Measure Theory
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Lecture 44
Polar-coordinates

So, in this session we are going to look at applications of Fubini's theorem, immediate applications of Fubini's theorem. One is the polar coordinates in R n which you are familiar with. So, you must have seen at least this an R2 or R3 and so on and there is something called the convolution on L1 of R n which depending on the time we will define and prove some results. So, let us start with polar coordinates first. So, polar coordinates will, polar coordinates is so for any point x in R n, R n minus 0 in fact, you have the radius of that point or the modulus of that point times an Angular coordinate which is on the unit sphere of R n.

So, that gives us two parameters, an R n minus the point 0 can be viewed as a product space that is the product space of 0 infinity with the unit sphere. So, we will show that on 0 infinity and unit sphere, we can put certain sigma algebras and measures, so that the integral of R n, integral over R n minus 0 can be transferred to an integral over 0 infinity and S n minus 1. So that is the S n minus 1 is the unit sphere in R n. So, that is the aim of this lecture, let us start.

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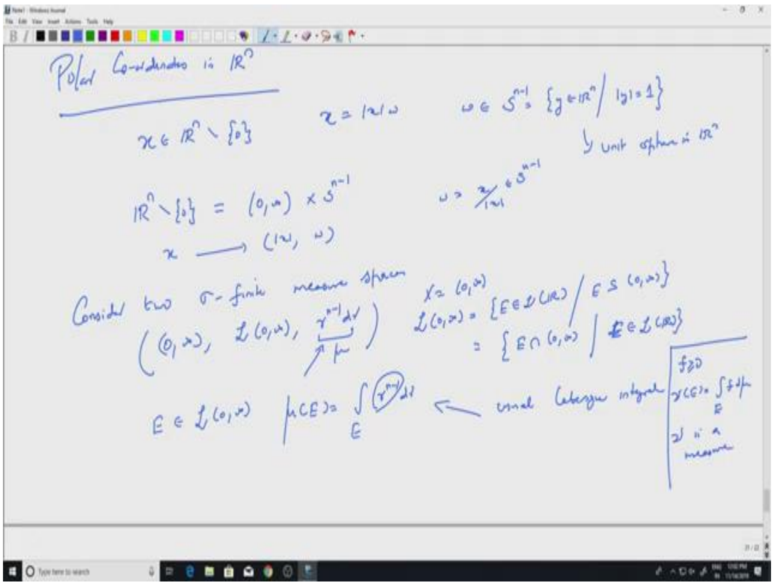

So, Polar Coordinates in R n. S, you start with any point x in R n minus 0. So we can write x as mod x times let us say omega, where omega is point in S n minus 1 so as S n minus 1 is the sphere in, so y in R n, such that modulus of y equal to 1. So, this is the unit sphere in R n. So in R2 it is the unit circle and we can identify R n minus 0 to be the product space of 0 infinity. Well where is the 0 infinity coming from? Mod x is a point in 0 infinity, cross S n minus 1, what is the identification? x going to mod x comma some omega, omega n S n minus 1. So, if you want omega is actually x by mod x that is a point in S n minus 1.

So, we put measures on the right hand side and then we will see that it gives us Lebesgue measure on the left hand side. So, consider two sigma finite measure spaces, one is 0 infinity, that is my space x, then I have a sigma algebra which is the Lebesgue sigma algebra of 0 infinity and the measure R to the n minus 1 dr.

So, I will explain what this means. So, let us call this Mu. So, 0 infinity is clear that is a space X, 0 infinity. What is the Lebesgue sigma algebra 0 infinity, well we have seen this before you can always restrict to a subset. So, this is the collection of all the Lebesgue sets in the real line, which are contained in 0 infinity. You can also write this as E intersection with 0 infinity, so intersect with 0 infinity, where E is the arbitrary set in the Lebesgue sigma algebra of R, both will give you the same space. So, that is one space.

