Measure Theory
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Lecture 52
Radon Nikodym Theorem II

Let us start, in this session our aim is to prove the Radon Nikodym theorem and Lebesgue decomposition. As I had mentioned earlier this is a beautiful proof due to John (())(00:43) which prove both the Lebesgue decomposition and Radon Nikodym theorem in one go.

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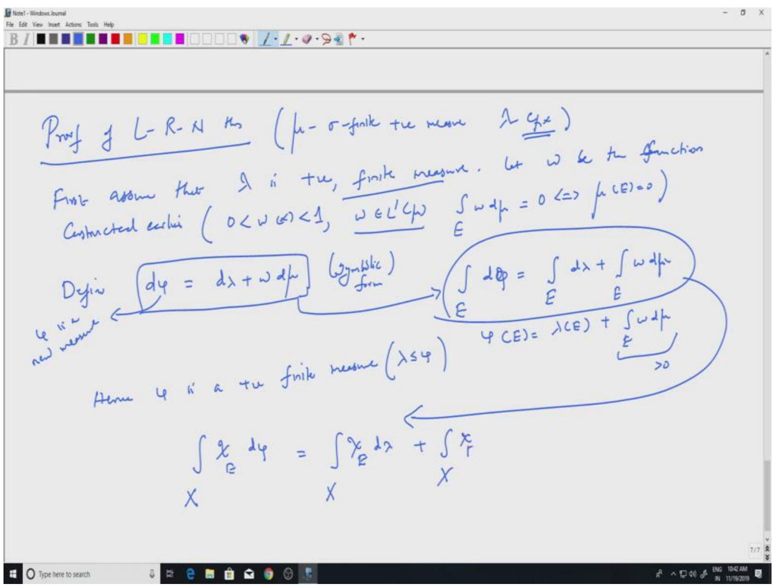

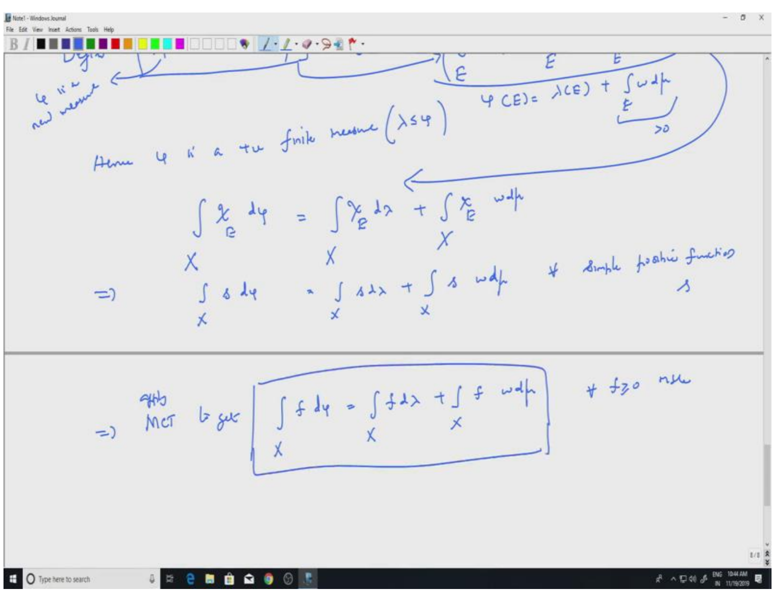

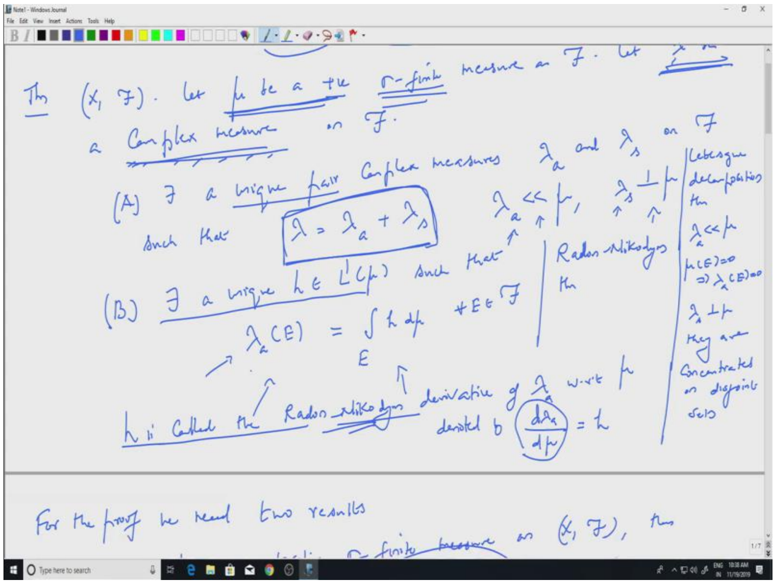

So, let us start, so proof of the Lebesgue Radon Nikodym theorem. So let us recall the statement of the theorem, So it had two parts, so given a positive sigma finite measure both are important and a complex measure lambda is a complex measure I have first decomposition. So that is the absolute continues part and the singular part and the absolute continues path is given by a Radon Nikodym derivative.

