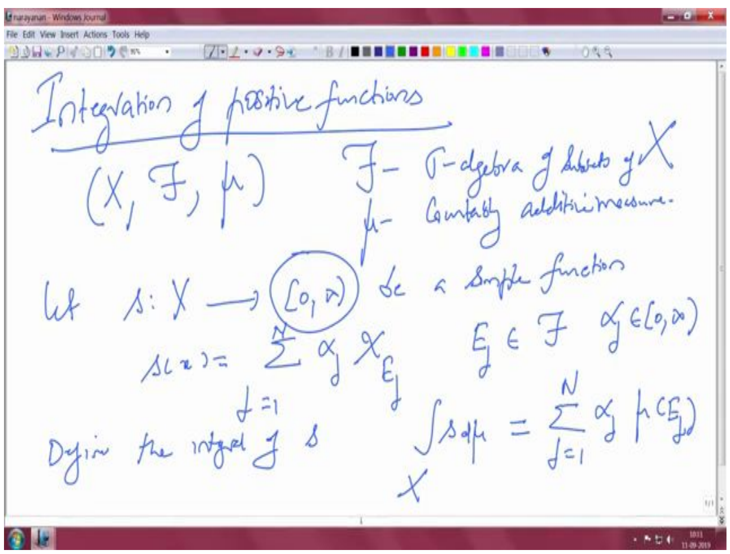
Measure Theory
Professor E.K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 05
Integration of positive measurable functions

So far we have seen, measurability concepts. Today we will be looking at, integration of positive functions. So we will start with integration of simple functions and then go on to positive functions after we define in the for positive functions and look at some properties of that, we would extend it to real valued functions and complex valued functions.

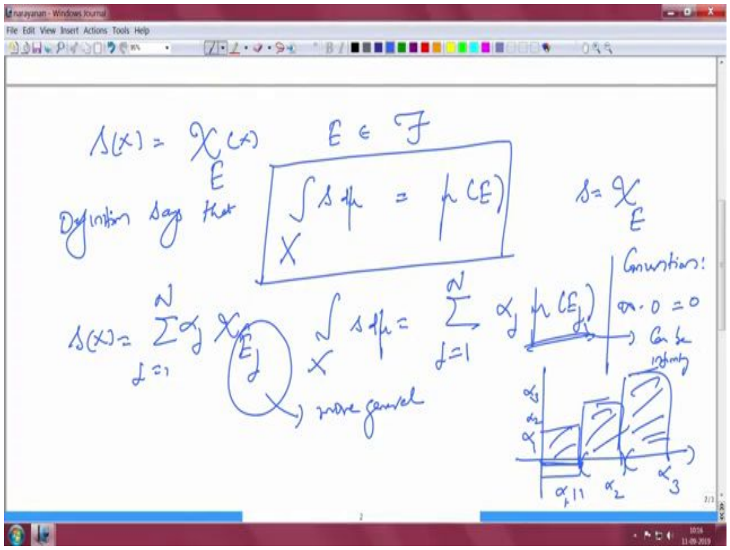
(Refer Slide Time: 0:58)



So we start with the positive function. So integration of positive functions. So our setup is always we have a space X, we have a Sigma algebra, and we have a measure, measure remember as accountably additive measure. So script F is a Sigma algebra of subsets of X, mu accountably additive measure will be defining integration with respect to a mu over the space X. So let us start with a simple function. let S be a symbol function, which takes values to the infinity be a simple function. The member symbol function is a measurable function whose range is finite. So I can write as of X to be equal to summation, alpha J chi Ej, j equal to 1 to N right.

What are Ej? Ej are measurable sets. So they are in script F, okay. Alpha j are scaler. So I will take this to be in zero infinity. Because I am taking now a positive simple function. I wanted to find the integral of S with respect to the measure mu, okay. So define the integral of S. So this will be denoted by integral, lower X S dmu. This is a symbol integral over X S dmu. This is defined to be, if s is given by this summation, J equal to 1 to N alpha J Chi Ej, then the integral is simply summation J equal to 1 to N alpha J mu Ej.

