Measures Theory
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Lecture no. 07
Monotone Convergence Theorem and Fatous Lemma

So far we have seen simple functions, measurable functions, and we have defined integrals for positive simple functions and taking appropriate supremum we defined integrals of positive measurable functions. And we looked at some properties like monotonicity and things like that. And if you integrate over a set of measure zero, you will get zero even if the function is infinity at that set. And if the function is zero you integrate you will get zero, even if the set has infinite measure.

So, today will prove one or two important theorems and integration, which will allow us to interchange the integrals and limits very easily. So this is one of the advantages of Lebasque theory of integration over Riemann integration. In Riemann integration we needed a sequence of functions to converge uniformly for us to interchange the limit and integrity. That is not necessary in the case of Lebesgue integral. So we will see that today.

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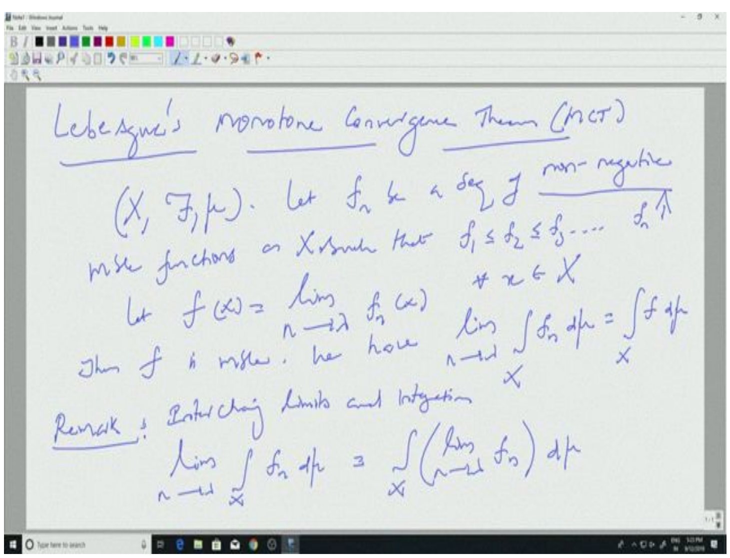

So the first theorem we want to prove is Lebesgue, Lebesgue monotone convergence theorem. So we will call this MCT for monotone convergence theorem. Let me write this in full detail. So, as usual we have a triple X, F, mu. X is any space F is a sigma algebra of subsets of X. Mu is accountable additive measure okay. So, let FN be the sequence of non-negative measurable functions, measurable functions defined on X such that FNs are increasing, that means a F1 is less than to F2, less than F3 etc.

So, FNs actually increase, So, FN of X at each X would be an increasing sequence of positive numbers so, they will have to converge, they may go to infinity but they will, they will converge to something. So, let us call the limiting function F. So, let F of X to be equal to limit and going to infinity FN of X for every X in X. So this of course exists, because if FNs are increasing, and then F is of course measurable, then f is measurable, it is a limit of measurable function.

So, that is not surprising, but what is interesting is, is that we have a limit of N going to infinity, integral of X, FN, D mu. So, remember, these are all well-defined now, because FNs are non-negative measurable functions, we know how to define the integral of FN which respect to a measure, this is actually equal to integral of X, F, D mu.

So, to remark, we are interchanging limits and integrations okay. So this is interchanging limit and integration right, because we are saying limit of N going to infinity integral over X FN D mu, this is actually equal to integral over X limit and going to infinity FN. So, that is our F, right and d mu. So we are saying this can be done, if you have a sequence of increasing measurable functions okay, non-negative increasing measurable functions.

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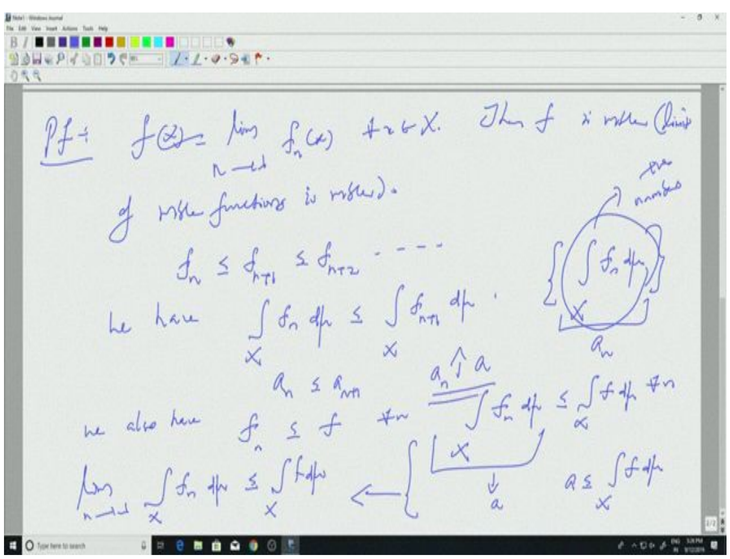

So proof, so, this is one of the most useful theorems you will see in measure theory, we have three such theorems which will allow us to interchange limits and integration we will see one more today and then when we define integration for real valued and complex valued functions, the next the third term will come okay. So, let us define again F of X to be equal to limit and going to infinity F and X right. So this is for every X and X. So then F is measurable.

Then F is measurable, because it is the limit of measurable functions, limit of measurable functions, is measurable okay. We want to compute the integral of F and see if it is the limit of integral of FX, okay? Now what do we know? We know that FNs are increasing, so FN is less than or equal to FN plus 1 less than or equal FN plus 2, etc.

