Measure Theory
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Lecture 50
Continuous Linear Functionals

So, let us start, in the last session we saw that every non empty closed convex subset of L2 has an element of smallest norm. So this will be crucially used in the next proof which state that every continuous linear map from L2 of mu to the complex plane is given by an inner product. So in the in the beginning of the last session, we saw that an inner product with a fixed function gives a continuous linear map from L2 of mu to the complex number. We are trying to prove the converse of it.

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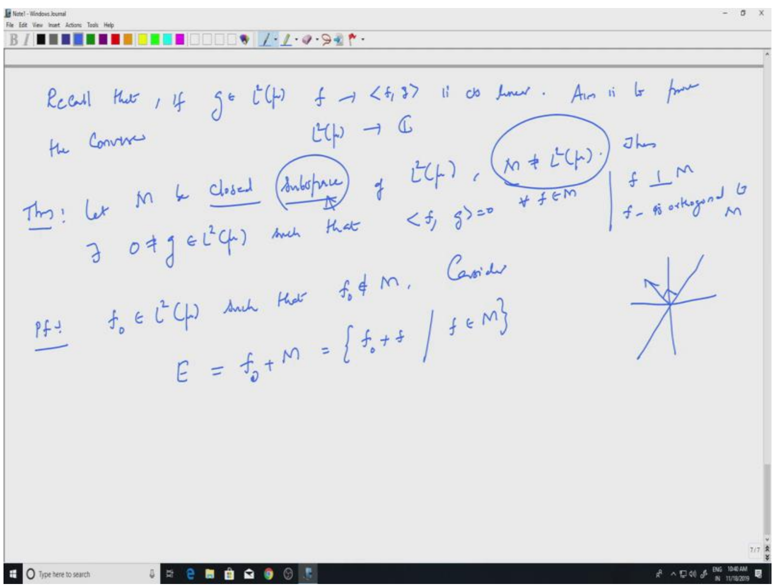

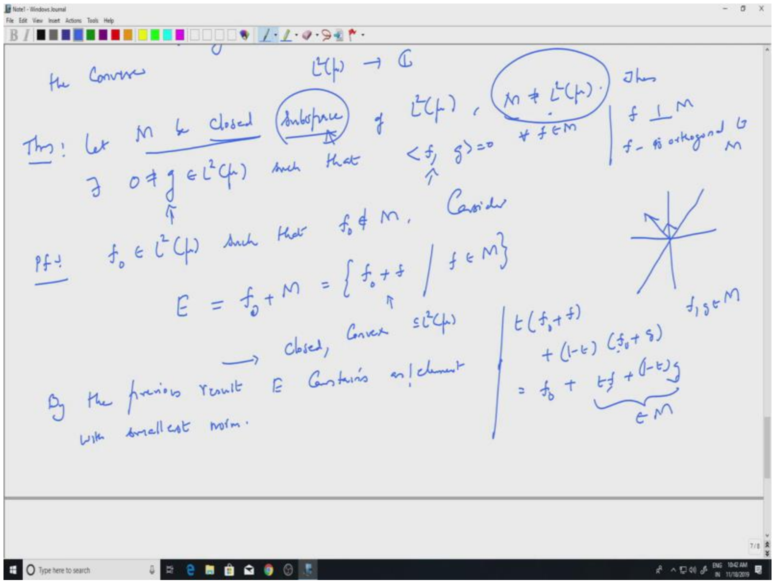

So let us start, so recall that if I fix a function g in L2 of mu, then the map f going to if inner product with g is continuous linear. So this goes from L2 of mu to the complex plane. Our aim is to prove the converse aim is to prove the converse. So this is easy to see infinite dimensional spaces and so on I will I will command upon it when it comes to that but let us start with a result which we will be using soon.

