Measure Theory
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Lecture 51
Radon Nikodym Theorem I

So, in the last lecture we saw properties of L2 of mu in particular we characterized continuous linear functionals. So when I say functional, it is a map from the vector space to the underlying field. So in the case of L2 mu, the underlying field is the complex plane and the linear functional is a linear map from L2 of mu to the complex plane and we saw that any continuous linear functional is a is given by an inner product.

So, we will use that in the proof of Radon Nikodym theorem. So that is what now we will do today, in this proof of Radon Nikodym theorem. In fact, we prove two theorems together. So the one is called Lebesgue decomposition of a measure of a complex measure with respect to a sigma finite positive measure and there is the Radon Nikodym theorem. So both the proofs are companion into one prove and this is beautiful proof due to for noman, which is what we should be able to do in the next two sessions.

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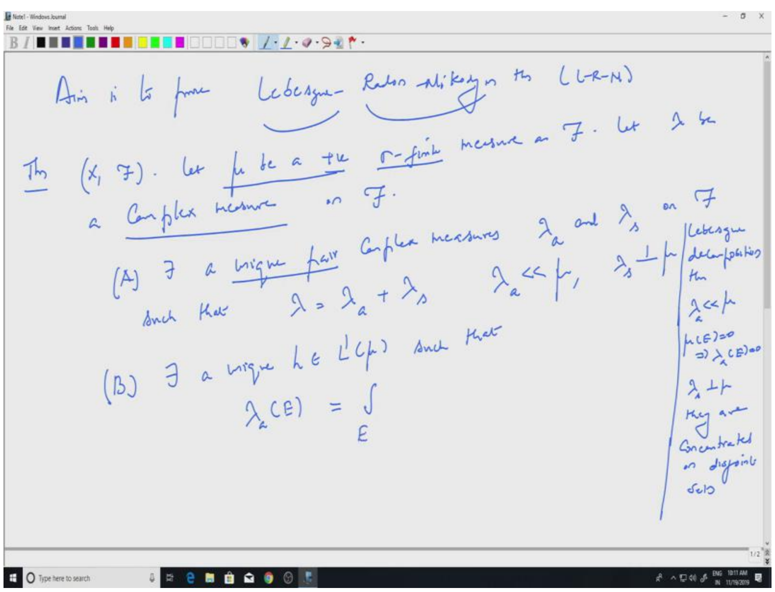

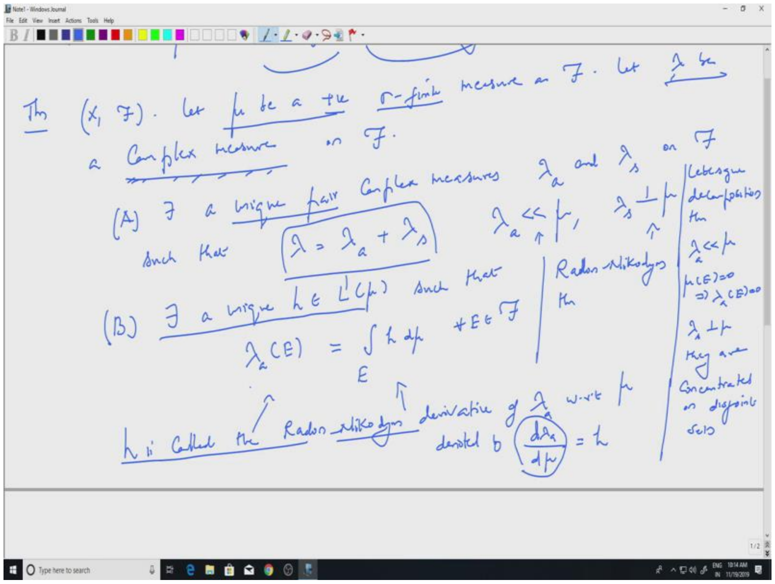

So let us start, so aim is to prove what is known as Lebesgue Radon Nikodym theorem Nikodym theorem. So these are two theorems, so one is the Lebesgue decomposition theorem and the Radon Nikodym theorem. So I will explain that. So let us so we will use L R and to use the short form for Lebesgue Radon Nikodym theorem. So make sure that you understand. These are two theorems one is the Lebesgue decomposition and other is Radon Nikodym theorem. So theorem so statement has two parts because there are two theorems. So as usual we have space and sigma algebra.