Because of the monotonicity property of the integral, we have integral over X, FN, D mu is less than or equal to integral over X, FN plus 1 D mu. So, if you look at the sequence of positive numbers. So remember FNs are non-negative measurable functions. And I am integrating over X with respect to a positive measure. So I am getting these are all positive numbers, positive numbers.

Sometimes they can be infinity. But let us not worry too much about it, we have a sequence of positive numbers, let us call AN. What we know is AN is less than to AN plus 1. So AN is an increasing sequence of positive real numbers. So it will converge, it may go to infinity, or some finite number, let us say A. Now, what do we know? We have, we also have, we also have FN to be less than or equal to F for every N, right? Because F is the limit of the increase in sequence of FN.

So for each N, FN is less than or equal N, and so integral of X FN, D mu will also be less than or equal to integral of X, F D mu for every N. But this is a sequence which converges to A. So, A is less than or equal to integral of X, F D mu. So, this is one inequality right. So, what we have? We are proving is that limit of N going to infinity integral over X FN, D mu that is my A that is less than or equal to integral over X, F D mu.

Okay this much did not use anything too difficult only, only thing we needed was the monotonicity of the integral and the fact that these numbers ANs were increasing and so, we got AN increasing to A okay.

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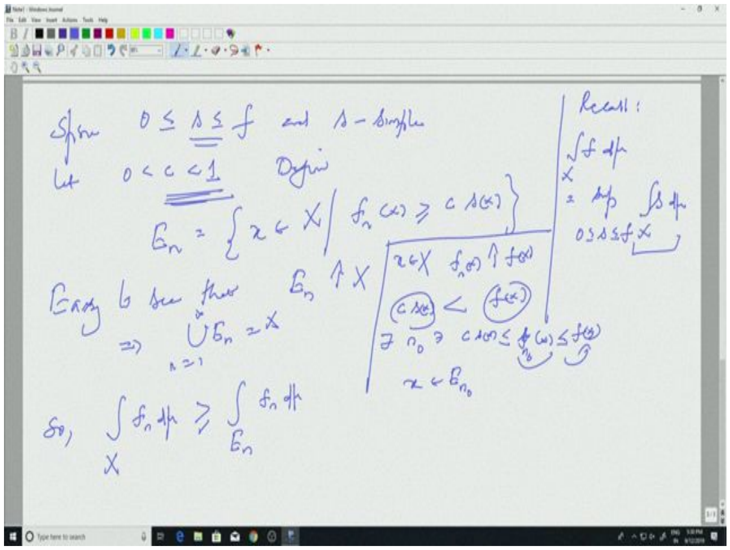

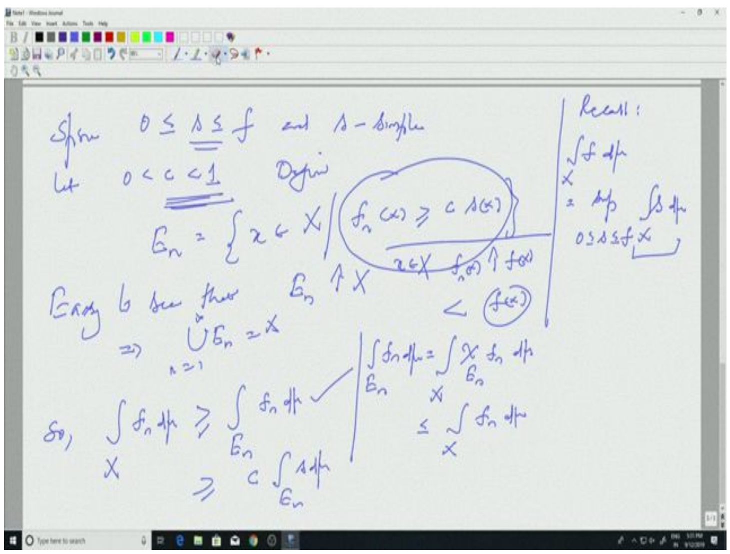

So, now, let us prove the other way. Suppose zero less than to S less than to F and S is a simple function. So recall that we defined integral of F. So that is something, let us recall immediately, they call integral of non-negative function, F is the supremum over all the simple functions less than or equal to F integral of X, SD, right. So SD mu is something which we know, we know how to define this, using the measure and the expression for S, and we take supremum over says things, we get integral of F.

So, I am going to use that now. So, take any single function which is between zero and F. Let us fix a number. C less than, strictly less than one, strictly less than one, okay. And define, define the following set EN, this is equal to all those points in the space X, such that FNX is greater than or equal to C times S of X.

So, remember S is a simple function less than or equal to F, C is something which is less than or equal to one. So C times S is something strictly less than F, and I am looking at all those X such that FNX will be greater than or equal to C times S of X. So, it is easy to see, you see to see that EN increases to the whole space X. Well, why is that? So let us Let us justify this. If I take any X in X, I know that FNX increases to F of X, right for at each X, we know this happens.

But if I look at C times S of X, this I know is strictly less than F of X. So, there X is some N naught such that C times S of X will be less than or equal to FN naught of X less than or equal to F of X right. Because the FNs converts to F. So, it has to be between this number and this number as and then becomes bigger and bigger and so, that particular X will be in EN. And so well because FNs are increasing ENs are increasing. So, this simply tells me that UN of EN is the whole space in equal to 1 to infinity. Moreover, it increases right, so UN is smaller than E2, smaller than E3 and so, okay.