So this two thing we have to proof. So in the proof, so we let us let us write it down mu is the sigma finite positive measure positive measure and lambda is complex. And we need to decompose lambda with respect to mu and also find the Radon Nikodym derivative for the absolutely continues part.

So first assume that lambda is positive, so lambda is positive since it is complex measure it also finite. So it is a positive finite measure. So this kind of reduction you have seen several times when we dealt with function you can always function is positive proof it and then then proof it for real valued function and then for complex valued function that is preciously what we are going to do.

Now, so let w be the function constructed earlier, the function constructed earlier, what was the property of this w? Well, this was strictly between 0 and 1. And w was in L1 of mu more importantly if I integrate w over E I get 0, that will happen if and only if mu of E is 0. So this was the property we needed. Now using this w we define another measure. So define, so first we will write in the symbolic form, so d fi equal to d lambda plus w d mu. So this is in the symbolic form, so I will explain what does it means in just the moment

It should be in whatever is written here and it should obviously suggest something, so what does this suggest? This suggest that if I integrate over E with respect to d fi, then should I get it to integral over E with respect to this measure on the right hand side. But that is the d lambda plus integral over E w d mu. So this is this is meaning of this identity, so I am defining a new measure new measure fi, so fi is a new measure (())(04:19). Well, what does this mean? This means that fi of E equal to lambda of E.

So lambda is my given is our given positive finite measure. We are assuming it is finite measure plus integral over E w d mu. We know that this also define some measure w is in L1, so that also a finite measure. So hence, I am adding two finite equation and hence fi is a positive finite measure but fi is fi dominates lambda because this is positive, so this is positive and so when I, I am adding positive things to lambda.

So fi is so lambda is actually less than or equal to fi at any expect is going to be true. So from this from this, so let me write this in slightly more familiar fashion, so I can write this as indicator of E d fi. So it is not theta it is fi that is a left hand side, right hand side is integral over X kai E d lambda and integral over X kai E w d mu, so w d mu you can view as one measure if u like.

So if something is true for indicator functions, we will immediately get that it is true for simple function. So this immediately implies integral over X s d fi equal to integral over X s d lambda plus integral over X s w d mu for every simple function simple positive function f positive function s. So it a positive simple function.

But then you can apply monotone convergence theorem. So MCT apply apply MCT. So we will get to get what you do, you take any positive function choose Sn simple function increasing to it and you will have convergence every were. On the left hand side you will have convergence to f d fi, right hand side because d lambda is a is a measure, so that will converge to f d lambda plus this w d mu another measure and so apply monotone convergence theorem there as well so you will get f t. So every f positive measurable of course. Alright, so keep this in mind this is something which we have to use several times. So that is the relations relation between fi and so may be I can write it down here again d fi is written as d lambda plus w d mu.

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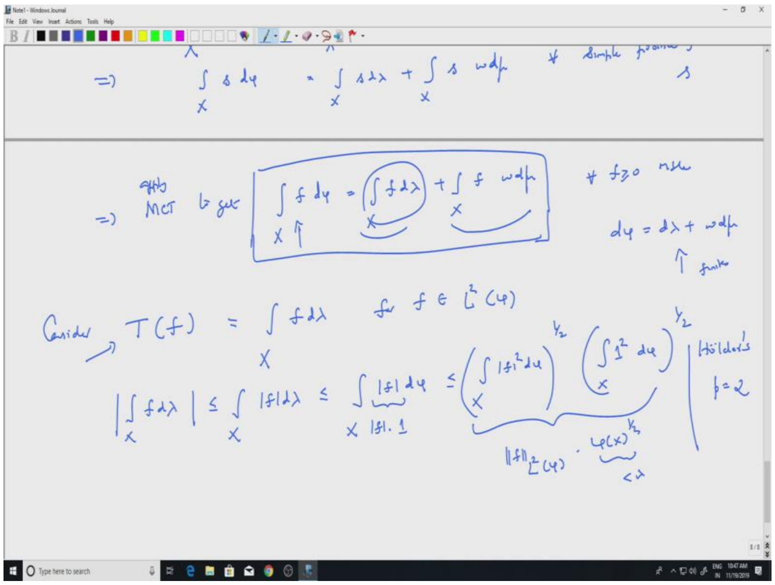