That is a sigma algebra, what is R minus 1? dr, R r to the N minus 1 d r that is a measure. So, I would not tell you how it is defined. So, if I take an set E in L 0 infinity that is the Lebesgue sigma algebra there, then Mu of E is defined to be integral over E R to the n minus 1 dr. So, this is a usual, so usual Lebesgue integral, Lebesgue integral.

So, you are using this function to define the measure. So, recall that exercise if I have a positive measurable function, I can define on some measure space. So I can define nu of E to be integral over E f d Mu, Mu is the original measure, then Nu is a measure, so Nu is a measure, so accountably additive measure. So, that is the measure we have defined. So that is one space.

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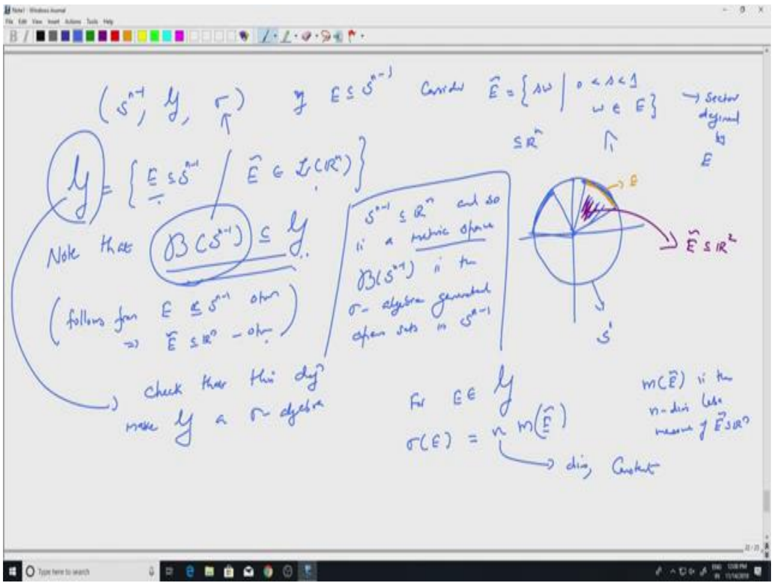

The other space is S n minus 1 and I have a Sigma algebra g and I have a measure sigma. So, I should tell you what these are. So, if E is a subset of S n minus 1, you look at consider E tilde. So, what is E tilde? So E tilde is s times omega where s is between 0 and 1 and omega belongs to E. So, this is a sector defined by, sector defined by, defined by E. So, let me explain this a little bit more. So, let us look at some examples, so that would be very clear. So, I look at the unit circle in R2 that is my S1, S n minus 1. So, in this case it is S1.

Then I take some set E. So let us take an arc E. So, this is this is my arc E that is a set in S n minus 1. Then what is E tilde? So, E Tilde will be you join these two endpoints to the origin, origin is not included because S cannot be 0. So, whatever is inside correct, so whatever is inside.

So let us, this portion is E tilde that is a subset of R n, in this case it is a subset of R2. So, this is a subset of R n that is a sector defined by that particular arc. So, E can be an arbitrary subset of the unit sphere. Now, I want to define G. So, the script G the Sigma algebra of subsets of S n minus 1. So, this is all the sets E in S n minus 1 such that E tilde is a Lebesgue set, in L of R n.

So, if I look at the sector and if I get a Lebesgue set, then the boundary that is inside the Sigma algebra we want to consider. So, notice that, notice that, the Borel sigma algebra of S n minus 1 is contained in G. Well what does that mean? S n minus 1 first of is a metric space, S n minus 1 is contained in R n and so is a metric space, we have the usual distance on S n minus 1 which is coming from the distance on R n, unusual metric.

So, any matrix space has a Borel sigma algebra. So, B of S n minus 1 is the sigma algebra generated, sigma algebra generated by open sets say. So, open makes sense because it is a metric space. So, open sets in S n minus 1. So, you take any open set in S n minus 1 that will be in the Borel sigma algebra and generate the Sigma algebra you will get the Borel set.