So theorem, so let M be a closed subspace, so closed subspace of L2 of mu, L2 of mu is a vector space, so subspace makes sense closeness makes sense because it is a metric space. And we assume that it is not equal to L2 of mu, so there are elements outside. Then, there exists a function which is not equal to 0 almost everywhere, L2 of mu such that the inner product of f with g equal to 0 for every f in M. So we write that f is orthogonal to m.

So, that is so for example, if you look at r2 or C2 a subspace is going to be some line which goes to 0. And you will have some element which is orthogonal to it. So this is a this is actually perpendicular to the line. So if it is not the whole space and if you are in an inner product space, which is complete you will always have something which is orthogonal to it. So this is this is written as f is f is orthogonal to m.

So, in the proof we use the previous result. So first of all take f naught in L2 of mu such that, you can take a non-zero f naught. So f naught, not in M, so that is possible because the subspace so it is a subspace remember that subspace meaning if I it is a vector subspace, so in particular it is convex. So it is a closed convex set M any closed subspace is a closed convex set. So we can apply the previous theorem. So take a function 2 which is not an M, because M not equal to the whole space, so this is possible. So now consider E, so this is going to be our closed convex set. It is f naught plus M, well what is that? That is f naught plus f where f is an M.

So we have seen this, so translation by n element in RN and so on. There we used only the vector space structure so you can do this in any vector space you like. So in particular we are translating M. So this is M is close, so f naught plus M is also closed, m is convex, so f naught plus m is also convex.

So for example, if I take two points, so let us say t times f naught plus f plus 1 minus t times f naught plus g. What would be this? So were fn g are an M. So that is how the elements are. So this is simply if I look at f naught I will simply get t f naught plus 1 minus t f naught which is f naught plus t f plus 1 minus tg. And this is of course an M because M is a subspace, so any linear combination of f and g will be in M in particular m is convex.

So if you translate a convex set you are going to get a convex set. So if E is a closed convex set contained in L2 of mu. So by the previous result by the previous result E contains an element with smallest norm. It is a unique element with smallest norm. So let us go back to the theorem m is a closed subspace and it is not the whole space. We are trying to find a g which is orthogonal to M.

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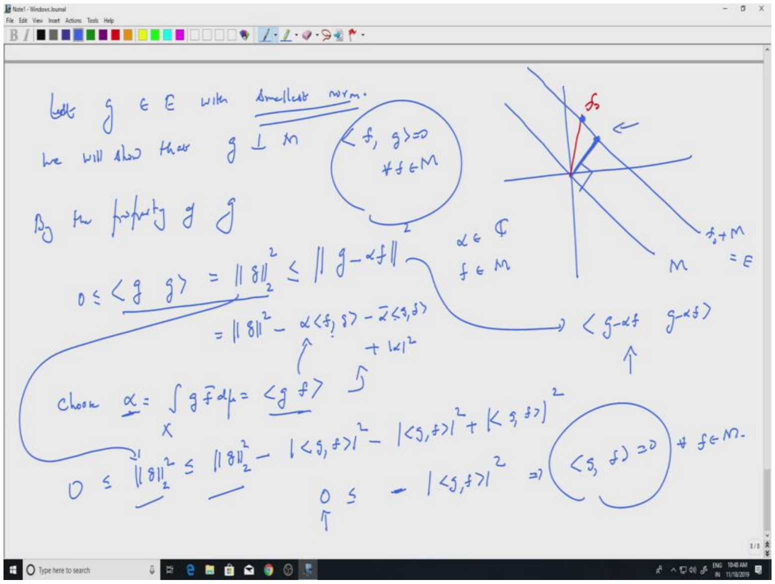

So let us call that g, maybe let me draw a picture for you to understand. So let us say this is M, so this is my set m. So what did we do? We took something which is not in m. So let us say this this vector call that f naught that is what we did. Now, if I add f naught to M, I am going to get this line, so it the f naught is here. So this would be f naught plus m. Now, what are we doing? We are looking at so that is our E so that is our E.