Let mu be a positive sigma finite measure on F or X. And let lambda be a complex measure complex measure on X. So first theorem says first part of the theorem says that you can decompose lambda into two parts one is absolutely continuous with respect to mu other is singular with respect to mu. So that is the first part. So this is called the Lebesgue decomposition, their exist unique pairs, so unique pair of complex measures.

So that is unique, complex measures lambda sub a, so a stands for absolute continuity and lambda sub s. So that is the singular part of lambda on same sigma algebra first, such that the original measure lambda is a sum of these two lambda a plus lambda s. This is not very, it is important but you know writing a complex measure the sum of two other complex measure is not got us a big deal.

Big deal is that the components are so the lambda a is the absolute continuous part of lambda with respect to mu. So remember mu is the positive sigma finite measure on F and the lambda s which is the singular part. So that is mutually singular with respect to mu. So this is called the Lebesgue decomposition theorem, so Lebesgue decomposition theorem.

So, recall absolutely continuous means, so lambda a is absolutely continuous with respect to mu means mu of a equal to 0 implies lambda a of E equal to 0 and they are mutually singular means they are concentrated on they are concentrated on disjoint sets. So we had defined all these and looked at various elementary properties of mutually absolutely continuous and mutually singular measures and so on.

So the first part of the theorem is the Lebesgue decomposition, second is the Radon Nikodym theorem. So what is that? There exists a unique h in L1 mu. So mu remember is the fixed positive sigma finite, sigma finite (()(06:02) not important sigma finite measure, such that lambda a of E, so lambda a is the absolute continuous part of lambda with respect to mu is given by h h d mu (())(06:26) for every E script F.

So this is the Radon Nikodym theorem, so Radon Nikodym theorem. So if I have a complex measure which is absolutely continuous with respect to a positive sigma finite measure, then the complex number measure is given by an integral with respect to E positive sigma finite measure. And there is an h, so there is a unique h in L1 mu which does this. The h is called the Radon Nikodym derivative. So h is called this is called the Radon Nikodym derivative derivative of lambda a with respect to mu.

So sometimes we also denote this by the derivative of lambda a with respect to mu in the usual dy by dx notation. So you are differentiating y with respect to x you use the notation but this is just a symbol. So remember that this is just a symbol, symbol for h, so this is h. So h is called the Radon Nikodym derivative.

So two parts to the theorem, the first part is so one could have stated this has two theorems. You can take any complex measure lambda and you can decompose it in this form. Where one of them is absolutely continuous with respect to mu. Another is a mutually singular with respect to mu and as a second theorem, which is the Radon Nikodym theorem if I have a measure, which is absolutely continuous with respect to mu that is the case with lambda a here.

So lambda s may not access. So for example, lambda could be absolutely continuous with respect to mu. Then you will have a Radon Nikodym derivative. So h so keep in mind the assumptions lambda is a complex measure. So mod lambda, which is the total variation would be a finite measure. So if lambda is positive, then it has to be a finite measure, otherwise, it is not a subset of the collection of complex measures. So there the theorem can be proved for two sigma finite two positive measures lambda and the mu. I will write it down after this is proved. So I hope this the statement is clear.

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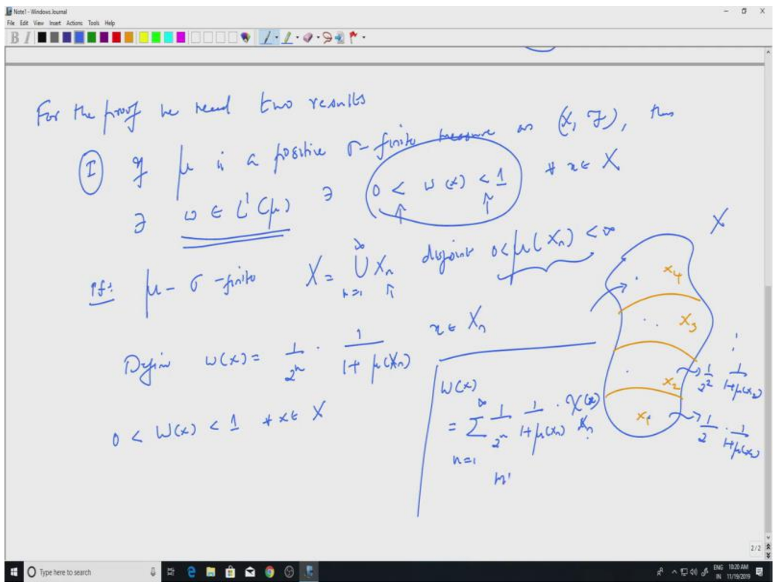