So, to show this what do you need to show? You need show that open sets are inside G. But that is trivial and if E is open so that means, it would be an arc like this or some other open set, the sector will be open. So, that follows so this follows from, follows from E open E contained in S n minus 1 open implies, E tilde which is a subset of R n minus 0 that is open.

So, any open set E gives me E tilde, which is an open set and so it will be in the Lebesgue sigma algebra which means that any open set is in G. So, the Borel sigma algebra itself will be in G. So, you have to check that G is a sigma algebra first. Check that this definition, this definition makes G a sigma algebra and so if open sets are contained in G, since G is a sigma algebra, sigma algebra generated by the open sets will also be in G, so we are fine.

So we have we have the space S n minus 1 we have the Sigma algebra G. But we have not defined the measure. So what is the measure? So, for E in G sigma of E. So, small sigma is the measure. So, I am denoting it by small sigma, sigma of E is well, you go to E tilde. So, E tilde is this sector and E is in G will imply E Tilde is a Lebesgue set. So, measure of that makes sense. This is the n dimensional, so m of E Tilde is the n dimensional Lebesgue measure on of E tilde, E Tilde is a subset of R n and you multiply this with n, n is the dimension. So, that is a constant. So, n is the dimension it is a constant.

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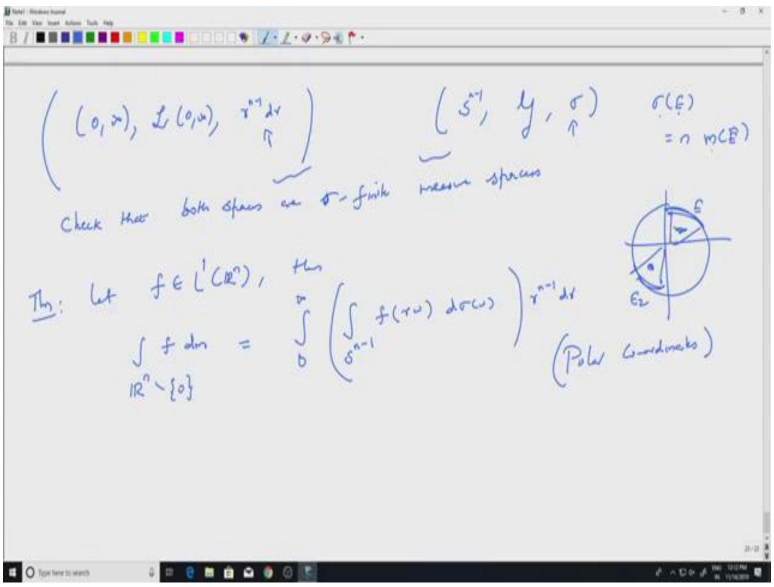

If you compute it in R2, you will see why sigma is defined like that. So, we have now two spaces. So, let me write it again. I have 0 infinity as space, I have the Lebesgue sigma algebra on 0 infinity, I have the measure R to the n minus 1 dr. Remember how it is defined? So this is one space and I have the space S n minus 1, I have the Sigma algebra G.

So, a set is in G if its sector is a Lebesgue set and I have the measure sigma which is sigma of E is n times sigma of E tilde, E tilde is a sector defined by E. So, you can check that both spaces are, so check that both spaces are, both spaces are sigma finite measure spaces, sigma finite measure spaces. So, you have to check that sigma is a measure that is because. So, I made a mistake here. So, let me write it again. So, this is n times Lebesgue measure of E tilde. Since capital M is a measure, so small m is a measure, sigma is also a measure.

So, because if Ejs are disjoined here then Ej tildes will be disjoined, the sectors will have to be disjoined. So, again look at some pictures. So, if this is E1 and this is E2, they are disjoint the sectors are also disjoined. So, this portion and this portion are disjoint. So, disjoint sets will go to disjoint sets in the sector and so it will add up.

