclusions are true only for almost all X or Y depending on which space you are on, not for all the. So, let me also give another exercise, which I will sort of explain in the case of real line or R n.

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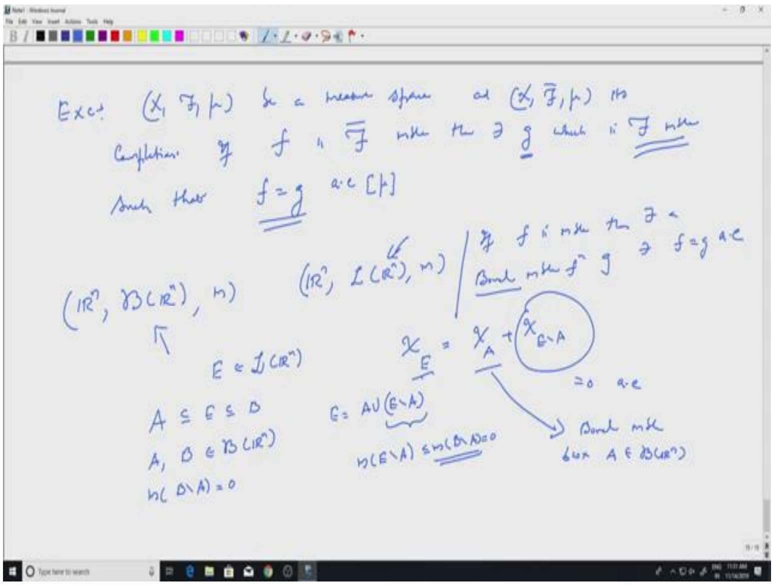

Exercise, so let us say X, F, Mu be a measure space and X, F bar, Mu, its completion, its completion. So, if F is script F bar measurable. So, that means when I pull back open sets, I am going to get sets in F bar, I may not get sets in script f. Then there exists g which is f measurable such that f equal to g almost everywhere, so that is a point.

So when I complete I am only putting certain sets and the measurable, set of measurable functions even though you will get a larger set of measurable functions, you will have g which is measurable with respect to the smaller sigma algebra script f such that you have equality almost everywhere.

Since integrals do not change when functions are equal, almost everywhere you can always work with f measurable functions instead of F bar measurable functions. So, let us look at this in the case of Borel sigma algebra and so consider Borel sigma algebra on let us say, R n and the Lebesgue sigma algebra.

So, I have R n and the Lebesgue measure here similarly R n, when I complete, I get Lebesgue sigma algebra, this is the completion. So, if I take a function which is measurable in the usual sense, I want to say there is a Borel measurable function. So, the assumption here is that if, assertion here is that if f is measurable then.

So, measurable remember is with respect to the Lebesgue sigma algebra, then there exists a Borel measurable function, Borel measurable function g. So, Borel measurable would be measurable with respect to the Borel sigma algebra. Such that f equal to g almost everywhere.

So, that is what those exercise is. The exercises in the abstract setting. How does one prove this? So, let us look at indicator functions first. Suppose, I take a set E in Lebesgue sigma algebra of R n and my function is Chi E. Because E is in L of R n, we know that there exists A contained in E contained in B.

What is the property of A and B? Well, A is an F sigma set, B is a G Delta set. But more importantly, they are Borel sets and the measure of B minus A is 0, we know that it exists. In other words, those I can write as Chi of, so E is written as A union E minus A and this has measures 0. Measure of E minus A and E minus A is a Lebesgue set. So, its measure make sense.

Measure of E minus A is less than or equal to measure of B minus A which is 0. So, I can write Chi E as Chi A plus Chi of E minus A. Because A and E minus A are disjoint. But this is a function which is 0 almost everywhere. Because its measure is 0. So, Chi E will be equal to Chi A almost everywhere. But Chi A is Borel measurable, because A is a Borel set, because A is a Borel set. So, for indicator functions or characteristic functions this is trivial. So, from indicator functions.