So, well, how do we use this? We start with integral over X, FN D mu number, these are positive numbers increasing we know it converges. We need to show that actually converges to integral of F. Well, this is of course less than or equal to, sorry greater than or equal to integral over EN, FN D mu, we call that integral over EN, FN D mu is integral over the whole space X indicator of EN times F.

So, let us, let us recall that part right, integral over EN, FN D mu. We know by definition this is integral over X, indicator of EN, FN D mu, which is of course, less than or equal to integral over X, FN, D mu right. Because Chi, Chi EN is less than or equal to one. So, I can by monotonicity of the integral I can do so, we get this right. This is of course, greater than or equal to.

Well on EN, we have and inequality, right? So we can use that using monotonicity of the integral again, we get the constancy that will come out of the integral, integral over EN as D mu right. Because S, C times S is smaller than FN on the set EN. So we use the monotonicity of the integral again there okay.

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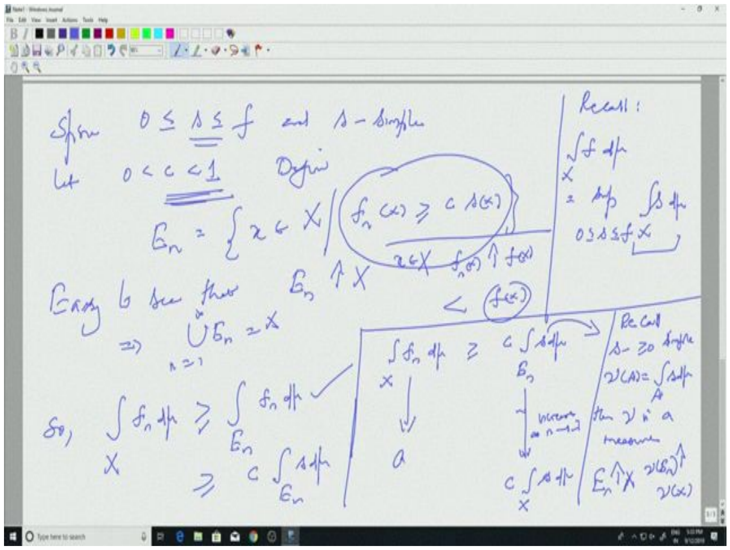

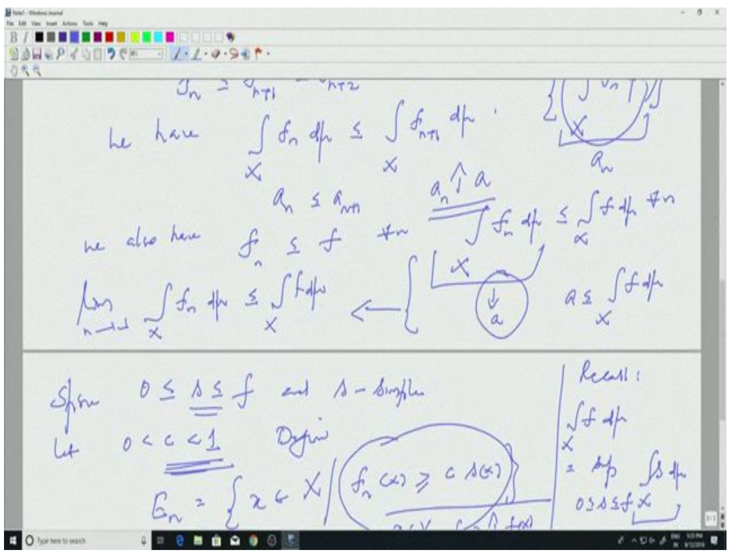

So now we are in good shape because the right hand side now. Well, what will happen to this so let us, let us write down those again? I have integral over X, FN D mu to be greater than or equal to C times integral over EN, S, D mu okay. So let us recall the result we proved in the last class. So we proved that if S is non-negative and simple. Then if I define U of A to be equal to integral in over A has D mu, then mu is a measure, we prove that okay. So this is a measure and ENs increased to the whole space, ENs increased to whole space okay.

So mu of EN, we know by the property of the measure will have to increase to mu of X. So this increases as N goes to infinity to C times integral over X, S, D. But for each and I know this is bounded by integral over X, FN D mu, right? So this is of course, less than or equal to sorry, let us put this. So, I know that this, this goes to our number A right. This is what we call A. let us see if it is here, yes. So, the A is simply the limit of these integrals.

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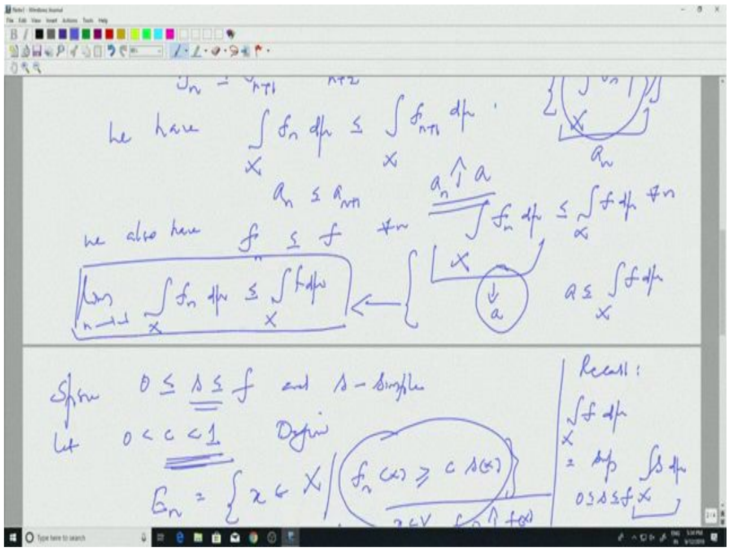