So, that is why it is a measure, so that part is easy, check that both are sigma finite. In fact, sigma of S n minus 1 is a finite quantity. So, it is a finite measure, but this is infinite measure but it is sigma finite. So, theorem let f be a measurable function on R n. Then, so if the integrals is axis, so I guess that part I will not go there, you can take a positive function orfF is in L1, so maybe let us put the assumption clearly.

Let f belongs to L1 of R n, then integral over R n, well R n minus 0 strictly speaking. But integral over R n and integral over R n minus 0 are same. Because 0 the point has measure 0. fdm this I want to write as integral over 0 infinity integral over S n minus 1. So, I am taking the product of this and this, S n minus 1, f of.

So, x is written as the product of two things. So, it is R omega and on S n minus 1 my measure is sigma, so d sigma omega. So, omega is the integral first and I have the measure R to the n minus 1 dr that is this measure. So, this is the claim, so this is called polar coordinates, so polar coordinates. So, you must have seen this several times in the case of R2, R3 and so on, it is true in general.

So, there are several things which are implicit in this which I will not bother to explain because you have seen this several times now. To write the inner integral this has to be measurable and so on and the integral will be finite for almost all R, etc, etc. So, such things I whatever we saw in Fubini's theorem will be applicable here. So, we will just prove the identity in general that is all we will do, all the other justifications are done as in Fubini's.

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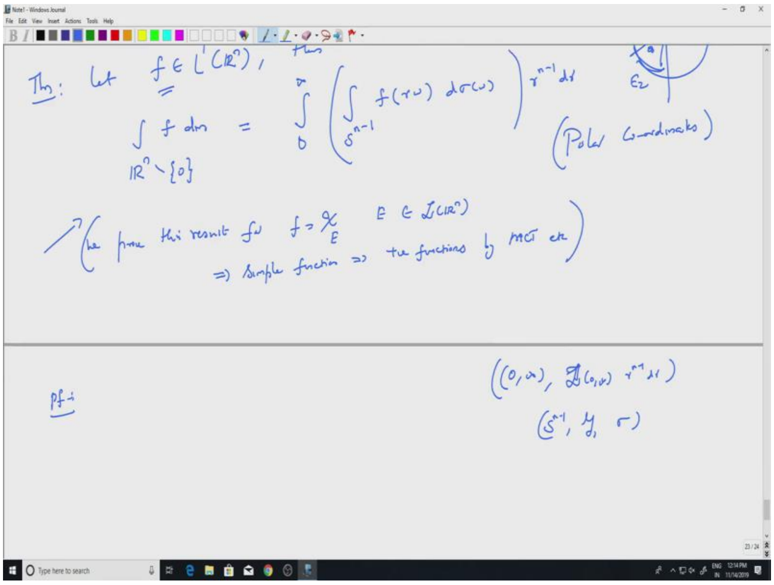

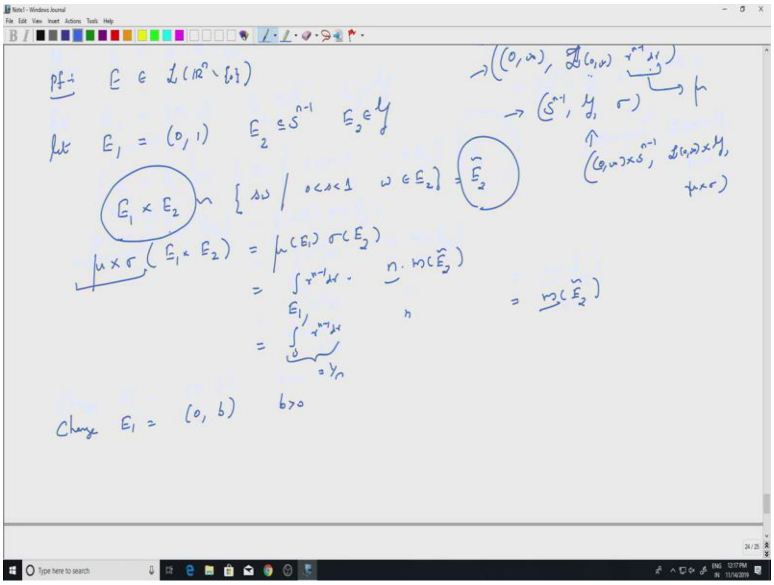

So, proof. So, keep the identity in mind. So first take, so I have two spaces. So, this is so the notation becomes a bit ugly. So, you keep track of the sets we are writing. So, this I have a sigma algebra. So, this is L of 0 infinity and I have r to the n minus 1 dr and I have S n minus 1, I have g and sigma. So, that we will keep.