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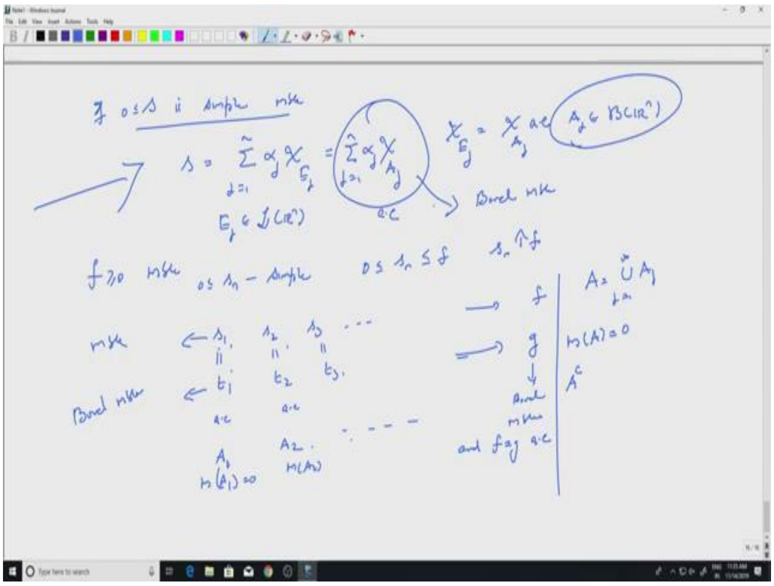

So, if I have a simple function. If S is simple, so I can take positive simple function measurable. Then I can write s as alpha j Chi Ej, j equal to 1 to n. But each Chi Ej I know is equal to Chi Aj. So, Ej are Lebesgue sets here, Chi Aj where Aj are Borel sets that is what we just saw.

So, this would be equal to summation j equal to 1 to n Alpha j Chi Aj almost every. Because they differ only on a set of measure 0. So, this is true almost everywhere. But this is Borel measurable, this is a simple function which is Borel measurable. Because the sets Aj are in B of R n. So, I am taking the linear combination of such sets. So, they are Borel measurable.

So, for simple functions also we have something which is Borel measurable. So, now, if I take an arbitrary positive function which is measurable. So, f is positive and measurable. So, this means, measurable with respect to the Lebesgue sigma algebra. Then I have s n symbol functions.

So, 0 less than equal to sn less than equal to f and sn increases to f, so this is we have. But each sn. So, I have s1, I have s2, I have s3, etc and they converge to f. So, each sn I can replace by let us say t1, t2, t3, etc, these are all measurable and this would be Borel measurable, why are they Borel measurable? Because of whatever we have just said here. For any simple function I have a Borel measurable function which is equal to that almost everywhere.

s1, s2, s3, etc are measurable with respect to the Lebesgue sigma algebra. t1, t2, t3 are measurable with respect to the Borel sigma algebra and they are equal almost everywhere. So, that is a very crucial thing here they are equal almost everywhere. Well, what does that mean? So, here I have a set A1, whose measure is 0, I have a set A2.

So, measure of A1 is 0, so we will need a bit more space here, measure of A1 is 0, measure of A2 is 0, etc, etc. Outside A1, s1 is equal to t1, outside A2 s2 is equal to t2, outside A3, s3 is equal to t3 etc. So, if I take A to be union Aj, j equal 1 to infinity, these are sets of measures 0, countable many of them. So, countable subadditivity will immediately tell me that measure of A is also 0.

What happens outside A? All this simple functions are equal to the Borel measurable functions. So, when it converges, it will converge to g which is Borel measurable, which is Borel measurable because tjs are Borel measurable, its limit will be Borel measurable and f equal to g almost everywhere and f equal to g almost everywhere. So, this is happening outside A and A has measure 0. So that is all is needed.

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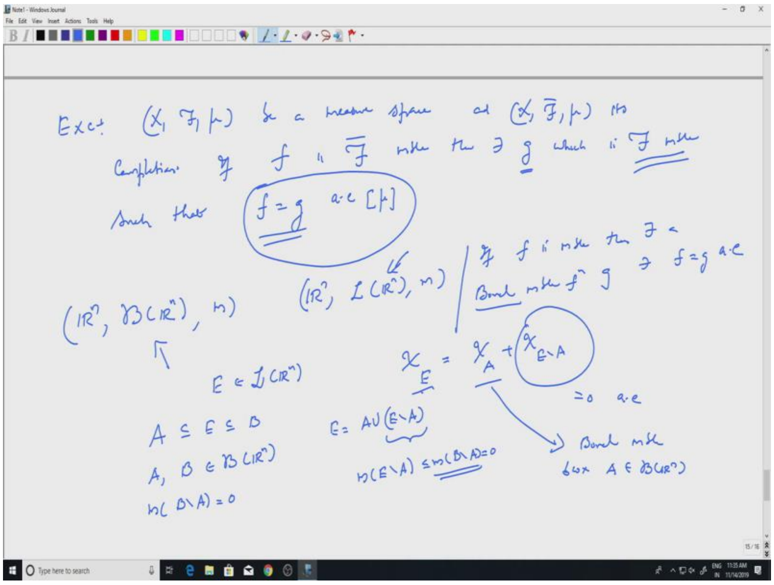

So, I will be able to get a g which is measurable with respect to the smaller sigma algebra and equal to f almost everywhere. So, let us continue this discussion of Lebesgue measure and let me state a theorem. So, we will prove that theorem. So, that will sort of explain all the Sigma algebra involved in full detail.