(Refer Slide Time: 3:24)

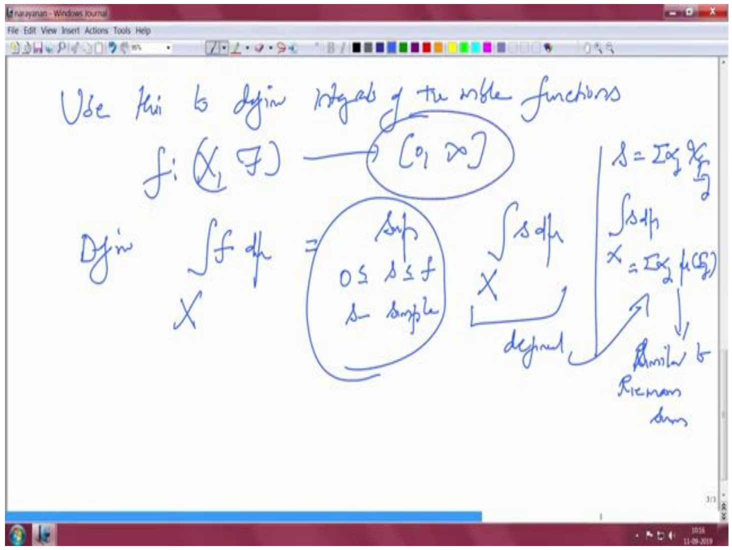


So let us, just look at it again. Suppose my simple function is simply an indicator function. I take indicator or characteristic function of E. E is some set in script F, measurable right. So our definition says that integral over X S d mu, this is simply mu of E, measure of E. So this is the crucial definition. So if S is given by the indicator of E, then the integral is simply the measure of E. And when you have more values for S, so that is same as saying it as a, it is a linear combination of a measurable sets from script f. Then of course the integral is going to be the sum of integrals of each piece actually. So we are forcing linearity in some sense here, but I will fire j mu Ej.

So think of simple functions like a step functions. Then this is what you will do in the case of a Riemann integration, right. Suppose my simple function is like this, the integral of this, the Riemann integral of this, is simply the value. So let us say this is alpha 1 and alpha 2 and alpha 3 you will multiply alpha 1 with alpha 1 into measure of this length of the intervals.

Then you take this interval and multiply it with alpha 2 then you take this interval, multiply with alpha 3 and then add abs. So that is how you get the area under the curve, right. So this is the integral value of the integral of the step function. Alpha 1, alpha 2, alpha 3 taking values in these intervals. That is precisely what we are doing except that now we have these sets Ej which are more general than more general than intervals, okay. So that is the concept we needed. We needed a length for sets like Ej, which is given by the measure of Ej, okay. So that is how it is defined.

(Refer Slide Time: 5:48)



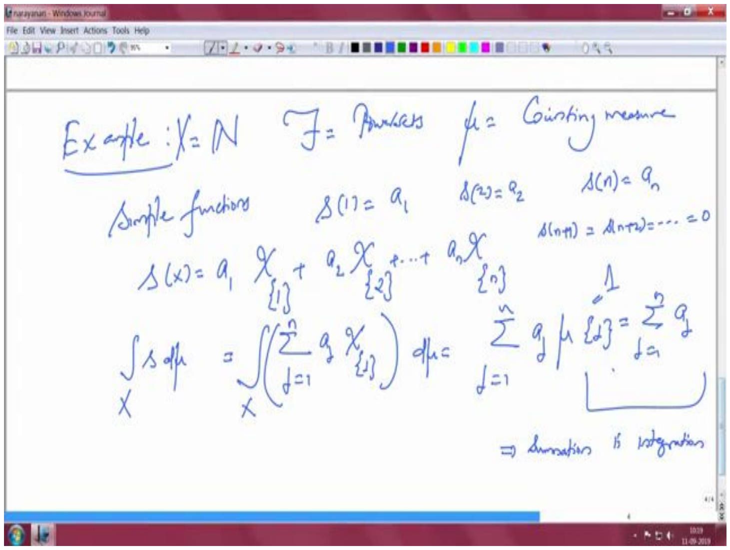
Now using the symbol functions, one can define use this to define integral of positive measurable functions okay. What does that mean? You take a measurable function, which takes values in zero infinity and you define integral of F d mu to be, well what you do is you take the supremum of all those simple functions less than or equal to F and you take the integrals of these simple functions.