So first, as all these proofs any theorem like this since f is in L1, what we do is, we proof this result for f equal to Chi E, where E is a Lebesgue set and then so that will imply for simple functions, which will imply for positive functions by MCT etc, you can complete it for all functions by taking real functions and then taking linear etc, etc. So, that part I leave it to you.

So, we just need to show it for a arbitrary set. So, first we will take. So, I need to look at sets which are inside L of 0 infinity, L of for n, r n minus 0 strictly speaking, those will not include 0. So, first take E1, so let E1 to be equal to open interval 0, 1 and E2 any set inside G. So, there we are taking an arbitrary set in G.

But here we are taking the interval 0, 1. So, that is very specific. So what is E1 cross E 2? E1 cross E2 can be identified with, this is the set sw, where 0 is less than s less than 1 and w is in E2 and this is simply the definition of E2 tilde. So, now let us look at the product measure. So, we call this Mu. So, that I do not have to keep writing R to the n minus 1 dr.

So, you look at Mu cross sigma. So, Mu cross sigma makes sense, I have two sigma finite measure spaces, I know the product make sense, so 0 infinity cross S n minus 1 that is my R n minus 1, I have L of 0 infinity cross G and I also have Mu cross sigma. So Mu cross sigma of E1 cross E2, this is a measurable rectangle.

So, the measure just is the product, so that is Mu of E1 into sigma of E2, sigma of E2 well which by definition is what is Mu of E1? Mu of E1 is integral over E1 R to the n minus 1 dr times sigma of E2 is n times measure of E2 tilde that is the definition. But E1 is 0, 1. So, what is this, so this is integral 0 to 1 r 2 n minus 1 dr, which is 1 by n. So, this is equal to 1 by n and I have n here0 so that gets cancelled.

So, this is equal to m of E to tilde. So, what we just proved was, even in the cross E2 identified with E2 tilde and we have Mu cross sigma of E1 cross E2 equal to m of E2 tilde. So, on E1 cross E2, these measures agree, that is what we want say. But even as a specific set 01. So, now change E 1 to 0 B where B is a positive number and let us do the same thing again.

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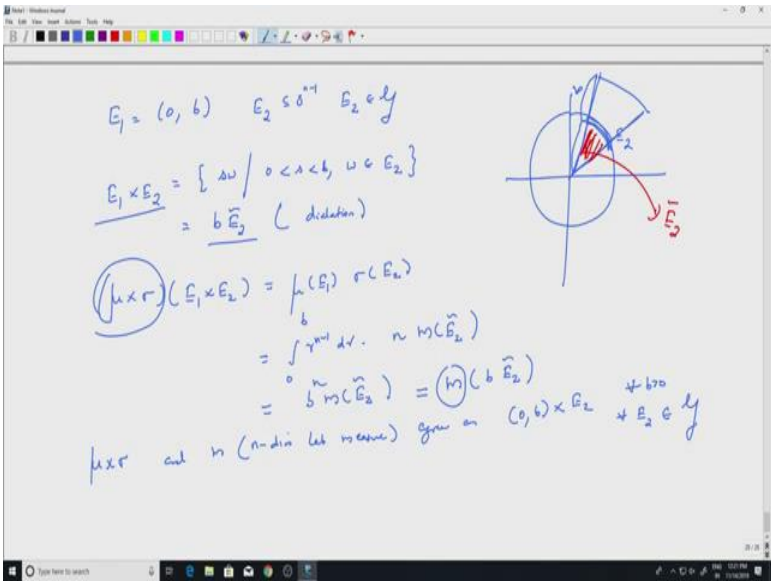