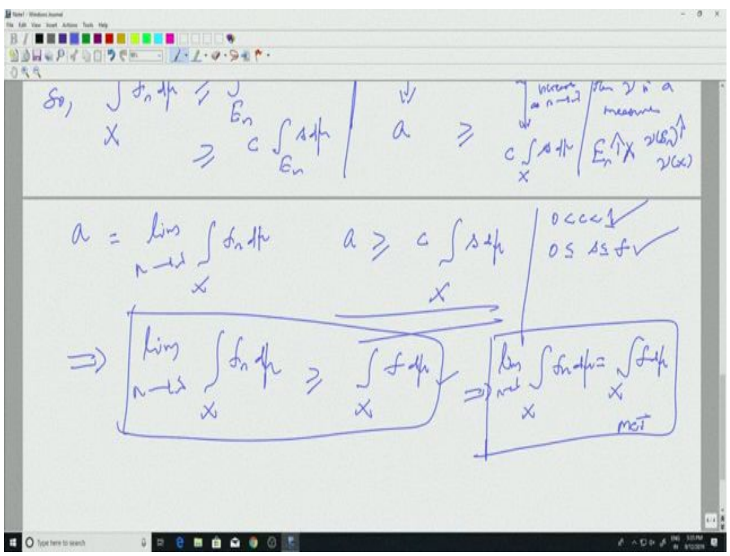

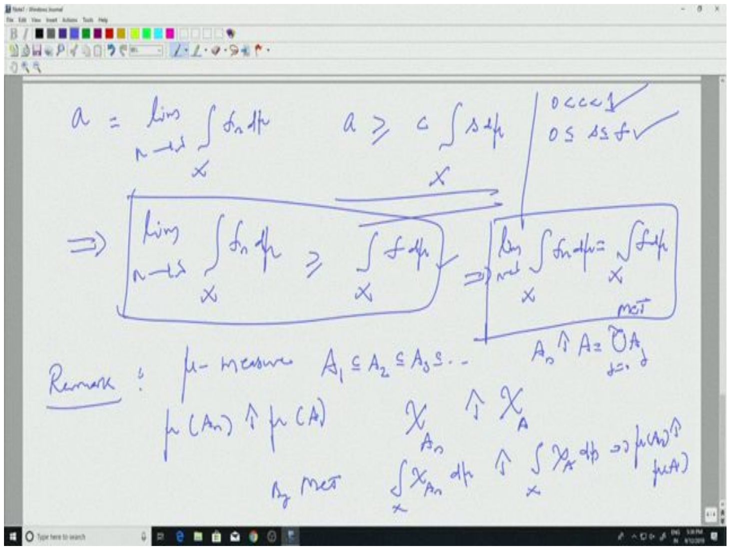

So, this goes to A. So, what we have is A greater than or equal to C times. So, let us, lets write that separately, so, that it is clear. So, let us recall A. A is simply limit of N going to infinity integral X, FN D mu, what we have just proved is that A is greater than or equal to C times integral over X, S D mu. So, what was C? C was less than one some fixed constant, what was S? S was less than or equal to F.

So, this is true for all C and all S like this right, any C between zero and one any S between zero and F, we have this inequality. Left hand side is independent of everything. So, I can take supremum over C and supremum over S less than or equal to F and I will on the right hand side I will get the integral of F. So, this implies that A which is the limit of FNs. FN D mu. I know is greater than or equal to integral over X, F D mu right.

So, this is one inequality and the other way inequality was the easy part which we did right, which uses only the monotonicity. So, we had this inequality. So, limit of FNs to be less than or equal to integral of F, and now we have limit of FNs to be greater than or equal to limit. So, this tells me that limit of N going to infinity X FN D mu is actually equal to integral of X, F D mu okay.

So, this is the monotone convergence there, if you have increasing sequence of measurable functions. Then you have convergence in the integral okay that is what monotone convergence theorem tells you, but you have seen. So, let me remark, you have already seen an instance of this when we started with a measure, so, we call. Suppose I have a measure mu and if I have sequence of measurable sets A1 smaller than A2 contained in A3 etc right.

So, this is what we call AN increasing an increases to A. What is A? That is the union AJ, J equal to infinity. In this case, you know that mu of AN increases to mu of A. This was a property, countable additivity property of mu. Well now, this is part of the monotone convergence theorem because what is mu AN, we are looking at Chi A, AN. I know AN increases to A. So Chi AN will increase to Chi A.

And by MCT, the monotone convergence theorem, we know that integral of Chi AN, D mu will increase to integral over Chi A, D mu right, over the whole space X which is say I am saying mu of AN increases to mu of A. So, you have seen an instance of the monotone convergence theorem for indicator functions okay. So, this is a much more general theorem, which is applicable to all positive measurable functions.