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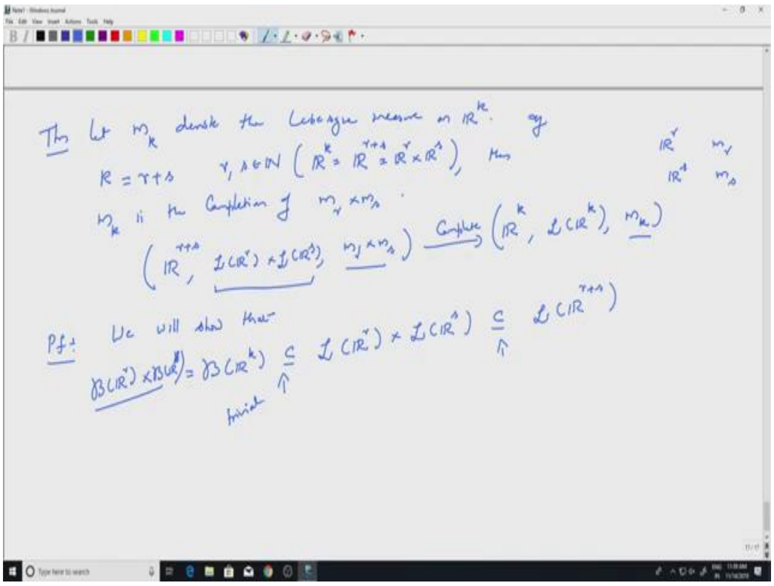

So now, since we need Lebesgue measures on different R n, we will use a different notation. So, let m sub k denote the Lebesgue measure on Rk. So, k is a dimension. Recall, remember that m sub k will denote the Lebesgue measure on Rk. So, I want to write Rk as a product space.

So, if k equal to R plus S, where r and s are positive integers. So, that is same as saying Rk equal to R. So, that is the real line to the r plus s which is same as Rr cross Rs. So, I am viewing Rk as a product space. Then mk is the completion of, well mk is the Lebesgue measure on Rk. But Rk is a product space. So, on Rr we have the Lebesgue measure m sub r, on Rs we have Lebesgue measure m sub s. You can take the product, you will get the product measure. So mk is the completion of the product measure. So, mr cross ms.

So, to write it again Rr plus s, so r plus s remember is k. L Rr cross L Rs comma mr cross ms this makes sense, this is my product space and the product measure. This if you complete, if you complete, you will get Rk. So, because k is r plus s, so the space does not change. The Lebesgue sigma algebra of Rk.

So the Lebesgue sigma algebra of Rk is the completion of the product of these two and mk, mk is the k dimensional Lebesgue measure, so our aim is to prove this. So, that sort of explains all the sigma algebras in detail. So, all that we need to look at is, the Sigma algebra is very closely and see how this measure and this measure are related, but that is not very difficult.

So, we will show that the Borel sigma algebra of RK this we know is the Borel sigma algebra are Rr cross Borel sigma algebra Rk, Rs. For Borel sigma algebra, this is a trivial equality, we will show that this is contained in the Lebesgue sigma algebra of R cross Lebesgue sigma algebra of Rs which is trivial. Because this will be contained in this and we want to say this is contained in the Lebesgue sigma algebra of r plus s. So, this is the non-trivial part, so this is trivial.