So remember these are defined, the integral of symbol functions are already defined and we are taking those values and then taking the supremum. So in, Riemann integration if you recall, in Riemann integration we looked at certain upper Riemann integral that is by taking the supremum over a certain partitions and taking the integrals, those integrals are precisely the integrals we just talked about, right, that we are integrating over, we are integrating a step function over an interval.

You will get an expression like what we have already see. So, s is a summation, alpha J chi Ej, then integral over X as d mu was summation, alpha J mu Ej. So this is, more or less like a Riemann sum, some similar to Riemann sum, okay. And then you are taking a supremum over all such simple functions, less than or equal to f. So remember this is already defined using this one. And then you take the supremum over, all simple functions less than or equal to F.

So this is what defines, so this defines integral for positive measurable function. So remember that, so I should also tell you that in this here, the convention is that infinity times zero is zero. So some of the, some of the measures can be infinity. We will look at examples, so it will be clear. But if the simple function takes the value zero there, and if the set has infinite measured, it would, it will multiply to be zero. So let us look at some examples okay.

(Refer Slide Time: 8:45)

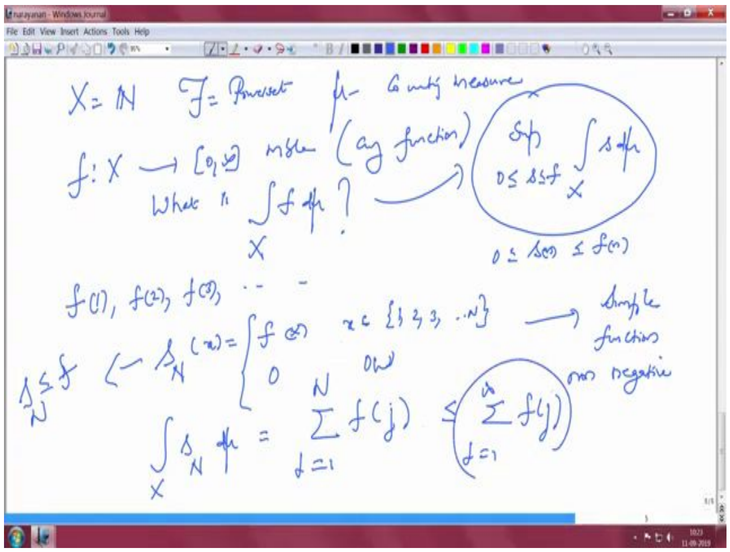


So example, easy example, so far we have seen only counting measure as a measure. So we will simply look at that. So let us look at natural numbers. So this is my space X. The Sigma algebra is the power set. So all sets are measurable and mu is the counting measure, okay. So what would be a simple function? Simple Function is something which, which takes finitely many values, right. So I can define as S to be.

So let us say, so one to be some value, let us say a one s of two, a2. And it could be zero for some values and so on. But it can take only finitely many value. So I will have to stop at some point Sn and some an and then everything else is zero. So S n plus one equal to S n plus two equal to etc. etc. is zero. So how does this look like? So the symbol function S is simply s of X, I can write as on one it takes the value one. So it is a1 times indicator of the set one plus a2 times indicator of the set 2 plus etc. etc. plus an times indicator of the set N right plus zero zero zero, right.

So it does not, so this is like aj chi Ej. So according to our definition, the integral of S d mu is nothing but, this is so I can write this as integral over X. So the function looks like this aj chi indicator of J. This is my function s d mu. These by definition these summation j equal to 1 to n you multiply aj with the measure of the set. So that is measure of the singleton J. But the singleton J has just one element. So the counting measure is simply one, right. So this is some summation j equal to 1 to n aj. So you see summation is an integration okay. So what this implies that summation it is actually integration or more generally integration is generalized submersion you are adding up certain things.