So, I have E1 equal to some set 0 b, where b is some positive number, E 2 is still a general set inside s n minus 1, but it belongs to the sigma algebra G. Then how it E 1 cross E 2 look like. So, let us try to understand this. So, this is my unit circle, let us say this is E 2 and b is some number. So, E1 cross E2 is identified with all those s w where 0 is less than s less than b. So, instead of one now go till B and w is in of course E2. How does it look like? Well if b is greater than 1, you will have a sector like this.

So, this distance is b, which is nothing but the dilation of this portion. So, you look at this portion which is E2 tilde and you multiply it with b. So, this is nothing but b times E2 tilde the dilation, so remember dilation. So, now let us compute the measures again. So, Mu cross sigma of this measurable rectangle E1 cross E2, where E1 is 0 b.

This is nothing but well, I have this to be equal to Mu of E1 into sigma of E2 by definition. So, Mu of E1 now would be integral 0 to b, r to the n minus 1 dr, because E1 is 0 to the b and Mu is defined by R to the n minus 1 dr times sigma E2 is of course, n times measure of E2 tilde. Well what is this?

Here, this will give me b to the n by n, so by n and into n will get cancelled. So, I will have b to the n times measure of E2 Tilde. But remember the dilation invariance properties of Lebesgue measure on R n. So, this is E2 Tilde times b and you look at the measure of that. So, notice that E1 cross E2 is identified with this and we are saying this measure and this measure equal on those sets.

So, on measurable rectangles of this form, E1 cross E2, they agree. So, now you can take limits. So, Mu cross sigma and m which is the Lebesgue measure on n dimensional space n dimensional Lebesgue measure, agree on, agree on 0 B cross E2 for every b positive and for every E2 inside S n minus 1 which is a measurable.

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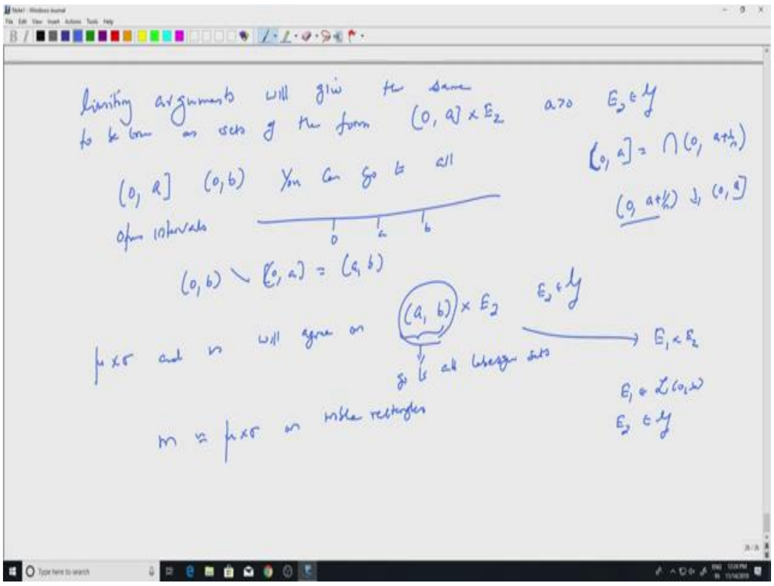

So, that tells me. So, now I can take limits. So, take limits, so this will be, from here onwards, I will be somewhat brief in telling you what exactly needs to be done, because it is all limiting properties. So, limiting arguments, limiting arguments will give the same to be true, same to be true on sets of the form 0, a closed cross E2, a is positive E2 is in G.

Well, how do you do that? You write, open 0, closed at a to be intersection of open 0, a plus 1 by a open and so when I take product with E2, I will be taking product with each of them and then. So, what I want to say is the sequence of set 0 a plus 1 by n decreases to 0 a closed and use limiting. So, because on this sets cross E2 we know they agree. So, on the limiting sets also, they will agree.