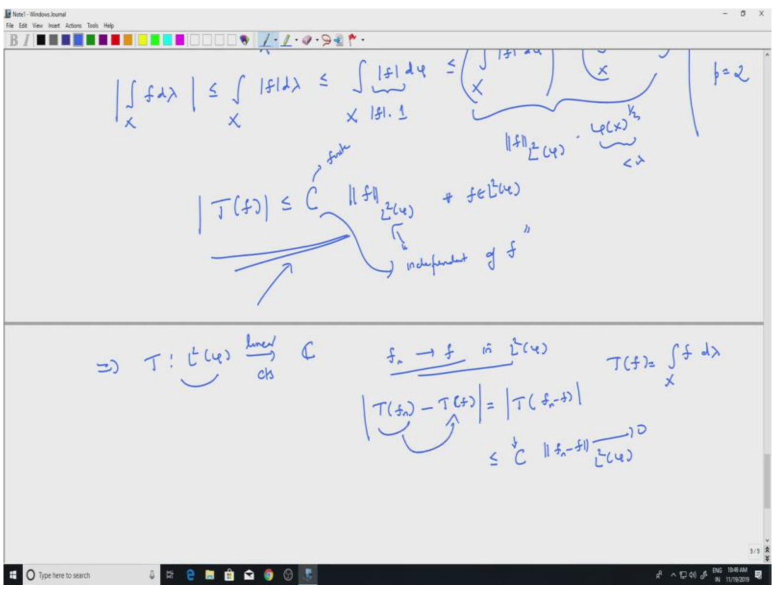

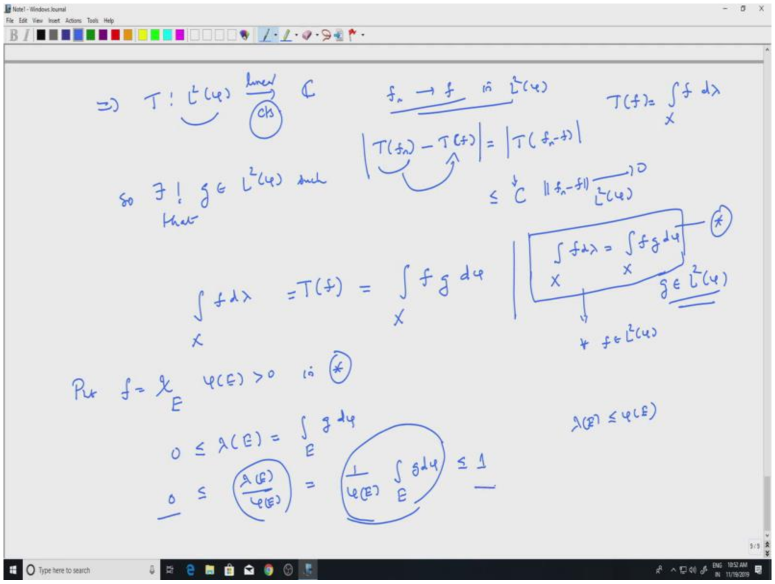

So next, consider T of f, so T of f so you have seen that this is going to be linear functional, I should give you a definition. So this is integral over X f d lambda for f in L2 of fi. So remember the measure fi lambda and mu, now there are three measures mu is the positive given positive sigma finite measure, we transferred it to w d mu so that that is a finite measure. Lambda is assumed to be positive and finite. So fi is something which dominates both lambda and w d mu. So I take fn L2 fi and I define integral over X f d lambda as Tf. I do not know if it is finite etc we will prove that.

Let us let us do that. So integral over X f d lambda, take the modulus inside mod f d lambda I want to say this is finite. This is of course less than or equal to integral over X mod f d fi because if f is positive, so mod f is what is here so that is a positive quantity. This is a positive quantity. So you are adding a positive quantity. So this this guy here will be greater than the integral of integral with respect to lambda.

Now, fi is a positive measure, so I can apply Holder's inequality or Cauchy Schwarz inequality to get mod f square d fi to the half integral over x. So I am taking mod f to mod f times the constant function 1. So 1 square w d mu sorry d d fi because I am applying Cauchy-Schwarz inequality to d fi to the half. So that is the Cauchy Schwarz inequality or Holder's inequality Holder's with p equal to 2.

So, the right hand side, so what happens so this? Well, this is the L2 norm of f. So with respect to the fi measure times 1 square is 1, so I have the total measure of x to the half, but this is a finite quantity, so finite measure because you are adding two finite measures. So here, so this is a finite measure.

So, what did we prove? We prove that the expression for Tf, which is the integral over X f d lambda. So modulus of Tf is less than or equal to some finite constant C. So this is a finite constant times L2 fi norm. So this is true for every f in L2 fi that is what we have proved. So that is say T is continues.

So this implies that T from L2 of fi, so it is a finite quantity, T makes sense, T is a map from L2 of fi to the complex plane. It is of course linear L, what is T? T after all is simply an integral. Tf is equal to integral over X, f with d lambda not with fi remember that it is a linear map of course and it is continuous. Why is it continuous? Because of this inequality.

Let us see why it is continuous. So if fn converges to f in the domain in L2 of fi. Then I look at Tf n minus Tf I want to say this is this goes to 0 in the complex plane. That is what continuity means but T is linear, So T of fn minus Tf is T fn minus f, but now I can apply this inequality so this is less than or equal to some constant which his finite it has nothing to do with f. So the constant here is independent of independent of f that is important.