So E is a closed convex set and we are trying to prove that looking at we are trying to look at the element with smallest norm smallest norm is the distance from the origin. So if I look at this perpendicular here that is the element with the smallest norm in E and so that that is actually perpendicular to M. So that is how we find the element. So let us call that g, so let g be the element in E with smallest norm.

So, we will show that, we will show that g is orthogonal to M. That is f inner product with g is 0 for every f in M, this is what we want to show. So this follows simply from the fact that it is the smallest norm. So by the property of g. So what is the property of g? It has the smallest norm in E.

So you look at g inner product with g of this is equal to L2 norm g square, which is this is the smallest norm. So for any other element in E you will have norm bigger. So if I look at g minus alpha f square for alpha a complex number and f in M, this would be bigger. So because of that. So this is so let us let us compute this. So if I if I expand this I will get norm g square.

So how do we expand this? So this is simply the inner product of g minus alpha f with g minus alpha f. So you can expand that using linearity and n in linearity in the second variable. So you will get norm g square minus alpha come out as a constant minus alpha coming from the other right hand side will give me alpha bar gf plus mod alpha square.

So this is simply expanding this inner product. What is, so now so this is true for any alpha alpha and (())(09:56). So we are going to choose alpha appropriately so that we get the result we want. So choose alpha to be integral over x g f bar d mu which is simply the inner product of g and f. So that is a complex number, so we plug this here you will get 0, 0 greater than or equal to or less then of equal to I have norm g square minus so I am putting alpha equal to g inner product at f here.

So g inner product with f and f inner product with g are complex conjugate to each other and if I look at these two terms, they will cancel each other. So you will get that. Now let me let me write it down. So g inner product f with if I if I substitute there. I am going to get modulus of g inner product with f whole square minus this is also modulus of g inner product with f whole square plus mod alpha square is again mod g in of inner product with f whole square modulus square.

So one of them cancels and there is, so I should have written one more step. So here I have norm g square this is what is greater than 0. So this, so if I looked it this so I have not done any extra work here. So I have started with norm g L2 norm g square, so that sort of written it down here again that is what I have done the other part is just computation. So if you looked at this this and this will gets cancel.

Here there are three terms of which 2 will get cancel and you will get a negative sign. So 0 less than 2 minus modulus of g f whole square, but this is possible only because the right hand side is modulus square, so that is positive and there is a negative sign and here I have 0. So this implies that g f is 0, which is precisely what we want, for every f in m. So that is all we want so this is the that is the theorem. So the whatever is there in the picture, so that one actually works.

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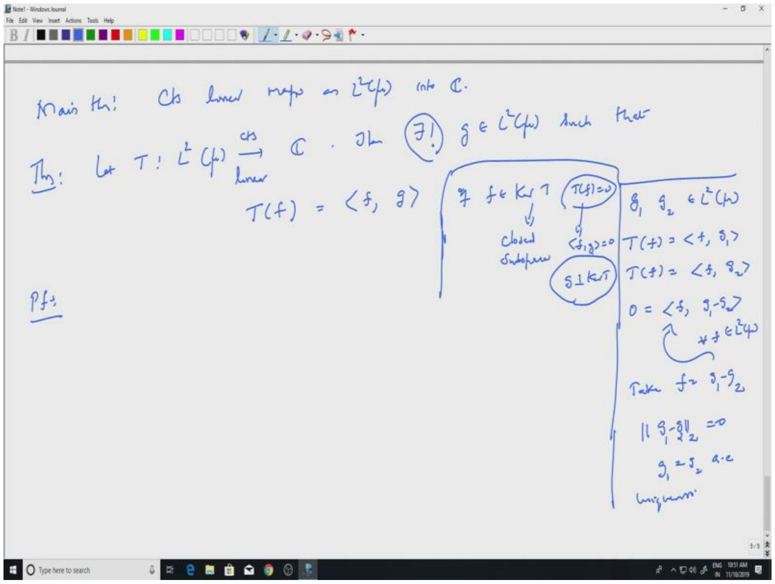