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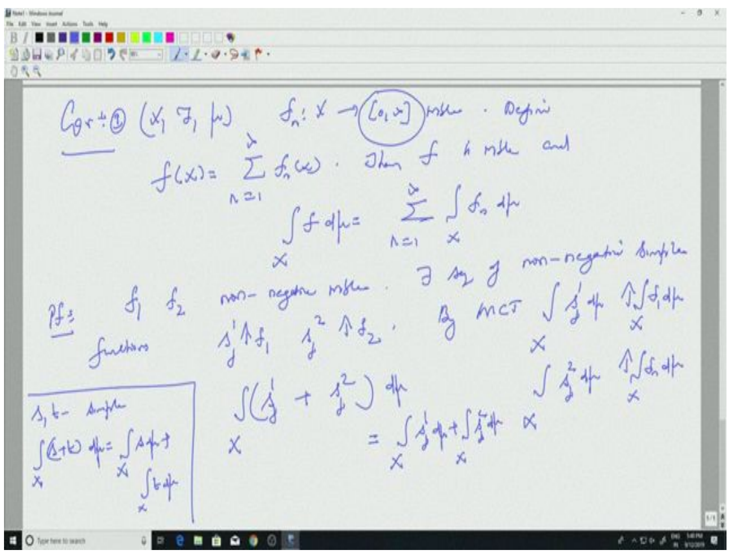

So, there are immediate corollaries to this, which are very useful and you will also see something which you have probably seen earlier. So, corollary. So, again, we have this triple and let us say we have a sequence of measurable functions to infinity measurable okay. Define F of X equal to summation FN of X in equal to infinity. So I am adding positive numbers, so it will exist it may be finite or it may be infinite okay.

Then F is measurable and the more important part is integral of F, D mu is the sum of integrals of FNs okay. So let us corollary one, let us prove this. So, positive part is important otherwise, there are things that can go wrong, one has to be careful about convergence and so. Well, so how does one prove this?

So let us start with the simpler case. Suppose I have two functions, suppose F1, F2 non-negative measurable functions, negative measurable functions. Then I know that there are sequence of, so there X is sequence of non-negative, non-negative simple functions. So, we will call them SJ1 and SJ2 okay. SJ1 will convert F1, SJ2 converges to F2. So, we know this for any non-negative measurable function, I know there is a sequence of simple functions increasing to that function.

But now by monotone convergence theorem, I know that integral of SJ1 D mu will increase to integral over F1 D mu and integral of SJ2 D mu, the second sequence will converge to the second function. So, we know this by MCT okay. But for simple functions we already know linearity. So, remember that, so, we proved that, so let us recall that if S and T are simple. Then we know that integral of S plus T, D mu equal to integral over X, S, D mu plus integral over X, T, D mu, we did this in the last class for simple functions. So we use that.

So if I look at integral over X, SJ1 plus SJ2 D mu. I know that this is equal to integral over X SJ1 D mu plus integral over X SJ2 D mu. So we can use monotone convergence theorem again.

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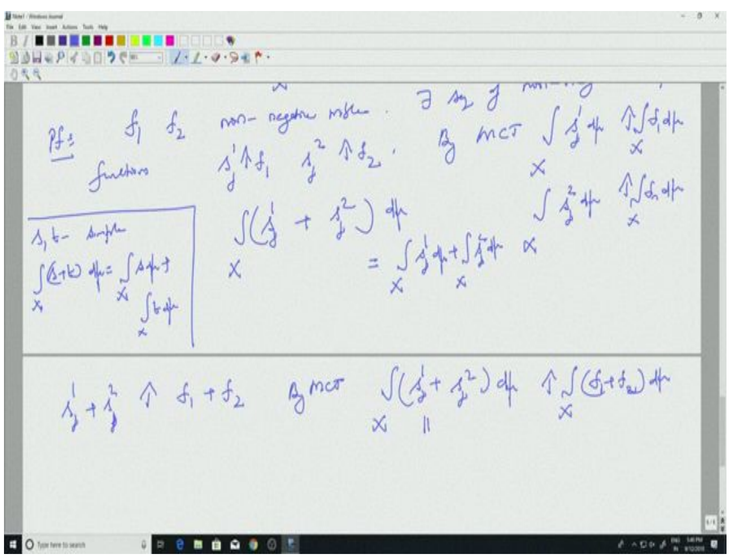