So I have some constant here which is independent of f times fn minus f the L2 fi norm. That is what we have here. But this is what which goes to 0 because fn converges to fn L2 fi that means T fn converges to Tf. So T is continuous, so continue three path is true. So now we can use our result. What is the result if I have a continuous linear map on some L2 space, then that is given by an inner part. So there exist a unique g in L2 of fi, such that T of f equal to integral over X f, instead of g bar, I will simply write g d fi that is what we have.

So you can think of g bar as the function so that g bar bar is g. So, we will have this but what is Tf? So recall that, that is integral over X f with respect to lambda no, not fi. So now, let us not bother about what in the middle. So we will simply write in the so what we have is integral over X f d lambda is equal to integral over X f g d fi, what is g? g is the unique function in L2 of fi which defines the linear functional on the left hand side.

So remember, this equality, we will need we will that again. So let us start with so we need to find understand what is g a little bit more. So put out f equal to indicator function of E, with fi of

E fi of E is measure positive fi is a measure. I am looking at set E such that fi of E is positive and use this.

So let us call this star. So in star you put f equal to kai that is allowed because that is in L2. So this is true for every f in L2 fi, g is a unique function in L2 fi. So if I put f equal to kai E which is in L2, then I will get, what do I get on the left hand side? That is Lambda of E on the right hand side I will get integral over E because it is kai E, f is chi E, g d fi, g d fi. So which is positive of course, so this is greater than or equal to 0.

So if I divide by fi E, so 0 less than or equal to lambda E by fi E equal to 1 by fi E integral over E g d fi. These are averages of g with respect to the measure fi, which is finite, but what do I know about lambda E by fi that is less than or equal to 1, lambda E is less than or equal to fi E fi E is bigger than lambda. So this is less than or equal to 1, the question and so the averages of the function g are between 0 and 1. So now, you can recall what we have done earlier.

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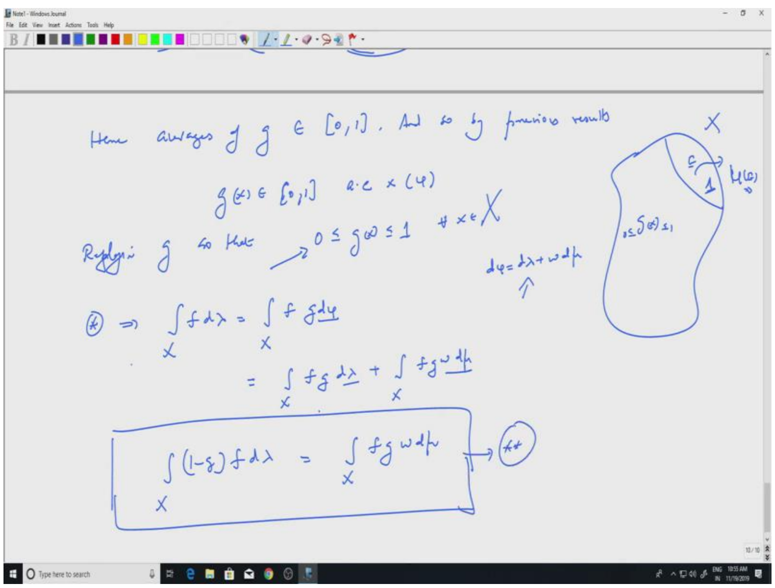