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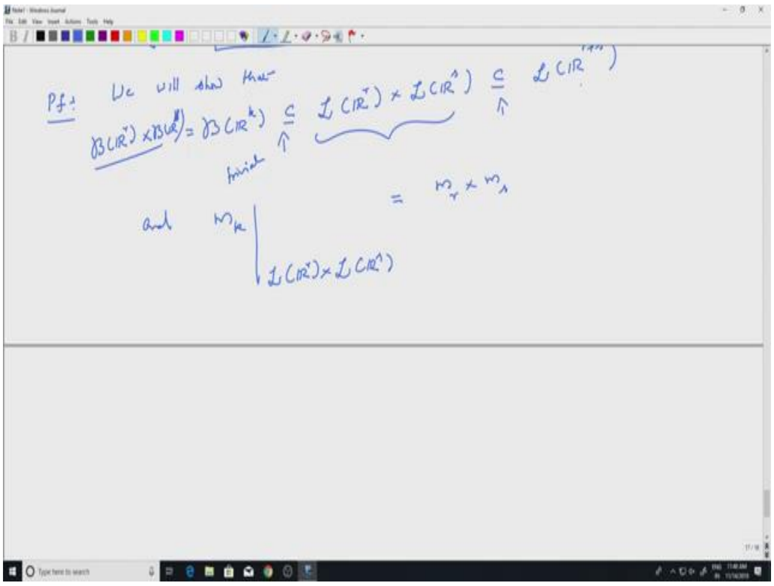

And so one more sentence here and the K dimensional Lebesgue measure. So, that is on r plus s, you restrict that to the smaller sigma algebra. So L Rr cross L R to the s, then this is equal to the product measure on the space m r cross m s. So, that is now very easy equality to see because that is how the product measures are defined that you, you look at measures on each component and then multiply.

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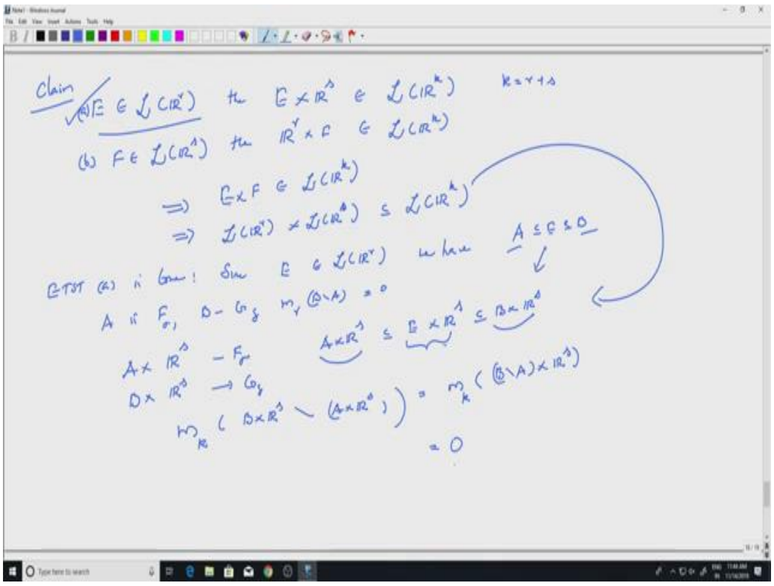

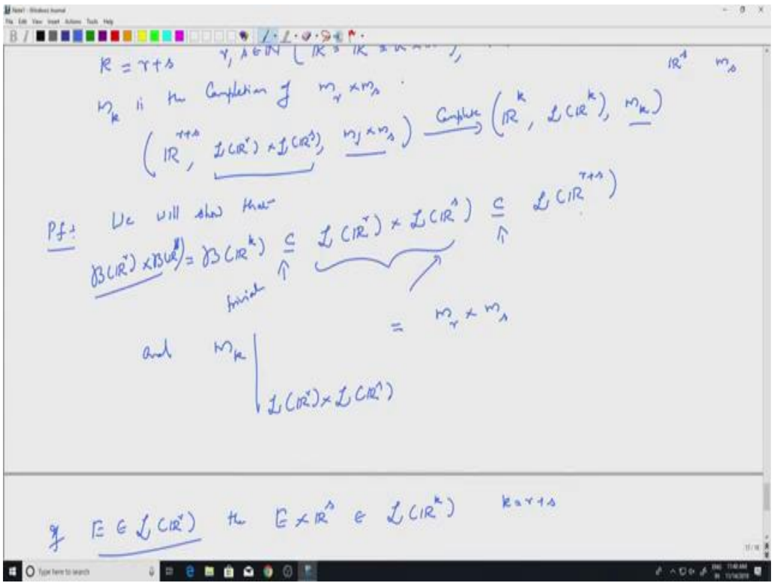

So let us start with that, so if E is a Lebesgue set in Rr. Then E cross R to the s belongs to L of Rk. So, remember K is r plus s. So, I am saying if I take a Lebesgue set here and then I look at the Cartesian product of that Lebesgue set with Rs. Then it is in the Lebesgue sigma algebra of the product set.