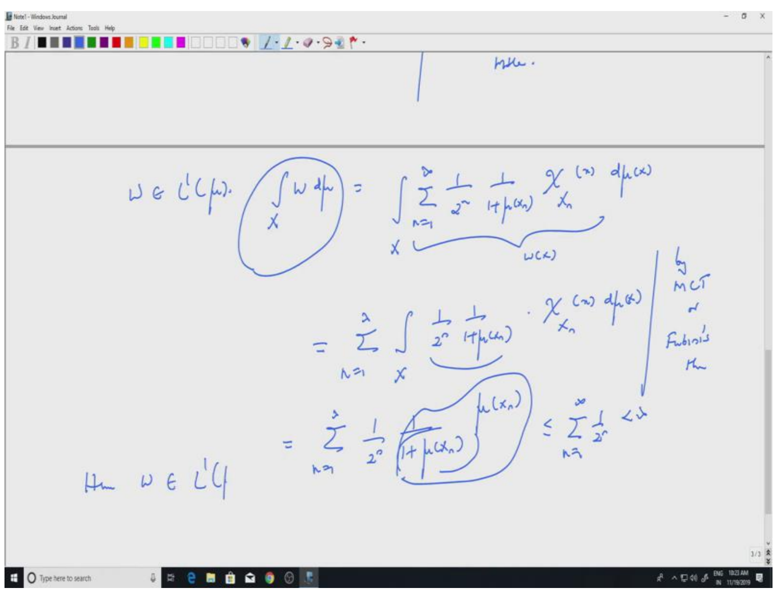

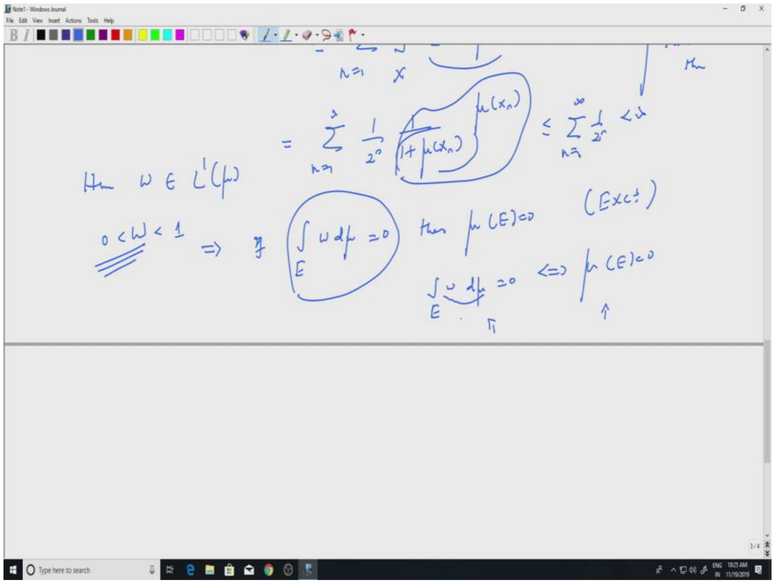

So for the proof we need several things. So for the proof we need, at least two auxiliary results. So let us write this as two results actually three but one of them was proved in the last two lectures where continuous linear functionals on L2 mu was given by inner products. So the first result we need is, if mu is a positive sigma finite measure on X, F so mu is what the positive sigma finite measure is the theorem as well. Then, their axis omega in L1 mu, such that 0 less than w let us say w x less than 1 for every x in X. So it is a function which is between 0 and 1 and it is in L1 it strictly so note the strict inequalities.

Especially that strictly greater than 0 something which we will be using. So let us note that, it is very easy to construct this there is nothing very surprising about this but we will we will use this particular w to convert mu into a positive finite measure, so remember this is a positive function which is in L1. And so when I use w to define a measure it is going to have finite the total measure would be finite.