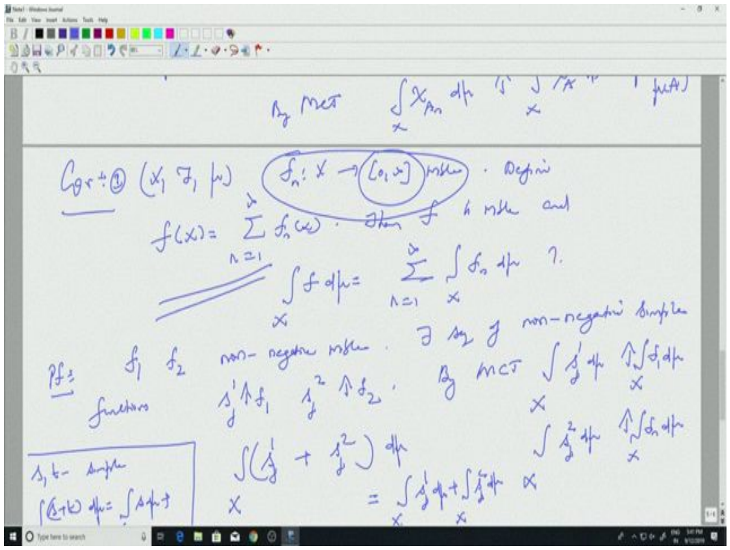

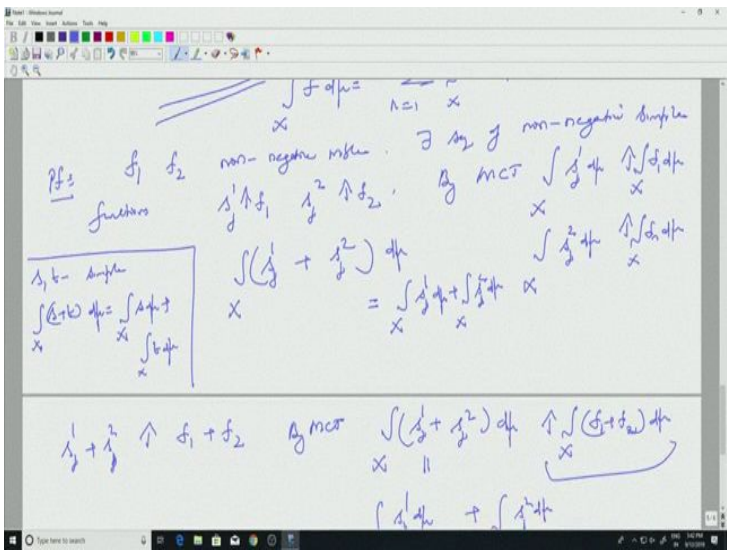

Right? Because SJ1 plus SJ2 will increase to F1 plus F2. So by MCT, we have integral over X SJ1 plus integral over sorry, SJ2 D mu will increase to integral over X, F1 plus F2, this is by MCT. But I know that this is equal to because it is linear on simple functions. This is simply the sum of two things plus integral over X SJ2 D mu. And I know that this increases to integral over X, F1, D mu plus integral over X, F2 D mu.

So, remember we started this also converges to this. So, we have what we have proved is that linearity of integral over positive function. So, whenever you have positive functions measurable, you can add them up. So, of course, if you have two, three functions of, you will get three pieces and so on right. Finitely many functions you can always do this.

So, now, let us get back to the proof of, proof of this. So, we are looking at countably many simple, measurable positive measurable functions. We are, we are looking at F of X equal to the sum of these things and we want to say we want to prove this right, this is what we want okay. So, we use monotone convergence theorem again okay.

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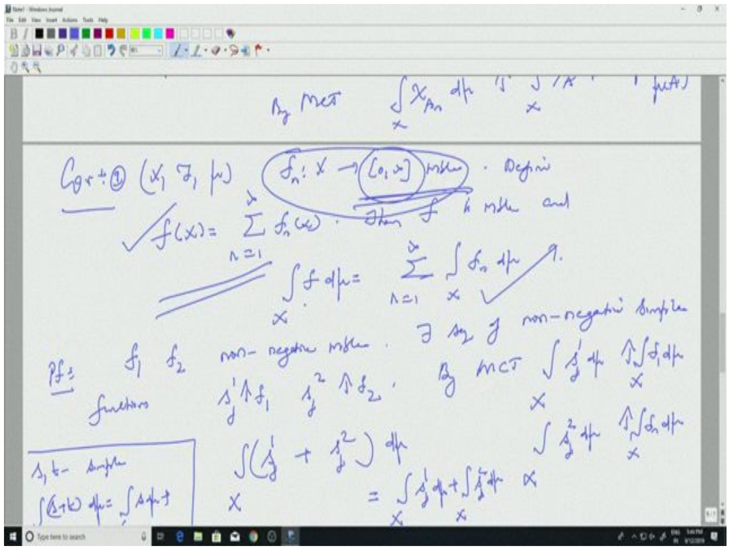

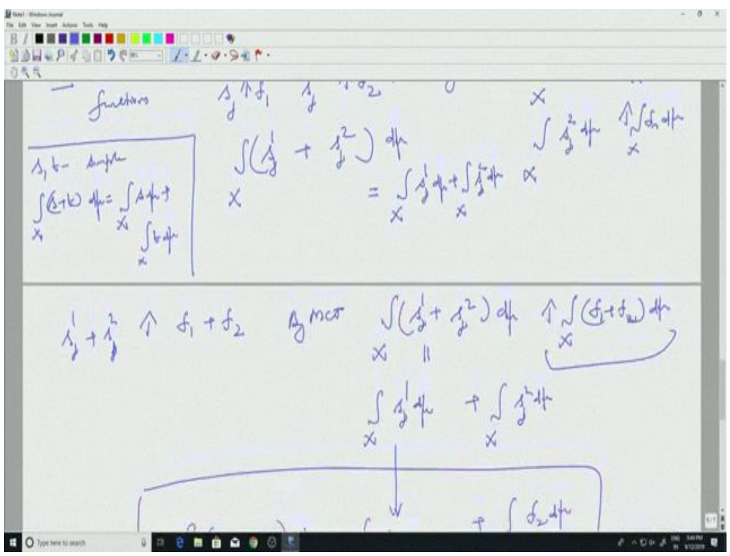

So, we use monotone convergence theorem again okay? So recall we have F1, F2, F3, etc right? This is what we call F. So let us take the partial sums, so let us call that GN. GN of X to be F1X plus F2X plus F3X plus etc, etc plus FNX right. Then what we know is FJs are all positive. So GNs are increasing right? Where will they increase to? They will increase to the whole sum which is F. So GNs increased to F. So, by MCT, we will have integral of GN D mu will increase to integral over X, F, D mu.

But what is this GN is finite sum, we just proved that if I have two functions, I know how to the linearity of the integral or two functions we know. So, similarly, for finite, finite sum we have the same result. So, this one would be simply summation J equal to one to capital N integral over X, FJ, D mu right. But as N goes to infinity what will happen to this? So, as N goes to infinity this goes to the sum J equal to one to infinity integral over X, FJD right.