(Refer Slide Time: 11:36)

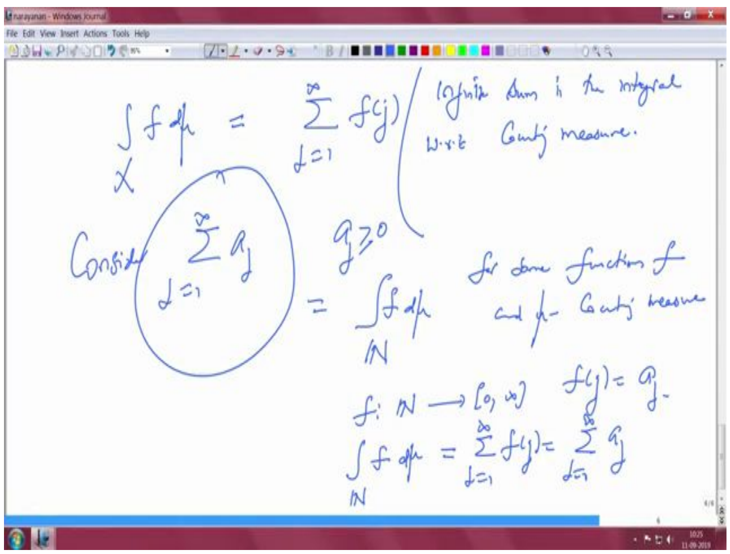


In the same example, let us look at what happens if I take a function. So again, X is N this is the power set. So if the sigma algebra is power set, any function is measurable. Remember that mu, the counting measure. And let us take a function F from X to zero infinity, which is measurable. So any function, so this is measurable would be in any function, right. Because the sigma algebra is the power set. I want to find out the integral of F, what does integral over X F d mu, okay, well what is this? This by definition is supreme over all simple functions less than or equal to F and then you integrate the symbol function.

This, we know how to do and you take the supremum, right. So how does a function look like? Function on N is simply F of one. I know what it is F of two, F of three, etc. right. So this is the collection of values of F and that defines the function F. Now, how would a simple function look like? Simple function takes values, only finitely many values. And so it would be, and it has to be less than or equal to F right. So the maximum value of S, S of N will have to be less than or equal to f of n, okay. So let us take these functions. So let us define S capital N of X equal to F of X. If F X belongs to the first N elements, okay. And zero otherwise, okay.

So I am taking that particular simple function. So this is a simple function, non-negative because F is not negative. What does the integral of SN this simple function with respect to the accounting measure. So we have seen this earlier. This is simply the values of SN added and that goes from J equals 1 to N and you look at F of J okay. But we are supposed to take the supremum over such simple functions less than or equal to F. right? So this SN is of course less than or equal to F. Now if I take the supremum, of course this is going to be less than or equal to summation J equal to one to infinity F of J. But I can go as close as I want to this summation by taking large N if this converges. If it does not converge it is going to go to infinity.

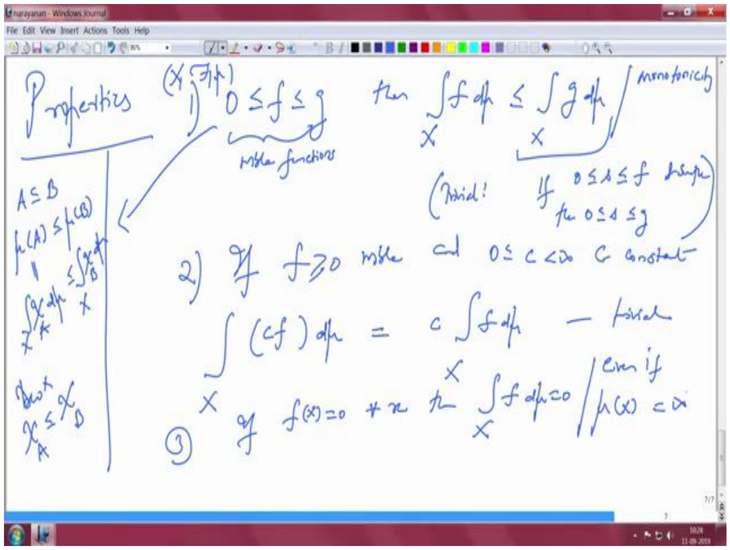
(Refer Slide Time: 14:42)



So this tells me that the integral of F over the space X, where X is N, this is simply the summation. So the infinite sum is actually the integral. So the moral of the story is an an infinite sum is the integral with respect to counting measure okay.