So let us prove this first, so this is sort of easy. So mu is sigma finite mu sigma finite which means the whole space I can write as union of xn n equal to 1 to infinity disjoint if they are not disjoint you can disjoint (())(11:38) them such that mu of xn is, I can say it is strictly positive and less than infinity.

So you can write it as a countable union of sets with finite positive finite measure that is what sigma finiteness means. So now you simply so this is your X and you are having Xn. So let say these are these are xn, so I have x1 I have x2 x3 and x4. And there can be countably many countably infinitely many. So all you have to do is to get a w in L1 is to define w to be constant in each of these components, so that some integral is finite and make sure that this is true, that is all we need to do.

So simply define define w of x to be equal to 1 by 2 to the n, 1 by 2 to n because you wanted to add up times. So I am going to so n denotes the n of the xn. So on xn and I am going to define some constant. So this is 1 plus mu of xn this is for xn capital xn so capital xn appears here as well.

So mu of Xn is a positive finite quantity. So all this makes sense 1 by 1 plus mu of Xn is a finite number positive number. I multiplied with 1 by 2 to the n for x and X so that is my quantity, so w is defined to be so here it is 1 by 2 into 1 by 1 plus mu of x1 here it is 1 by 2 square into 1 by 1 plus mu x2 etc etc.

So in each of these sets x1, x2, x3 it is a it is a number is a positive number. So obviously w is strictly positive, so w of x is strictly positive and of course it is less than strictly less than 1. So this is true for every x an X. it is clearly measurable because this in each exercise are measurable and you are multiplying the indicator function of xi with some constant.

So you can write so let us write w bit more. So what does w so, this is summation n equal to 1 to infinity 1 by 2 to the n 1 by 1 plus mu of xn times indicator of xn. The characteristic function of Xn and X. So if I take an X, it would be in one of the xn. All the other ones will be 0 and I will get 1 by 2 to the n 1 by 1 plus mu xn has the value. So this is obviously measurable.

But there are other assertions that w is actually in L1 of omega, so let us justify that but it is very simple, it is a linear combination of indicator function so you can simply integrate. So let us see why is w is in L1, so w is in L1 of mu. Why is that? You look at integral over X d mu I want to compute this.

So this is equal to w is given by an infinite sum, so integral over x summation n equal to 1 to infinity 1 by 2 to the n 1 by 1 plus mu of xn, so you will see why 1 by 1 plus mu xn was used kai xn next d mu x, so this is my w, so this is w of X. Everything is positive, so you can apply monotone convergence theorem to interchange the summation and integral or since everything is positive you can apply Fubini's theorem because I have summation summation as an integral. So I have two integrals and I want to inter change. So this is equal to summation n equal to 1 to infinity integral of 1 by 2 to the n 1 by 1 plus mu of xn.

So these are constants times the function indicator of xn at X d mu x. By so you can apply by monotone convergence theorem or Fubini's theorem because we have positive functions we have two integral so you can interchange the integrals. Which is equal to, so this is a constant so that comes out n equal to 1 to infinity 1 by 2 to the n 1 by 1 plus mu of xn mu of xn is some number positive number. Then I am integrating the characteristic function of xn, so I will get the measure of xn there is nothing else.

So there now you see why I put a 1 by 1 plus mu xn instead simply mu xn, you can take mu xn also. So this this quantity is less than or equal to 1 and so this would be less than or equal to summation n equal to 1 to infinity 1 by 2 to the n which is finite. And so w being a positive function has integral finite. So it is integrable. So hence w belongs to L1 of mu, but what is important for us is that, so here is an exercise. So w is strictly positive and strictly less than 1 strictly less than 1 will be used perhaps some point but strictly positive is important this implies that if integral over E w d mu is 0 then mu of E will have to be 0.

So I will leave this as an exercise should think about it. It is not difficult to see if integral over E w d mu is 0 see because w is strictly positive. So if mu E this is not 0 then there would be some n for which w is greater than 1 by n and you will get the integral to be strictly greater than 0 but if the integral is 0 of a strictly positive function then the set of the measure has to be 0. So that is that is easy exercise we will so this is what more important to us.

So in fact integral over w sorry integral over E w d mu equal to 0 if and only if mu is 0. So this the point is whenever the left hand side is 0 right hand side is 0, whenever the right hand side is 0 I am integrating over a set of measure 0 so I will get 0 so that is the easy part. And w d mu I can think of as another measure. So this measure on the left hand side and this measure on the right hand side has same set of measure 0 sets, so the collection of measure zero sets are same mu of is 0 if and only the left hand side is 0 that is the important part we will use.