So that is one thing I want to show, so to show this part, I take measurable rectangles from here and then show that it is here. So that is what I want to prove for that I take. So, let us let us write this as a claim, then it will be clear to claim.

So let us say a, b is if F is in the other component, proofs are same, then R to the r cross F also belongs to L to the k and so both imply if I intersect, E cross F will be in L of to the k. So, that means measurable rectangles are inside L of R to the k. So, the Sigma algebra generated by E cross F that is the product sigma algebra cross L of R to the k, R to the s will be contained in L of R to the k, k is the sum of r and s that we will do.

So, it enough to show that A, so enough to show that A is true. So let us do this in detail so, how will you show? So, since E belongs to the Lebesgue sigma algebra on R r, we have a set A which is contained in E contained in B, where are A and B? A is F sigma, B is g delta and B minus A has measure 0, but which measure?

The R Lebesgue measure, R dimensional Lebesgue measure is 0. But I am trying to look at E cross R to the s. So, obvious candidates are A cross R to the s and B Cross R to the s. So, look at A cross R to the s. So, this is of course F sigma because R to the s is closed. Similarly, B cross R to the s is g delta and A cross R to the s of course contains, is contained in E cross R to the s is contained in B cross R to the s because of this.

So, now to show that this the middle one belongs to the sigma algebra in the K dimensional space, we need to look at the measure of this minus this. So, we need to look at the K dimensional Lebesgue measure of B cross Rs minus A cross Rs. This is what we need to look at.

If we show that it is 0, then by definition this would be in the Lebesgue sigma algebra of k dimensional space. Well, this is mk of B minus A cross R to the s and I want to say this is 0. So, I need to justify this, so I leave this the computation part to you, how will you do this?

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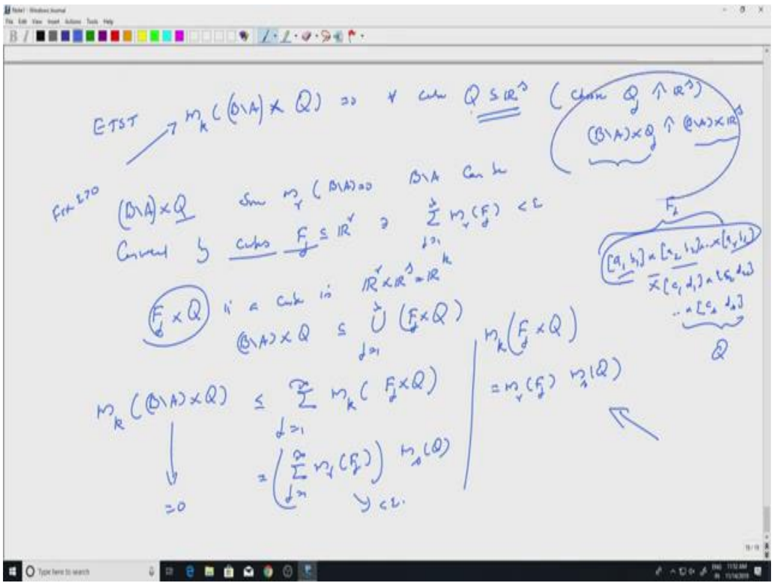

Well, so enough to show, enough to show that mk of B minus A cross Q is 0, for every cube Q contained in R to the s, that is by monotone conversions. If I take any cube Q such that Q is contained in R to the s, then mk of B minus A cross Q has measured, so this is the K dimensional Lebesgue measure of B minus A cross Q that is 0. Well, how do you do this, if I choose Qj going to increasing to R to the s, then B minus A cross Qj will also increase to B minus A, so that is easy to see B minus A cross R to the s and if these have measures 0, then this will also have measure 0 by the convergence theorems.

So, we will simply justify that part. So, how do we do this? So, you start with B minus A cross Q, since the R dimensional measure of B minus A is 0, B minus A can be covered by, can be covered by cubes. So, I will use cubes Fjs, cubes Fjs. So, these are all contained in Rr such that summation mr of Fj that is the volume of Fj, j equal to let us say 1 to infinity, is less than epsilon. So, you start with fixed epsilon positive, etc. Then you can do this. Now, the important part is, these are cubes and Q is also cube. So, Fj cross Q is a cube in Rr cross Rs, this is Rk.