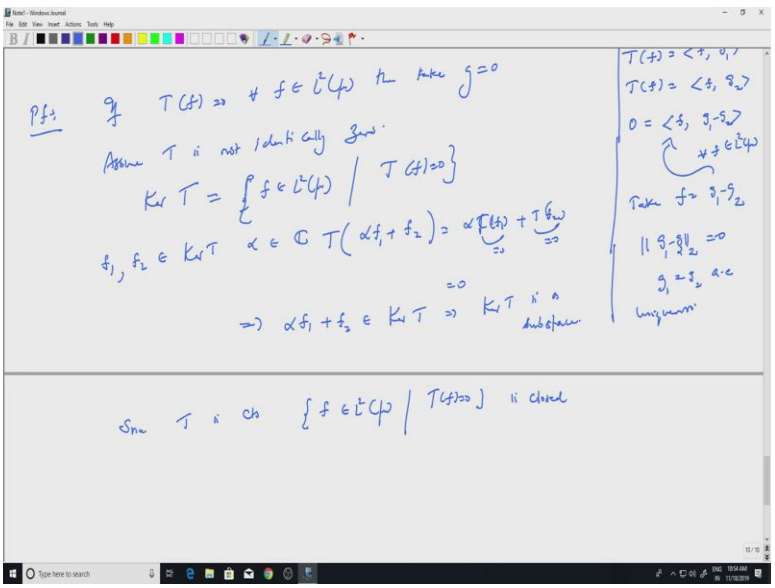

So now we are into the main theorem, so main theorem. So this is the continuous linear maps on L2 of mu into the complex plane C. So then it let T be such a map. So maybe I should write it as a stated as a theorem, so theorem. Let T is a map from L2 of mu to the complex plane is continuous and linear.

Then there exists the unique element g in L2 of mu, such that T of f, so this is a complex number that is given by f inner product with the fixed function g. So the unique element g. This is associated to T. For any f T of f is f f inner product of g. So converse we have already seen any such inner product gives a continuous linear map and what we are saying is any continuous linear map on L2 and inner product space in fact which is complete you actually an inner product the with a fixed vector. So we use all that we have proved just now. So proof again uniqueness is a sort of trivial. So I will let me write it here. So I will if I have two functions g1 and g2 in L2 mu defining T. So that means Tf equal to f inner product with g1 and Tf equal to f inner product with g2, then I can subtract. So I will get 0 equal to f inner product with g1 minus g2. But this is true for every f in L2 mu. So in particular take f equal to g1 minus g2 and plug it in here we will get that L2 norm of g1 minus g2 is 0, so g1 equal g2. So g1 equal to g2 almost everywhere.

So that is the uniqueness. So uniqueness. Uniqueness is easy, we need to show the existence that actually there is g which works. How will I get the g? If you look at this if the theorem is true any if so this I will rub off after writing down if f is in the kernel of T. That means T of f is 0, kernel of a linear map is a subspace.

So this is a closed subspace. Why is it closed? Because T is continuous. So it is zero set has to be a closed set and it is linear so it is a closed subspace. So we will I elaborate on it so soon. If f is in kernel of T Tf is 0, which means if Tf is actually f g then what we have is f inner product with g0. That means g is orthogonal to kernel of T.

So we need to find and g which is orthogonal to kernel of T and see if it works. That is (())(16:13) would be an idea. So let us try to do that. So if T Tf is 0 for every f will L2 mu that means T is 0. Then take g to be 0. Then take g to be 0. So we can assume Tf is not equal to 0. So assume T is not identically zero, T is not identically zero.

And consider kernel of T. So, what is Kernel of T? This is all those f in L2 of mu, such that T of f is 0, all those so it is like the null space of a of a linear transformation, but now this space this space is nice, kernel of T is a subspace first of all, so if I take f1, f2 in kernel of T and alpha complex number. Then if I look at alpha f1 plus f2 and look at T of that, this is alpha of T f1 plus T f2 because of linearity.