Now from 0 a and open interval 0 b, you can go to all open intervals, all open intervals by subtraction. Well if you subtract for example, if 0 is here, a is here and b is here, you subtract 0 b and 0 a, you will get a b. So, you can go to all open intervals. So the measures Mu cross sigma and M will agree on, agree on sets of the type open interval a b cross E2, where E2 is the general set in G.

So, we did not bother too much about E2. So, from open intervals who can go to all Lebesgue sets, go to all Lebesgue sets. So, this we have seen from opens sets going to Lebesgue sets.

So, apply the same argument. So, the measure m is identified with Mu cross sigma on measurable rectangles. Because from a b you can go to, so from here we are going to E1 cross E2, where E1 is a general Lebesgue set in 0 infinity and E2 was some general set in G. So, those are the measurable rectangles.

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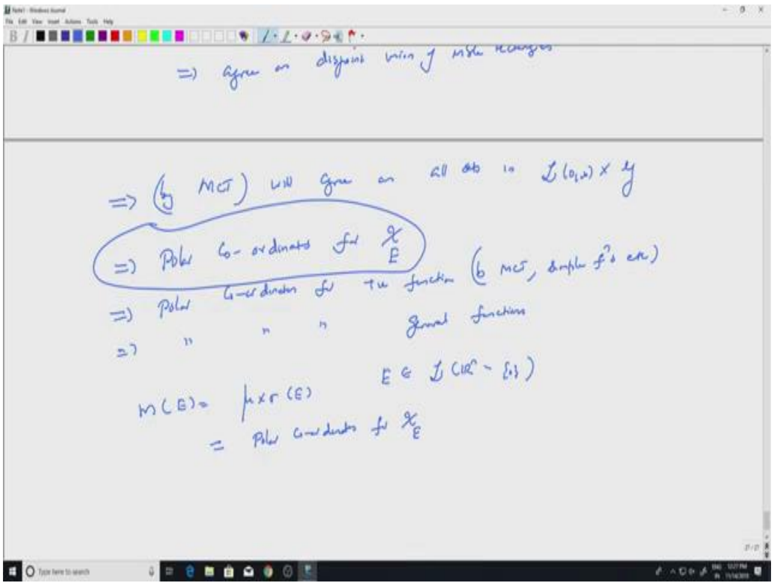

So, these two measures agree on measurable rectangles. So, this would imply they agree on disjoint union of measurable rectangles, because they are measures, so they add up and then going from disjoint union of measurable rectangles to everything is by taking appropriate limits.

So, this by MCT will agree all sets in L of 0 infinity cross G, in that Lebesgue sigma, in that sigma algebra in the product sigma algebra. So, this is what is polar coordinates is for, so this is the part I will leave it to you, coordinates for indicator functions. So, once it is done for indicator function, this will imply polar coordinates for, polar coordinates for positive functions by MCT and simple functions etc.

So, that is the standard argument I will not bother about explaining this and then from positive functions you can go to general functions, polar coordinates for general functions. So, we finish with this, the point I have not explained is that the measures agree is same as polar coordinates for Chi.

So, that is something I will leave it to you, you write it down, what does it mean to say that? m and Mu cross sigma agree on a indicator on a set. So, I mean this was the starting of Fubini's theorem if you like. So, if I know m of E is same as Mu cross sigma of E, where E was a Lebesgue set in R n minus 0. So, 0 is not needed there. Then, if you write down the right hand side, this is actually polar coordinates for, or whatever the theorem said the integral polar coordinates for Chi E.

And once it is true for Chi E, you can go to symbol functions, positive functions and then general functions in the usual manner. So, we will stop here. So we have just seen one application of the Fubini's theorem, which is the polar coordinates on R n which you are familiar with. There are lots of other applications, so we will see at least one more in the next lecture before we go on to new topics in Measure theory, especially complex measures and so on.