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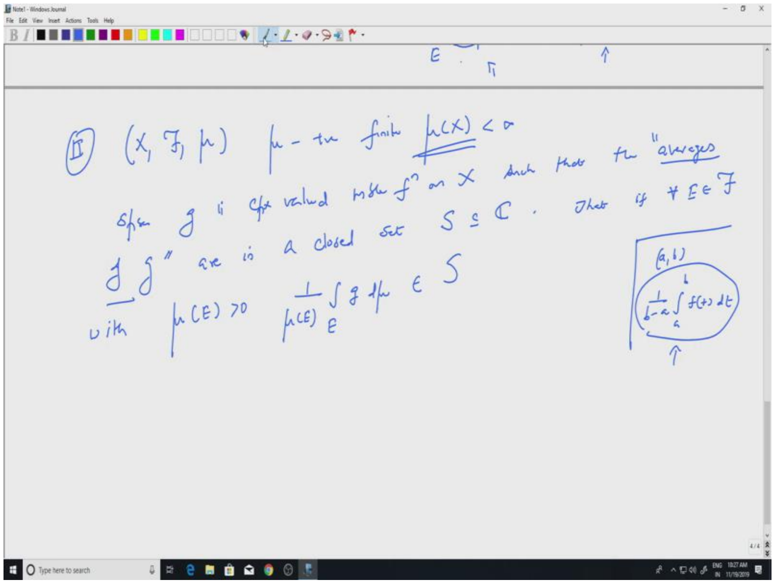

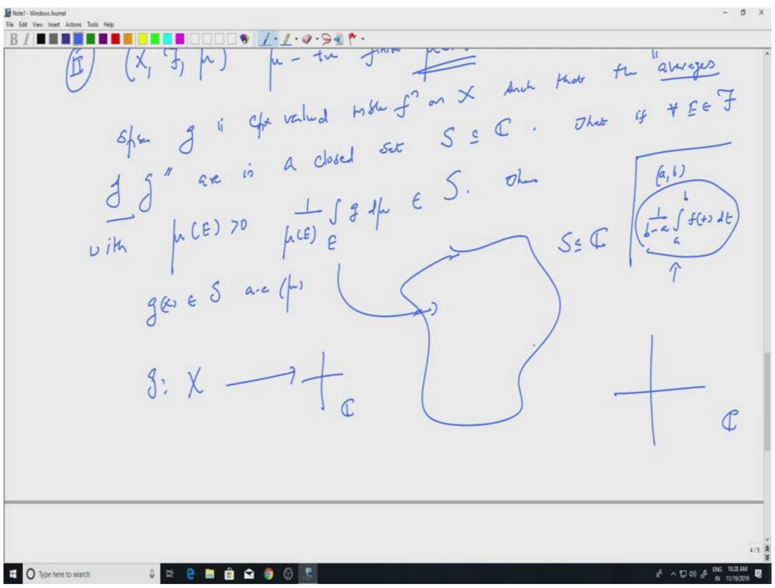

So the next so this is one of the results we need, second result is about averages of the function. So let us let us take positive measures so positive mu is positive and finite finiteness one can relax the restriction finiteness but let us let us not worry too much about it right now. So it is a finite measure. Suppose g is a complex valued complex valued measurable function on X, such that the averages of g so I will tell you what the averages are, averages of g are in a closed set let say S inside the complex plane.

So what are the averages of g? That is for every E in script F with mu of E positive. It will be finite because mu of Xn finite for for the capital X the total space as finite measures of mu of E will also be finite. The average, so what is the average? Well, you integrate the function over E and then divide by the total mass of E or the measure of E. So, this is what we mean by an average.

This is a complex number and that should be S, why do we say average? So let us look at the real line if I take an interval like a, b and I integrate a to be ft dt. So this is the integral of f but when I say average I will be dividing by the length of the interval. So that these are the averages you know. So this is the abstraction of the concrete things you have seen before. So averages of g are inside S, then conclusions is then g of x belongs to S almost everywhere.