So, this and this will have to be same. So, that is what precisely or statement of the proof, statement of the theorem right. If I have F to be the sum of FNs, then the integral of F is the sum of the integral of FNs, right. Remember, they are all positive measurable functions when you apply monotone convergence theorem, you have to be careful, you apply it to increasing sequence of functions.

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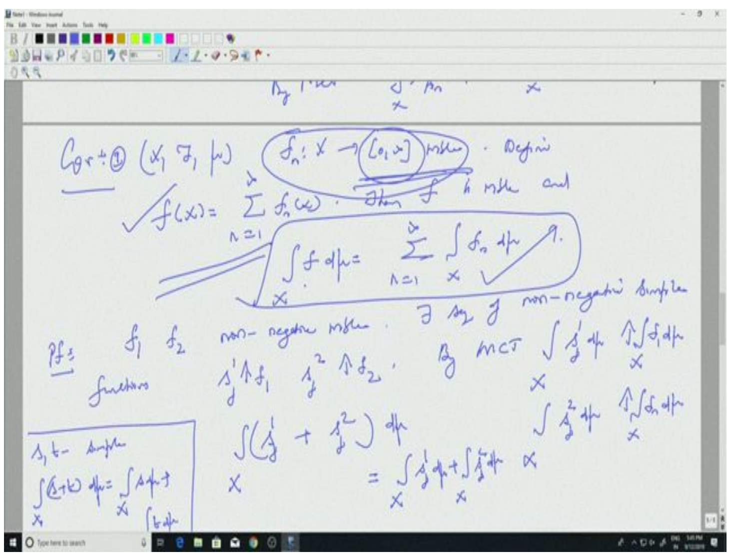

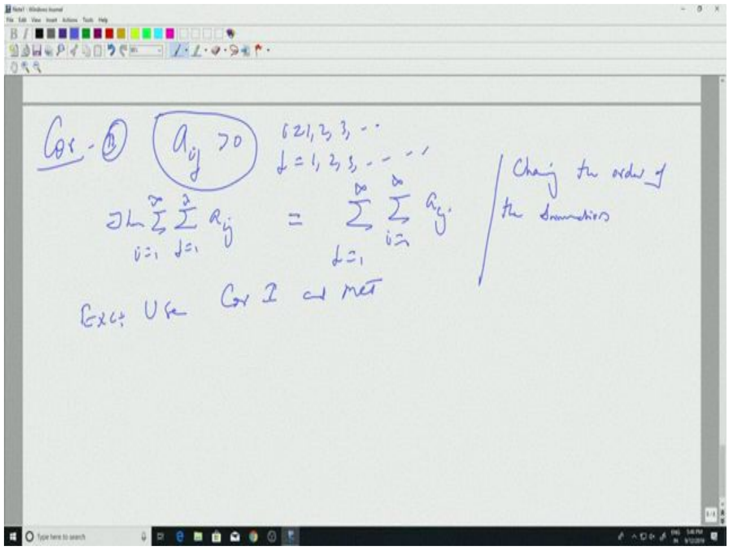

Okay, so, let us look at another corollary. This is something which you have probably seen when you studied series of a positive numbers and so on. Suppose I have numbers AIJ positive I equal to 1, 2, 3 etc. similarly, J equal to 1, 2, 3 etc. okay. Then summation AIJ, J equal to one to infinity, summation I equal to one to infinity, this is the same as summation J equals one to infinity, summation I equal to one to infinity AIJ.

So, remember we are changing the order of the summation, changing the order of the summation right. So, whenever AIJ are positive, you can do that. So, that is an easy consequence of the monotone convergence theorem. So, I will leave it as an exercise to you. Well, how do you see this? So, this is remember summation is an integral right and we just proved that if I add up functions I can change summation and integrals, right that is what we proved? So, use this use this result, use this result and make sure that you are integral over X is the summation.

So, remember the, if you look at natural numbers with counting measure, then each of those can be viewed as an integral. And so, you have one summation and an integral and you know how to interchange them using monotone converges. So, use corollary one okay and MCT.

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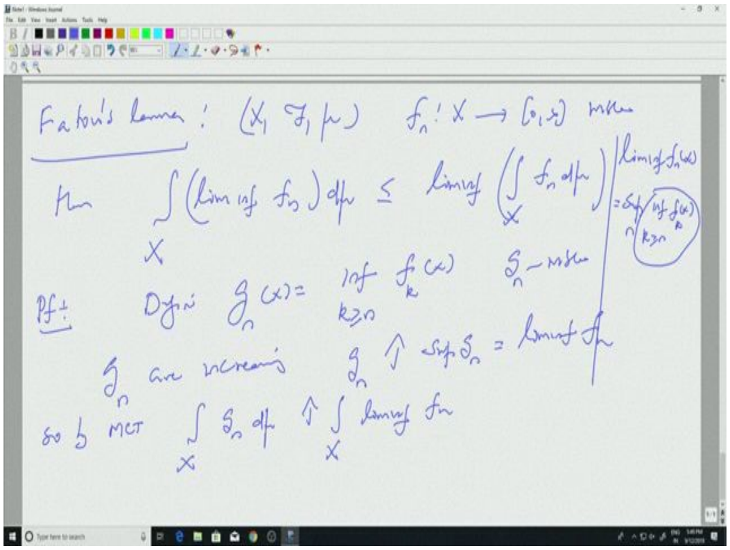

Okay, so the first part we will stop with the following lemma, which is another important result, it is called Fatous lemma. So, as usual, we have X, F, mu, and I have FN. Again, positive measurable functions, measurable okay. Then integral over X lim inf of FN, D mu, this is less than or equal to lim inf of integral FN, D mu okay.