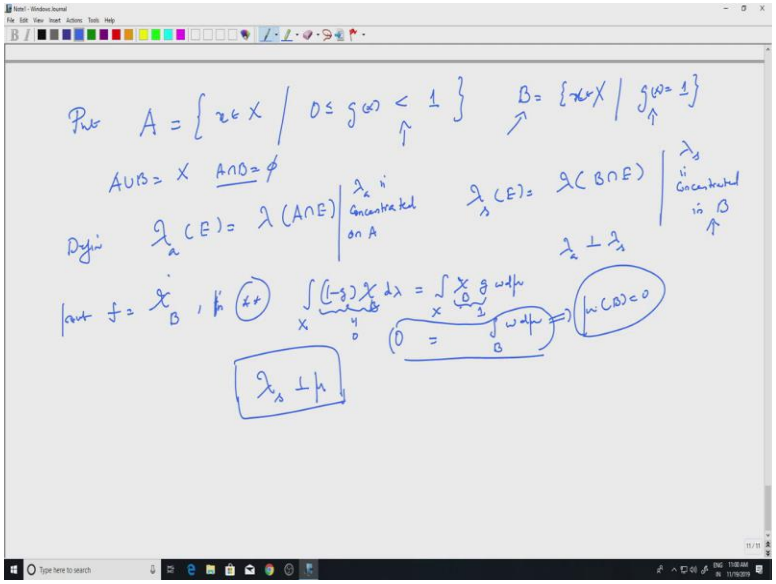

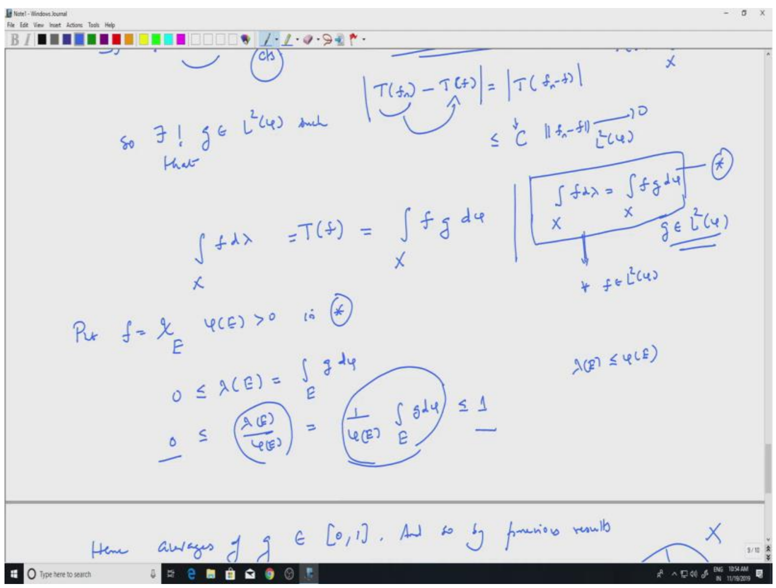

So hence averages of g belong to the closed interval 0, 1 which is a closed set in the complex plane and so by our results and so by previous results, previous results g itself takes values inside 0, 1. So g of x is in 0, 1 almost everywhere x with respect to the measure fi remember g is in L2 fi, so I have I have the space x and I have some set here and g, g of x here is between 0 and 1. This set has measure 0 with respect to fi.

So if I call E, this measure 0 and outside that set it has it is between 0 and 1. So here I will simply put g to be 1 or 0, 1 constant. So remember the space may not be complete. So you cannot define g to be arbitrarily there. So redefine g, so that 0 less than 1 to g of x less than equal to 1 for every x in X. So I can define g to be just one here and then we are done.

So we assume it for all x. Now let us go back to, so we have some information on g. So now let us use this again, star so what does star say? Star says that integral over x f d d lambda. That was our Tf that is equal to integral over X f g d fi, g is the function in L2 fi, but d fi d fi what was d fi d fi is d lambda plus w d mu.

So I put that in, so this is equal to integral over X f times g d lambda plus integral over X f g w d. So I am just using this summation. Now, there are lambda integrals on the left hand side and the right hand side. So bring it together and recall that g is between 0 and 1. So I will get integral over X 1 minus g f d lambda equal to integral over X f g w d mu. So this is another identity which we will use in an again.

Let us call that star star, two stars. So now, comes the magic of (())(18:32). So here he decompose the space, so put put the set A to be all those points X. So now everything is controlled by g, g is going to be this g g will give us a Radon Nikodym derivative and the decomposition. So all those points X, such that 0 less than to g of x strictly less than 1. So pay attention to the strict inequality and B equal to the set where g is actually equal to 1.

So remember we define g to be 1 on a set of measure 0, so it does not really the A and B will differ a whatever definition you take A and B will differ at most by sets of measure 0, so it is not going to change anything in the integrals. So you have A and B, so what do you know about A and B? A union B is the whole space A in the section B is empty.

Now, define, so this gives you the first result the define lambda sub A of E, so I am trying to give you the expression for the absolute continues part of lambda. So that is lambda that is where g is strictly less than 1. So that is A in the intersection E and I have the singular part where g is 1. This is lambda of B intersection E. So lambda A is concentrated on so this this tells me lambda A is concentrated on A and lambda s is concentrated in B. So the definitions itself says that there concentrated on A and B, but A and B disjoint.

So that immediately implies that they are mutually orthogonal to each other, I mean mutually singular. So lambda A is singular with respect to lambda s. So now if I take f equal to indicator function of B. So what is B? B is the place where g is equal to 1 and put this in, so put f equal to kai B in star star, star star is this. So if I put f equal to kai B, what do I get? So let me write one line integral over x 1 minus g kaichi B d lambda equal to integral over X kai B g w d mu.

What does this say? This says that, what is B? B is where g is 1. So g is 1, so 1 minus g times kai B this is equal to 0. So the left hand side is 0, right hand side is on B g is 1, so kai B times g is 1. So I have integral over kai B times g is kai B. So integral over B w d mu. So with respect to w d mu, the set B has measure 0, but then we know that the, this immediately implies mu of B is also 0, that was one of the results we mentioned when w was constructed, w d mu and d mu will have same 0 sets.