So let me rephrase it. So suppose we are considering… consider summation, AJ, J equal to one to infinity. Okay. AJ positive, because we are looking at only positive functions right now. I want to say this is actually, this is integral over X, Fd mu for some function F and mu counting measure. Where this whole space X is simply N, the natural numbers. Well, how do you do this? You simply define F to be y F of J equal to AJ. Then integral over natural numbers that is my whole space F d mu, mu is the counting measure, and the Sigma algebra is the whole power set. This I know is summation J equals one to infinity, F of J, which is summation J equals one to infinity AJ. So any infinite sum is actually an integral.

(Refer Slide Time: 16:46)



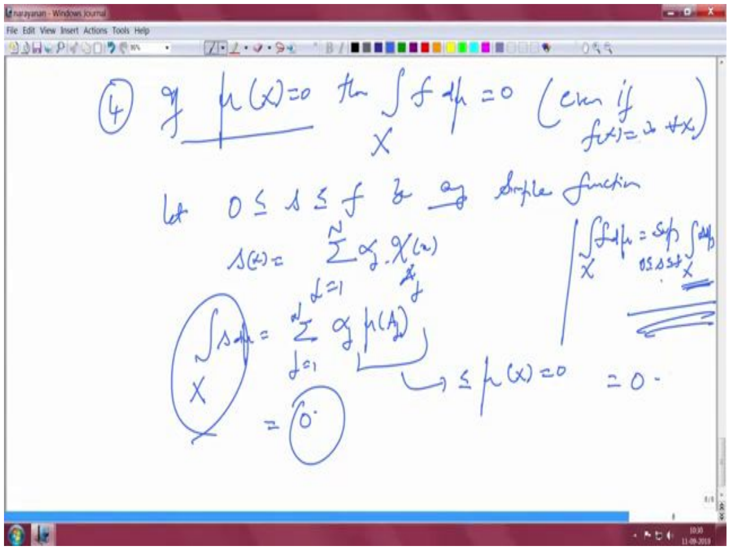
Now before we go further, we look at some properties. First Property:- Zero less than or equal to F, less than or equal to G then integral over X F d mu is less than or equal to integral over X G d mu. All these are measurable. So these are measurable functions. Otherwise the integral does not make sense. Well, this is trivial because if so this I will say is trivial. If zero s less than or equal to f S simple, then zero less than or equal to S less than or equal to G. so you will be taking supremum over a larger set for the definition of G and so it will be much bigger than this integral.

Two, if F is non-negative measurable and I have a constant C then of course C times F is also measurable, right. And the integral of C times F with respect to the measure mu would be this is equal to C times integral X Fd mu. So always remember we start with the triple X script Fd mu okay. We have a space X, we have a Sigma algebra subsets of X and mu is some measure accountably additive. And we defined integral of simple functions taking supremums we defined in the integral of positive functions and we are looking at the properties of that. So this is called the mono-tonicity property.

So recall that we had monotonicity property for the measure right. That same as this. So let us just recall that. So if A is contained in B, then mu of A is less than or equal to mu of B right. So remember mu of A is the integral of the chi of A with respect to the measure mu and we are saying that is less than or equal the integral of chi B right? Why is this true? Because chi A is less than or equal to chi B. Because A is contending B. So that is simply the mono-tonicity property for indicator functions. We just true for all positive measurable functions. That is what we know, okay. So this is also trivial. I leave it to you.

Third one, if F of X equal to zero for every X, then integral of F d mu is zero. So this is even if mu of X infinity, okay. So remember the convention is that zero times infinity is zero.