So remember these are number and I am taking the lim inf, here it is the function lim inf FN. So let us recall that lim inf FN at X is well, it is supremum over N infimum over K greater than or equal to N, FKX okay. So, the proof of this is two lines from using MCT. So, all we do is, we look at this as a collection of functions for each N. So, define GN of X to be infimum over K greater than or equal to N. So, for each N, you fix N and look at all K greater than or equal to N and look at FKX right, take the infimum of that. These are positive ones. So, you will have an infimum which may be zero does not matter, but you will have an infimum.

Now you are looking at infimum of various things. And as N increases the set you are looking at us various smaller and smaller. So the infimum will increase, so GN are measurable of course, it goes supremum and infimum of measurable functions are measurable and GNs are increasing, GN are increasing. So it will convert, where does it convert to the? GN increases to supremum of GNs right, because GNs are increasing, but supremum of GN is precisely the lim inf of FN.

So GN increases to lim inf for FN. So, by MCT, the monotone convergence theorem, integral over X, GN D mu will converge to integral over X lim inf of FN, okay.

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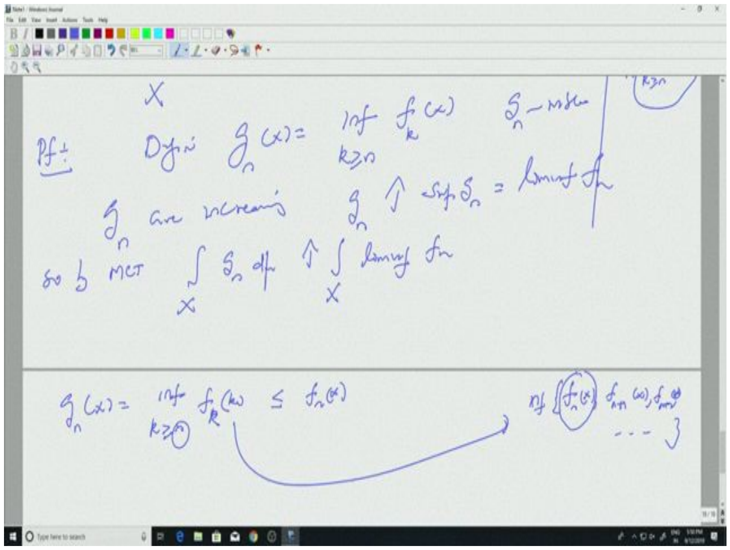

So, let us use one inequality here. So, let us recall them GN, so remember GN was GN of X is infimum of K greater than or equal to N FKX. And so this is of course less than or equal to FNX right? Because I am looking at, so what is this is infimum of FN, FNX, FN plus 1X, FN plus 2X etc. So, FNX is one such element and so infimum will be less than or equal to that.

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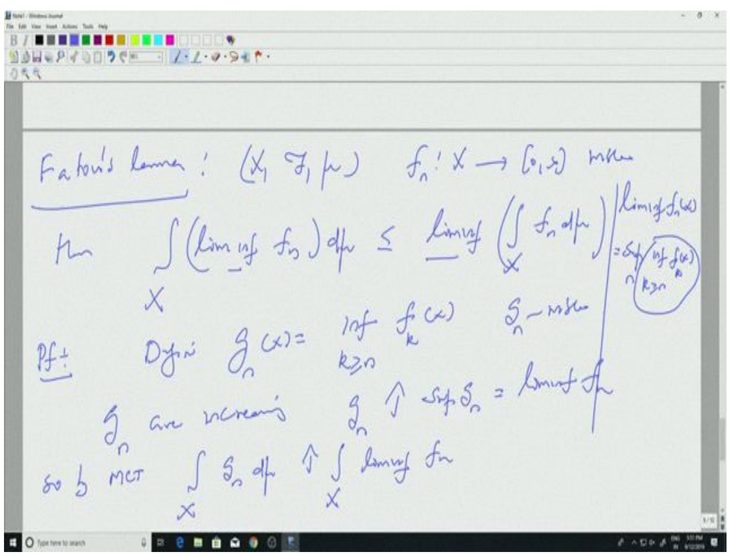

So this tells me again by monotonicity of the integral, I have integral over X GN and D mu to be less than or equal to integral over X FN D mu, I know that this converges to integral over X lim inf of FN D mu and so of course, I can say this is less than or equal to the lim inf of integral over X FN D mu right. These are so what am I using? If I have a sequence AN and so, this is a simple exercise AN and BN positive, positive sequences okay. I know that AN converges to A or AN increases to A. AN increases to A and AN is less than to BN okay. So this implies A is less than to lim of BN okay that is all we are using here.

So, this is called Fatous lemma. So Fatous lemma is an inequality. It is called Fatous in lemma, integral lim inf is less than or equal to lim inf of integral, this is another extremely useful result so. So you have just seen two results were interchanging of the integral and limits are in the changing of the limits and integrals are involved. First is the monotone convergence theorem, which allows you to interchange the limit and integrals, if you have a sequence of increasing measurable functions, okay. If you have a sequence of positive measurable functions, you have an inequality which is given by Fatous lemma okay.

So, so far we have seen monotone convergence theorem and Fatous lemma, which allows us to interchange integrals and limits in the. In the next session, we will look at complex valued functions and how to integrate them. So, far we have integrated only positive functions, starting from positive symbol functions, we have gone to positive functions, we know the integral is linear there. Now, we will extend it to real valued functions, and then to complex valued functions and then one more result of this kind which will allow us to interact limits and integrals will be put.