So mu be is 0, what does that mean? Lambda S is concentrated in B and mu of B is 0. So that means these two are mutually singular. So lambda S is mutually singular to mu, so this is one of the things and we need to show that lambda A is absolutely continuous with respect to with respect to mu. So that is the last part of the proof

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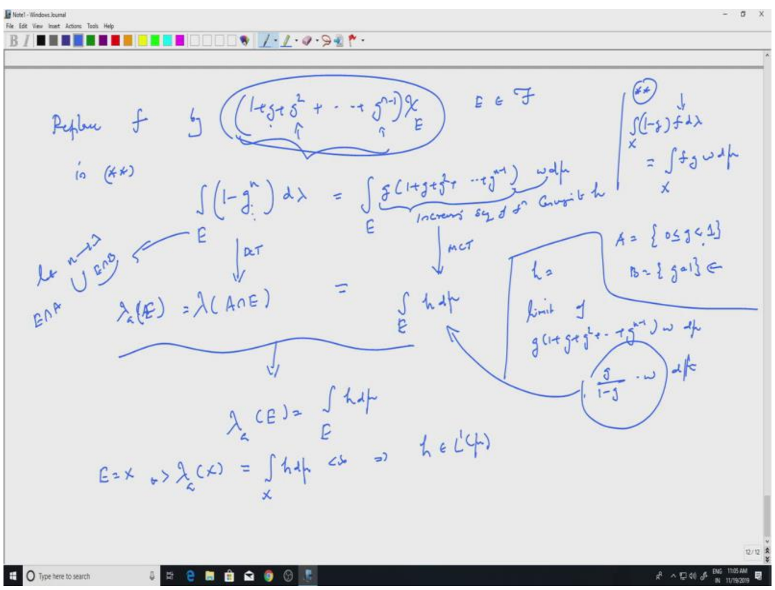

So this so this is another new idea will come here. So replace f by 1 plus g plus g square plus etc etc plus g to the n times kai E. Well, what is E is set F, in so we used star star for that. So in star star, star star I will write down again, so did it is clear in star star, so what is star star? So star star let us denote write it down here again. So that was integral over X 1 minus g f d lambda equal to integral over X f g w d mu.

So I am replacing f by this, so that is allowed because this is true for all f and L2, L2 of fi, fi is a finite measure g is bounded, so g is less than equal to 1. So all these quantities are less than all these functions are less than or equal to 1, so it will add up to some finite bounded function. So I can replace f by this, well, what will I get then? I have the whole summation here times 1 minus g that will give me 1 minus g to the n, let us take n minus 1 here, so that I get 1 minus g to the n, d lambda over the set E because there is a kai E.

So let us instead of x let E because f I am replacing f by this whole thing. So equal to on the right hand side f is replaced by, so kai E is there so integral mu will become over E f is replaced by this summation times w g d mu so g into 1 plus g plus g square plus etc etc g to the n minus 1 times w d mu. So we have this, now let n go to infinity. So let n go to infinity, if n goes to infinity what happens to the left hand side this converges to lambda of A inter section E. Why is that? Because what was A, A is the set where g is strictly less than 1 B is set where g is 1.

So I can write E as E in the section A union E intersection B. So the integral splits and on E intersection B this is 0 because g is 1 on E in the section A g is strictly less than 1. So g to the n will go to 0. So applied DCT whatever you want and this is by definition the lambda A of A in this A, sorry E lambda A of E by definition is lambda of A intersection A.

So the left hand side converges to what we want, what happens to the right hand side. This is a increasing sequence because g is positive and you are adding each time, so increasing and so you multiply by w you will have an increasing sequence of functions converging to something. So converging to let us say h h will be positive and by monotone convergence theorem. So by MCT this converges to integral over E h d mu.

So what is h h is the limit of g times 1 plus g g square etc g to the n minus 1 times w, so I can write that explicitly if you want h equal to limit of g into 1 plus g plus g square plus etc etc g to the n minus 1 into w d mu. If you write this as g into g by 1 minus g times w d mu, you will see what is actually happening whenever g is one this g by 1 minus g into w is infinite. That is the singular part. That is what kept the set B captures. That is the singular part because this is what behaves like Radon Nikodym derivative.

So whenever g is 1, the Radon Nikodym derivative is infinity. So that part used to the singular part remaining gives you absolutely continuous part. So if I write this separately what we have just proved is lambda a of E is integral over E h d mu and so this this of course and if I put E equal to x, I know lambda of a of x. So this is a finite quantity because lambda means assumed to be a finite measure lambda of x is finite. So lambda of a x plus lambda S suffix is finite and this is equal to integral over h d mu. So this is finite implies h is in L1 because h is positive so h is L1. So now we had proved all that we want.