But this is 0 and T f2 is also 0, so this is 0. Which means that alpha f1 plus f2 is in kernel of T. Which is same as saying kernel of T is a subspace. It is a vector subspace of L2 since T is continuous the set f in L2 of mu, such that T of f equal to 0 is closed. So it is a closed subspace so that is what we want.

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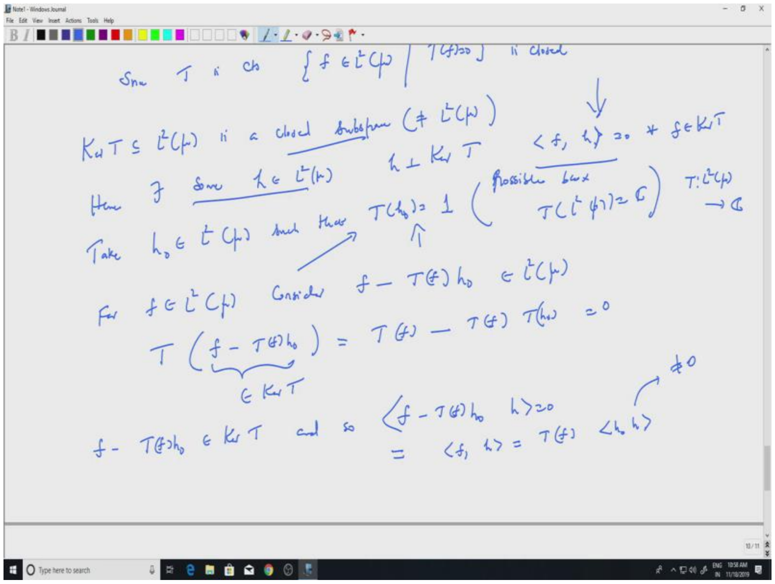

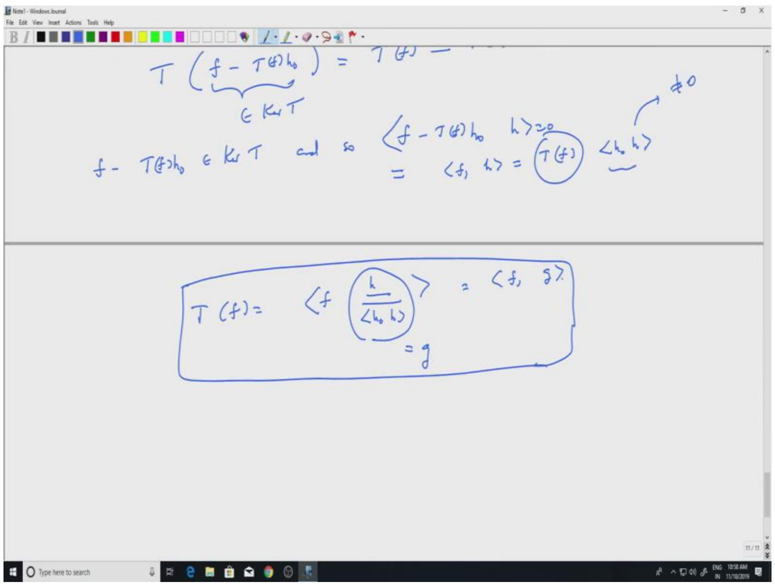

So let us start with that assertion again. The kernel of T contained in L2 of mu is a closed subspace and not equal to L2 of mu. So it is like the M we had earlier at the closed subsets which is not equal to L2 of mu because T is not 0. So there exist something which is orthogonal to it. So hence there exist some h in L2 of mu. Such that h is orthogonal to M. M is my kernel of T. So let me write kernel of T, which is same as saying f inner product with h equal to 0 for every f in kernel of T.