So let us I just in some tutorial form. So this is my S. So I am I am in the complex plane. So I have some closed set S and all the averages are inside. Then g itself takes values inside. Remember g is a function from X to the complex plane. And the set S is a subset of the complex plane. So if all the averages fall inside S, then g itself takes values inside S almost everywhere of course g may take one value here, which can be on a set of measure 0 etc etc.

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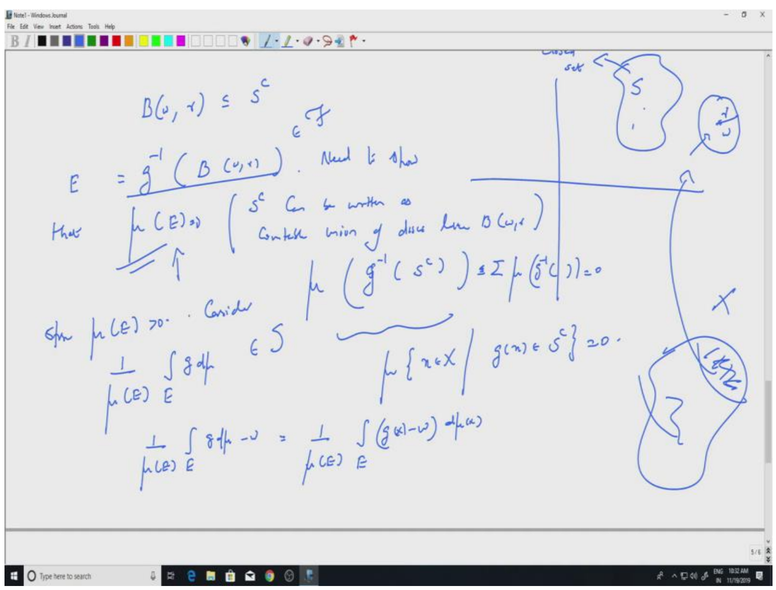


$\Rightarrow$ E

$\int_E w \, d\mu = 0 \iff \mu(E) = 0$

(II) $(X, \mathcal{F}, \mu)$ μ – the finite $\mu(X) < \infty$

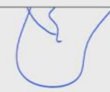
Suppose g is cpx valued msble f^n on X such that the "averages of g" are in a closed set $S \subseteq \mathbb{C}$. That is $\forall E \in \mathcal{F}$ with $\mu(E) > 0$ $\frac{1}{\mu(E)} \int_E g \, d\mu \in S$. Then

$g(x) \in S$ a.e (μ)

$g : X \longrightarrow \mathbb{C}$

$S \subseteq \mathbb{C}$ (a, b) $\frac{1}{b-a} \int_a^b f(t) \, dt$

$\frac{1}{\mu(E)} \int_E g \, d\mu - w = \frac{1}{\mu(E)} \int_E (g(x) - w) \, d\mu(x)$

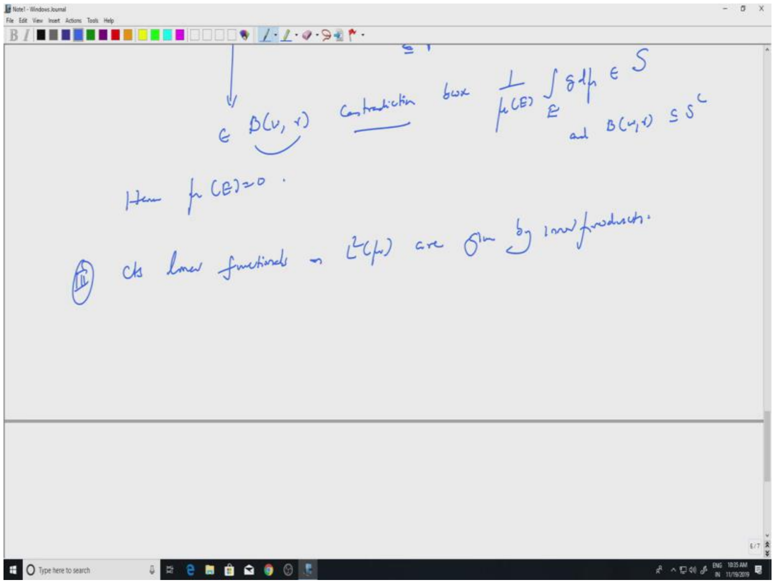


Hence $\left| \frac{1}{\mu(E)} \int_E g \, d\mu - w \right| \leq \frac{1}{\mu(E)} \int_E \underbrace{|g(x) - w|}_{\leq r} \, d\mu(x)$

$\leq r$

$\in B(w, r)$ Contradiction b/w $\frac{1}{\mu(E)} \int_E g \, d\mu \in S$ and $B(w, r) \subseteq S^c$

Hence $\mu(E) = 0$.