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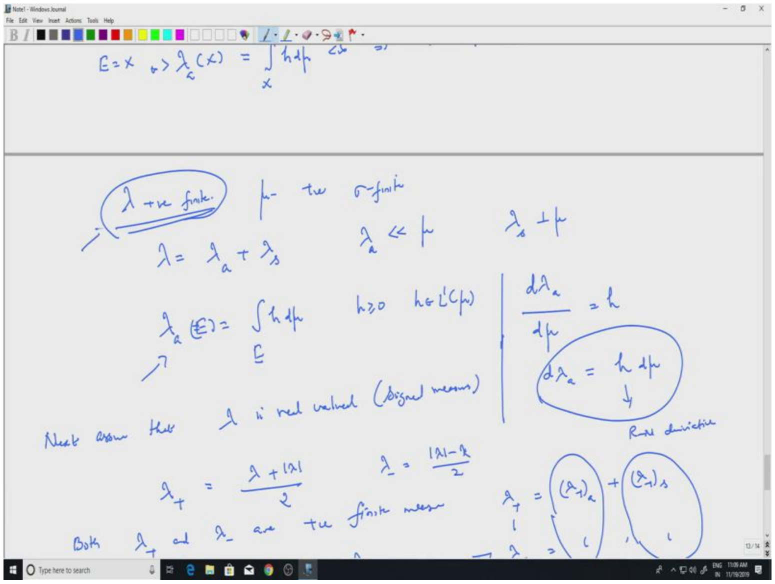

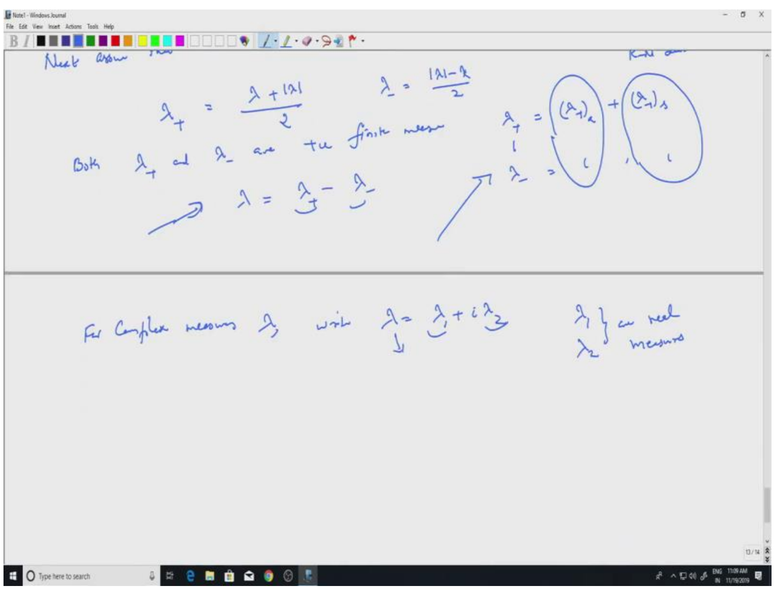

So let me let me recall that again and then we will conclude with some remarks. So what did we prove we have lambda positive finite mu positive sigma finite. Then I can write lambda equal to lambda a plus lambda s, lambda a is absolutely continuous with respect to mu lambda s is mutually with respect mutually singular with respect to mu.

And lambda A can be written as so lambda a of E can be written as integral over E h d mu where h is a positive function because we have positive lambda and h is in L1 mu. So if you want, so in some in in symbols Radon Nikodym derivative of lambda a with respect to mu. So this is h, that is the symbol we have or in other words d lambda a, so we have already used this notation h d mu.

That is what this equality means, so sometimes we symbolically we will write like this. So h Radon Nikodym derivative. So the proof is not at complete because we assume that lambda is positive and finite. So next thing is to assume so next assume that lambda is real valued instead of complex valued, so that is we what we call sign measures. Then we can define lambda plus to be lambda plus mod lambda by 2.

We have done this before and lambda minus equal to mod lambda minus lambda divided by 2. Then both lambda plus and lambda minus are positive finite measures. So the theorem applies, so I can write lambda plus as lambda plus the absolute continues part plus lambda plus the singular part.

Similarly, lambda minus also will have, so lambda is given by that is a real measure. So that is lambda plus minus lambda minus. So when I subtract I am going to get the difference of two absolutely continuous measures. Which is also absolutely continues and difference of two mutually singular measures with respect to mu, so that will still be mutually singular with respect to mu. So for real ones, the result will follow from positive ones and for complex ones it will follow from real ones.

For complex lambda for complex measures lambda write lambda equal to lambda 1 plus i lambda 2, where lambda 1 and lambda 2 are real measure a real valued measures. And for this we have decomposition for lambda 2 we have decomposition. So just like what we did here we will get a decomposition for Lambda as well, so that part is easy. So going from positive finite to complex measures is like what we do with functions for positive functions you go to real valued functions and then from there to complex valued function.