So, we want to scale h so that it works. Take h naught in L2 of mu, such that T of h naught is equal to 1. Why is does it exist? So T is the map so recall T is linear map from L2 of mu to the complex plane so the image has to be subspace of the complex plane. But if T is non zero there are nonzero elements in the image of T. And so you will get the whole subspace. So this is because possible because, so there is usual linear algebra, you know, the image of a linear transformation is a subspace.

And complex plane is one dimensional complex space. So it is either 0 or the whole space, if it is 0 then T is 0 but we are assuming T to be a nonzero linear function. So possible because T of the whole space is the complex plane C. So now we are all almost done, so we have just two three line left to complete the proof.

So for an arbitrary f an L2 of mu. Consider f minus Tf, Tf remember is a complex number times h naught, h naught is a is the guy we have chosen so T of h naught is 1. Now, of course this is also in L2 of mu I am just adding two function which are in L2, but what is T of f minus Tf h naught?

This because of linearity it is Tf minus Tf is a is a complex number that comes out then T of h naught, but T of h naught is 1 so this is 0, which means that this function belongs to kernel of T, but if it is not kernel of T, it is orthogonal to h. So, f minus Tf h naught is in kernel of T and so f minus Tf h naught is orthogonal to h. So h remember is because we have since it is a closed subspace there is some edge such that h is orthogonal to everything in kernel of T.

What does it mean? This mean that f inner product with h equal to minus Tf etc I take to our side Tf is a constant so that comes out h naught inner product with h. So this cannot be 0, because if it is 0 Tf will be 0, f will be inner product f inner product h will be 0 for every f, so Tf will be 0. So that is nonzero.

So we want to find out what is Tf. So T of f is simply f inner product with this I can take to other side this is a complex number. So h divided by h naught h, there is a complex (())(22:41). And that is your g, so that is all we want, g is this f inner product with g. So that proves the theorem because I have found that g for you which, such that T of f is inner product of f with g.

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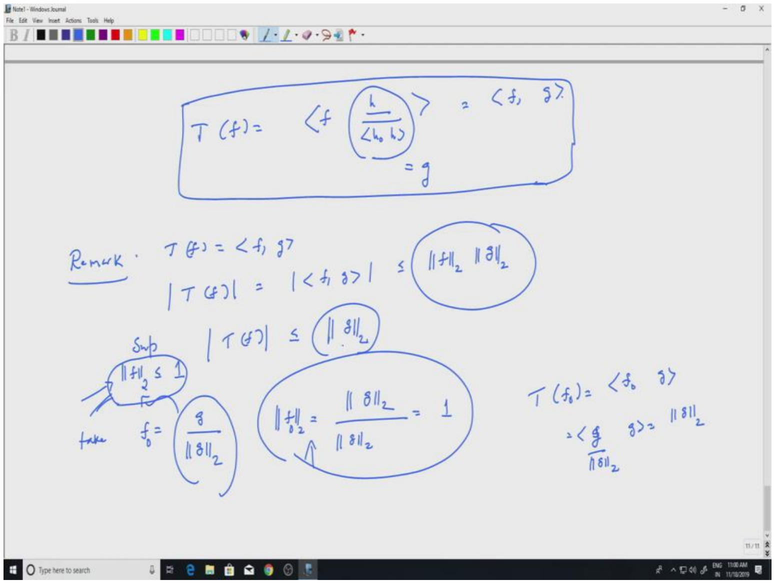