So, this is one cube in Rk and B minus A cross Q is contained in the union of Fj cross Q, j equal to 1 to infinity and if I take the Kth dimensional Lebesgue measure. So, mk of B minus A cross Q, this is a force less than or equal to summation j equal to 1 to infinity mk of Fj cross Q.

So, now comes the punch line, because if I take a cube. So, what is the Kth dimensional Lebesgue measure of Fj cross Q. So, this is a cube inside Rk. So, you simply look at the product of side length of each side. So, what I mean is Fj I can write as so, maybe A1 B1 cross A2 B2, etc etc Ar Br and Q is another product of intervals. So, I will write it as C1 D1 cross C2 D2 etc etc Cs Ds. So, this is, this is Fj and this is my cube.

So, the Kth dimensional Lebesgue measure is the product of side lengths. But if I take the first R of them, I am going to get the Rth Lebesgue measure, Rth dimensional Lebesgue measure of Fj. So, this is mr of Fj times ms of Q, which is a constant. So, ms of q comes out. So, this is simply summation j equal to 1 to infinity mr, this is very crucial try to understand this mr of Fj times ms of Q and this is less than epsilon. This I can do for every epsilon and so this would be equal to 0 and because of this, we have proved that this is 0.

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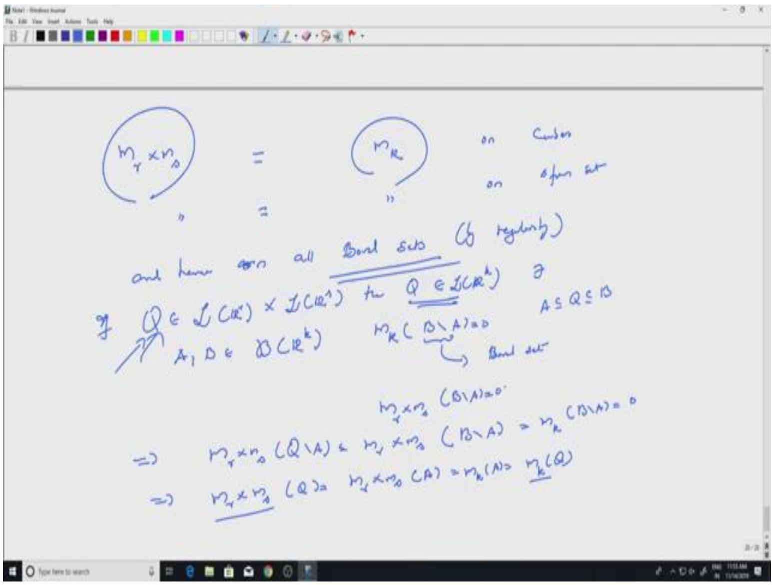

So, the only thing to be justified now is, if I look at mr cross ms as a product measure and mk as two measures. So, they agree on cubes that is what we just saw. So, they will agree on open sets. Because any open set is a disjoint union of, almost disjoint union of cubes. So, from cubes, you can go to open sets and hence all Borel sets and hence on all Borel sets by regularity if you like, by regularity or you can by hand you can compute this and see.

And so if I take Q in L of Rr cross L of R to the s. Then Q will be in L of R to the k. So, that we have already seen now. So, there exist A and B in the Borel sigma algebra of R to the k because Q is in L of R to the k such that measure of K and the Kth dimensional Lebesgue measure of B minus A is 0 and of course, A is contained in Q contained in B.

So, now Q is not a cube anymore, this is a general set. But where is this? This is a Borel set, and we have just proved that on Borel sets, they agree by regularity, which ones these this measure and this measure? So, that is mr cross ms and that has measure 0. So that tells me that the measures agree.

So, this is the, so mr cross ms, so I will just write one more line, Q minus A this is less than or equal to mr cross ms of B minus A, i know this is 0, this is equal to mk of B minus A, which is 0. This simply implies that mr cross ms of Q equal to mr cross ms of A which is equal to mk of A equal to mk of Q. So, they agree, so they agree on all Lebesgue sets, all sets in L R cross L S.

So we stop here, so we have essentially looked at how completion of product Lebesgue sigma algebras look like. So, we simply get the Lebesgue sigma algebra in the products space. Next we will look at what is known as polar coordinates. So, essentially an application of Fubini’s theorem. So depending on the time, we will look at another application of Fubini’s theorem.