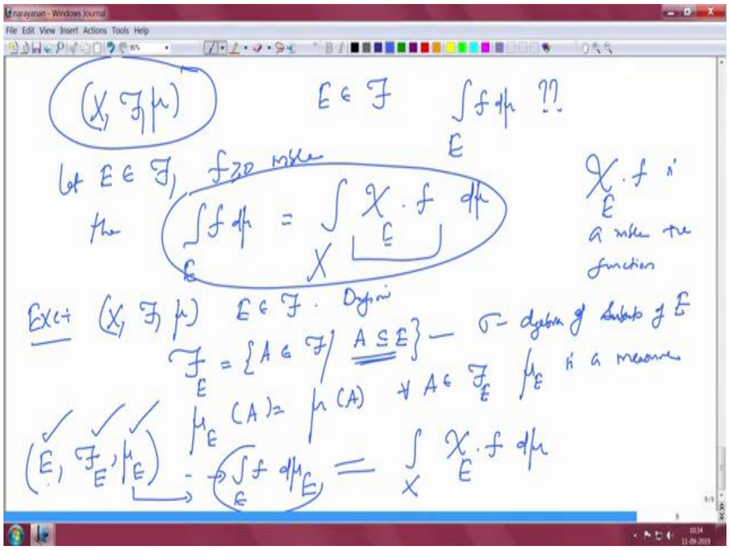
(Refer Slide Time: 20:07)



Forth Property- If mu of X is zero, then integral over X. Fd mu is zero, even if F of X equal to infinity for every X, okay. Why is this true? So let us do the forth property. Suppose mu of X is zero. So total measure is zero. Then let us take any simple function. Let F be any simple function which can be a return as F of X equal to a alpha J chi AJ, J equal to one to some capital N finite linear combination of indicated functions. That is what simple function is.

I am assuming this is less than or equal to F. Now Integral of F is the supremum over certain things, right. So supremum over S less than or equal to F integral over X Sd mu. Okay, So I want to calculate these things and then take the supremum. So if I look at integral over X Sd mu. Well, what is the definition? This is simply J equals one to N alpha J mu AJ. That the our definition, but what do we know about mu AJ. Mu AJ is less than or equal to mu of X, which is zero. So all this is zero. So for any simple function S less than or equal to F, the integral of this is zero. And so when you take supremum over this you will get zero. So even if F of X is infinity for every X, if the total measure is zero, then the integral is zero.

(Refer Slide Time: 22:16)



So now let us just slightly generalize this notion. So I have the space. So recall in the Riemann integration, we integrate over any interval reward, not just the whole real line or all the whole space. So we want to do the same thing here. I want to define if I take some measurable sets in F, I want to know what is this? Can we define this? Well one definition is this. So let E belongs to scrip, F, F positive, measurable, then integral over E, F d mu. Well we do the natural thing. So we know how to integrate over the whole space. But I want to restrict F just to E. So I simply multiply F indicator function of F, indicative function of E and then integrate.

So what does this, this chi E times F is a measurable function. It is a measurable, positive function. So I know how to integrate it, right? I look at all simple functions less than or equal to this, integrate and then take the Supremes. So that leads us to two definitions, in fact. So let me give this as an exercise. I will explain what that is. So you take X, F, mu and let us take some set in script E, script F. Define the following. Sigma algebra. Define script F, sub E. what is this? This is the collection of all sets in script F but which are contained in E. So now it is like the whole space is E.

We are not going out of E, look at all subsets of E which are measurable. Okay. So prove that this is a sigma algebra of subsets of E okay and define mu of E of A equal to mu of A for every A in script F, okay. Then mu of E is a measure, of course this is a measure because mu is accountability additive measures. So mu is also going to be accountable additive except that now we look at only sets which are content E. We are not going out of E. So now we have a triple. E is my whole space. Now I have script F sub E, which is a sigma algebra subsets of E and I have accountably additive measure. So whenever we have a triple, we know how to define the integral.

So I can define the integral using this, right. So I will have integral over E, F, d mu sub E, correct. This I know how to define, if I have a space, I have a sigma algebra script F, and I have accountability to measure, then I know how to define this, okay. Check that this is equal to integral over X, chi E times F d mu. That is our original definition, right. So this definition and this definition are same. So we could have started with any subset of X, which is measurable and defined integration, over that sets, or we could start with the whole space and then multiply F with any indicator function so that we can restrict everything to, that particular sets.