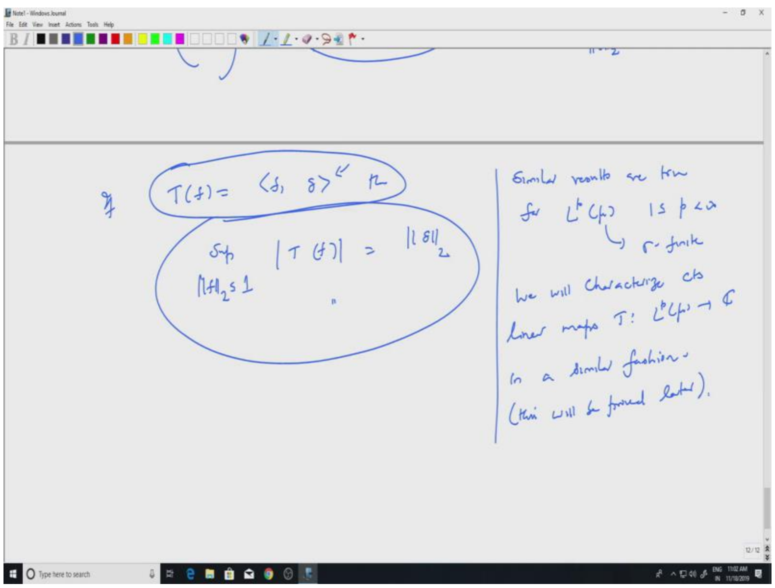

So, let me finish with one small remark here remark here. So Tf is, if inner product with g, any continuous linear function is given by this. So if I look at modules of Tf, this is of course modules of f inner product with g which is less than or equal to L2 norm of f into L2 norm of g by Cauchy Schwarz inequality.

So, if I take supremum over L2 norm of f less than or equal to 1 modulus of Tf. So I am taking supremum this on the left hand side. I have this quantity and I am looking at all those f such that the norm is less than equal one. So this is clearly less than or equal to the L2 norm of g. So not just that if I take f2 to be equal to g divided by L2 norm of g.

Then L2 norm of f is 1 because that is L2 norm of g divided by the constant L2 norm of g will come out, which is equal to 1. So this f is actually one element in the set. So let us compute. So maybe let me write f naught here, so that f f naught are different, so if I come f naught is an element then the unit ball at the all elements with norm less than or equal to 1.

So what is Tf naught? Well f naught is f naught inner product with g, which is equal to g by L2 norm of g inner product with g which is equal to L2 norm of g, square divided by L2 norm of g, so it is L2 norm of g. And f naught is something which has norm 1. So supremum over such set is less than or equal to this quantity and there is an element where that supremum (())(25:09).

So, if I put together what I get is, if T of f equal to f inner product with g then the supremum of L2 norm of f less than or equal to 1 modules of Tf is actually equal to L2 norm of g. Not just that it is less than or equal to is actually equal to because there is an element for which, so we have two things in that, so we have that all continuous linear functionals are given by this inner products and this equality.

So similar results are true for similar results are true for Lp spaces, So not just for L2 1 less than equal to p strictly less then infinity but mu has to be sigma finite. We did not knew any sigma finiteness in this case. So we will show that continuous linear, so will characterize continuous linear functionals continuous linear maps.

So let us call that T from (())(26:26) L2 of mu to the complex plane. In a similar fashion in a similar fashion, but that will come later, so this will be proved later. So we will stop here. So we just characterized continuous linear functionals on L2 of mu and this was slightly easier because L2 of mu had an inner product structure and that gave us concept of orthogonality and subspace which are proper had some vector orthogonal to it and that helped us in getting the function g we want that.

And this will be used so the continuous linear functionals on L2 of mu are given by inner products will be used crucially in the proof of the Radon Nikodym theorem. That is what we will be doing in the next two sessions. So, we will start with a complex measure which is absolutely continuous with respect to a positive sigma finite measure and we will show that the complex measure is actually given by an integral of the positive measure.

So, finding out that particular function which works for the complex measure that function is called Radon Nikodym derivative that will we will use this particular fact that continuous linear functionals on L2 are given by inner product. So we will somehow construct continuous linear functionals on L2 that is the first step and then from that we will have to deduce the Radon Nikodym theorem. So we will do that in the next two sessions. Let us stop.