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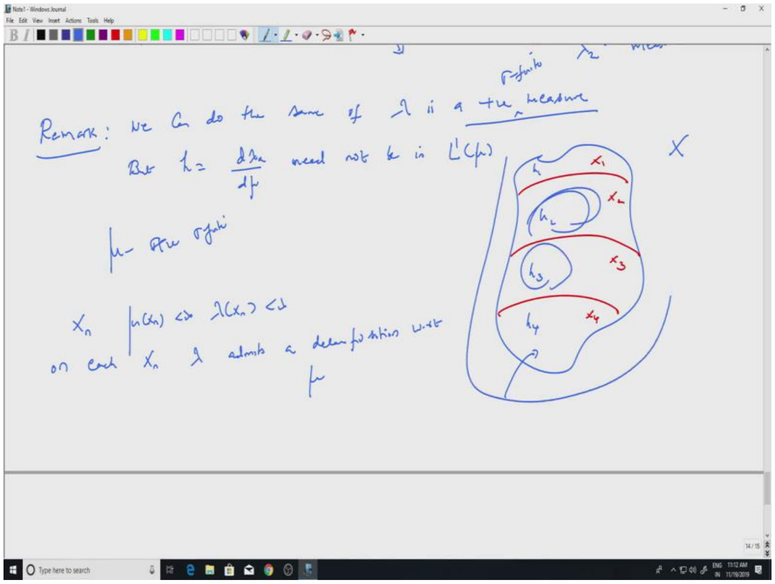

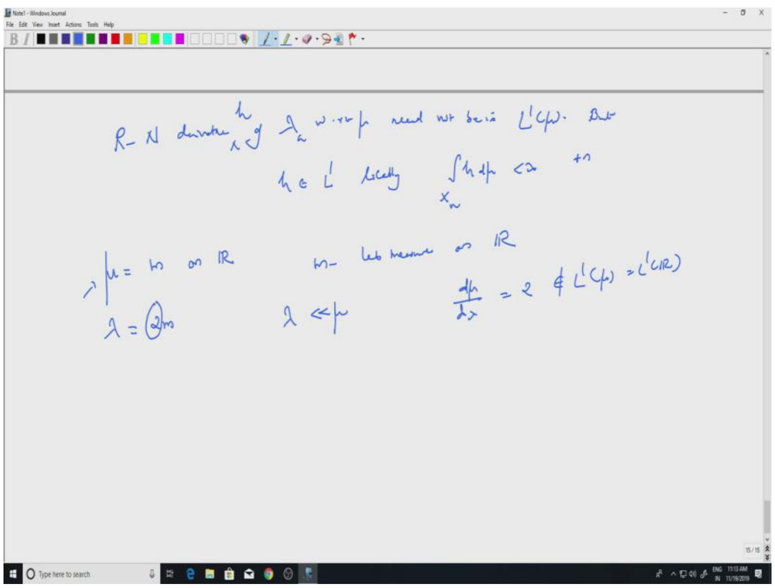

So one remark, we can do the same if lambda is a positive measure positive measure, may be sigma finite. Let us let us puts signal finite sigma finite measure, that means but Radon Nikodym derivative, but h equal to let us say d lambda by d mu need not be in L1 need not be L1 of mu. So what do I mean by this? So let us let us take two sigma finite measures and we so we have X as our space.

Now mu is sigma finite mu is the given positive sigma finite measure. So that decomposes X into various pieces. So x1 I have x2 I have x3 x4 etc. Now in each of them I can so I can restrict everything to Xn. So if I take Xn I have mu of Xn finite and lambda of Xn finite. So I can decompose it because both are sigma finite measures. So I can decompose it X such that both of them are finite and so in each Xn I will have a decomposition.

So on each Xn lambda admits decomposition decomposition with respect to mu and then I can add up adding up will give me decomposition of lambda with respect to mu only thing is you will have the Radon Nikodym derivative. We will have h4, h3, h2, h1 etc. they are all in L1 in in those spaces, but over the whole space X it may not be in L1. So the Radon Nikodym derivative.

So the R-N derivative of derivative of lambda or lambda a with respect to mu need not be in need not be in L1 of mu but it will be in L1 Radon Nikodym derivative let us call that h but h will be in L1 locally. What does that mean? Integral over h h d mu will be finite for every n for every n, so in each of these components you have a L1's but put together it will not be. So let us see a trivial example to see why this is true.

So let us take mu to be in the Lebesgue measure on R and lambda is the positive sigma finite measure. So I will take it to be two times Lebesgue measure, so m m is the Lebesgue measure on on R. So, of course lambda is absolutely continuous with respect to mu, whenever mu of E equal to 0 lambda is 2 times mu, so obviously lambda of E is also 0.

So restricted to any compact set I should be able to get a Radon Nikodym derivative, which is the constant function 2. So d mu by d lambda if you look at is actually 2, which is not in L1 of mu not in because L1 of mu is L1 of R R has finite say infinite measure. So constant functions are not there, but if I restricted to any set which has finite measure it will be there.

So this completes the proof of the Radon Nikodym derivative theorem. We will stop here. So we just saw the proof of Radon Nikodym theorem, first we did prove we did the proof for positive finite measures lambda and as usual we extend it to the real measure and complex measure and the same proof works for 2 sigma finite positive measures mu and lambda and you can get Radon Nikodym derivative which need not be in L1 globally but it is L1 locally.

That what you should understand. Now from next session onwards we will see various consequences of the Radon Nikodym theorem there quite lot of interesting consequences some of them we will see that. We will stop here.