So let us try to prove this. So maybe I draw picture again. So let us say this is my S this is it this is close set close set. So let us take some w outside, so w is not in S. So I can put a ball around w, which does not intersect S. So let us call that r, so I look at ball around w of radius r that is in S complement.

And look at g inverse of B w r. So let us call this set E. What do I want to show? I need to show that need to show that E has measure zero of course, this is measurable because g is measurable. So when I look at the inverse image of open set in the complex plane. I will get a set in script F. We need to show that mu E is 0, why? Because if I show this S compliment can be written as can be written as countable union of this like this like this like B w r.

So for each of them, I know it is inverse image has measure 0, so it will follow that if I look at g inverse of s complement, and take the measure mu. So that would be some of mu of g inverse of or less than or equal to if you like of certain balls, which have measure zero. So we if we prove this part then these things will have measure 0, so this would be 0. What does this mean? So this is, so if I write (())(25:00) expanded. So this is all those points x in X such that X is in g inverse of S complement that the same as g of X is in S complement.

This has measures 0. So if X is this all those points which go outside S that has measure 0, so the points here will go inside s as and this has measure 0 which means that g takes values inside as almost everywhere. So this is just two lines proof, it is very easy. So let us try to prove that. So we start with we start with w outside and look at g inverse of B w r.

Now suppose, so I want to show that mu of E is 0. So suppose mu of E is positive. It is finite, so consider the average consider the average integral 1 by mu E integral over E g d mu, of course, I know that this has to belong to S because all the averages belong to S. But let us look at the distance of this from w.

So if I look at 1 by mu E integral over E g d mu minus w, this I can write as 1 by mu E integral over E g x minus w d mu x that is because w is a constant that will come out and d mu E will give mu E mu E will get cancel, that is why it is called the average. So take modulus hence, so we last line in the proof modulus of 1 by mu E integral over E g d mu so this is some complex number minus w is less than equal to 1 by mu E integral over E modulus of g x minus w d mu x mu is positive measure so this makes sense.

Now on E what happens to, so what is E? E is the inverse image of B w r the ball, which is outside outside S. So g of x would be here. So if x is in E g of x would be inside this ball. This is my set E. Which means g of X and w are at a distance that most r, so this is less than or equal to R. So that R is a constant I can take it outside. I will have mu E mu E cancelling each other.

I have so I have another I have a complex number here whose distance from w is less than or equal to r. So then that complex number will be in this ball because that is the ball of radius r centred at w, but that is not possible because the average should be in S. So this will be in a ball of radius r centred at r is a contradiction, contradiction because the average is supposed to be inside average is supposed to be inside S and this and B w r is inside S compliment, so that is not possible.

So that is the contradiction and the contradiction comes because I can make this average by assuming mu E is positive. So hence mu E is 0 hence mu of E equal to 0. So that proves the second assertion. So second assertion was that, if I have averages inside a closed set in the in the complex plane, then g itself will take values in that close that set.

So we will see this of averages of g is let us say between 0 and 1 then g has to take values between 0 and 1 and things like that we will use. And the third one is of course continuous linear functionals on L2 of mu. So these three results we will use. So third one continuous linear functionals on functionals on L2 of mu are given by inner products.

So we have already proved this given by inner products. So we will stop here. So we just proved some auxiliary results which will be used in the proof of the Radon Nikodym theorem and the Lebesgue decomposition. So the three results which we need are one is if you have a sigma finite positive measure, you can get a function w in L1, which is strictly positive and the measure defined by that w, will have the same sets which are the same 0 sets as the sigma finite positive sigma finite major mu, that is the first thing.

So that is essentially shifting or changing the or transferring the positive sigma finite measure to a finite measure without changing the sets of measure 0. That is the a that is the usefulness of the first result. Second one says that the averages of a function belongs to some closed set then the function should itself be in that closed set almost everywhere. Third one is the continuous linear functional on L2 of mu are given by inner product. So these three will use in the proof in the next lecture. So we will stop here.