Design of Mechatronic Systems
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Lecture: 25
Modelling DC Motor with Loads

Now we will start with the next part of the mathematical modelling topic.

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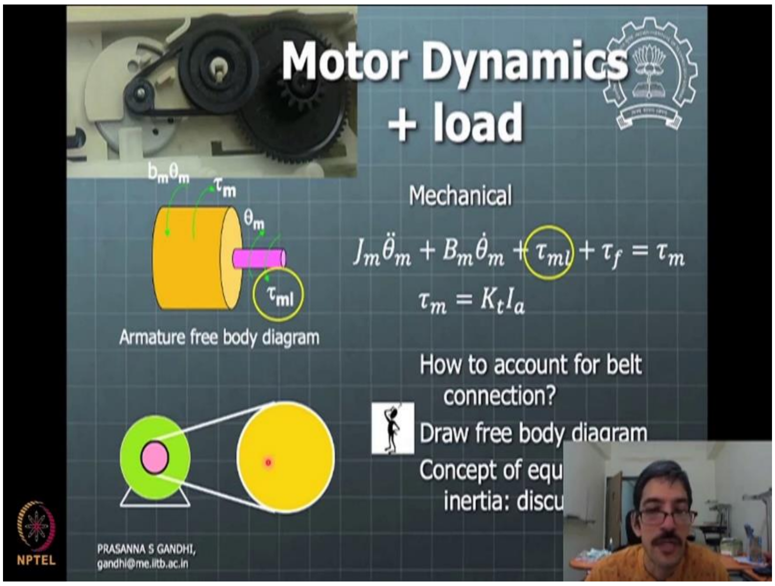

So, what we are going to look at is now, if the motor is added with the load. So, far we have seen the modelling of motor and friction, so different models of friction we have seen. Although models were defined for a block on the on a flat surface, those models can be easily like no seem to be translated into rotary kind of a joint, friction as the rotary joint as in the bearing case in the motor.

So, that part is this friction here and now what we are going to do is like if see typically motors are not alone like then have some load or some application to be driven and when that application is getting driven by the motor this torque corresponding to that on the motor side will come. So, this is a like load torque but which is seen on the motor side, so that is why this torque τ_{ml}, this m stands for torque and the motor side and l is the load.

So, now since we have this load as we have seen the CD round drive some kind of belt drive operated here, so there is a further that better is belt driving gear and then gear is driving like some kind of a tray but we are not kind of considering that for now. We are just considering simple load here, simple gear kind of the belt driven rotary disc here. For such a case how do you think about now getting this belt connection modelling.

If you plug in your kinematics and dynamics of machines fundamentals, this is pretty straight forward. Remember this is a rigid body motion, but this is a rotary motion for the rigid body, this is one rigid body and this is another rigid body and there is a coupling of belt in between. Now this belt is considered to be some kind of a rigid coupling and it is massless kind of coupling, normally belts if you see they are stretchable, one can stretch it and they will have some kind of flexibility we are not modelling that flexibility here.

So, it depends upon how deep you want to get into the modelling, depend upon what is your interest. Say, if you are interested in very accurately driving this output load and a belt is very-very flexible, it can get stretched when it is getting driven, stretches quite a bit then you may need to consider flexibility in the belt this as in some kind of a spring here.

But when it is now, let us consider to begin with a simple case where this is just driven in some kind of a rigid connection of the belt and belt is massless, so belt mass and belt weight inertia we are ignoring because it will be much lesser as compared to the motor and load inertias. So, now with this you can see that we need to draw a free body diagram to get these two rigid bodies considered, how would you draw this free body diagram?

Just pause here for a while and then draw it yourself and then we will see for that. So, we have to draw the free body diagram and then like a see the concept of equivalent inertia, we will what is this concept. But you draw the free body diagram here and then we can see in the next slide.

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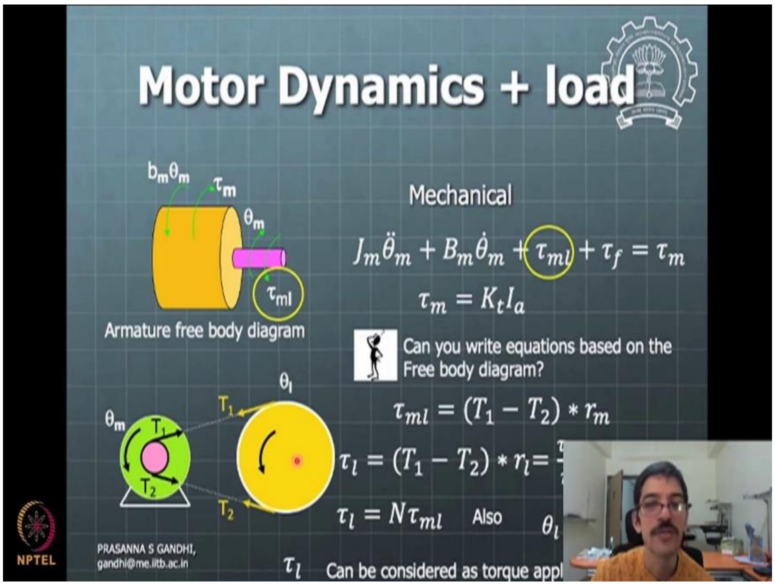

So, you can see their free body diagram is up here. So, you have these tensions in the belt which have different shear. So, if this is a motor which is getting driven on these, in this direction motor which is driving the belt in this direction, so the motor is rotating in this direction tension in the in this side is going to be higher than tension in the other side.

This T1 is more than T2, pretty straight forward to kind of see that T1 will be higher than T2. So, we are assuming the here the belt is not sleeping. So, this difference in the tensions is going to produce some kind of torque here. So, you remember these are our motor equations on motor side mechanical equations and in that we have this load which is we are considering right now load torque.

This load torque now can be written, so, you can you write these equations. How do you write this equation for τ_{ml} here? So, in terms of T1 and T2 one can see that these equations are going to be like this. So, T1 minus T2 times are radius, radius of the motor shaft will get you this τ_{ml}. Now, if we see this expression, this expression will be on the motor side.

What is the connection to the load side? Same tensions on the load side will be existing, T1 and T2 in the belt, belt is continuous and here this action and direction process will exist up there. Now, one can write this τ_{ml} is equal to, τ on the load side is equal, in terms of these two tensions on the load side. So, let us call that torque as τ_l. So, τ_l is a torque which will be applied by the motor to the load, it is like a driven like driving torque the load.

So, nobody is getting driven by this torque τ_l, and when load is driven by torque τ_l on the motor side we see the τ_{ml} as a load torque, τ_{ml}. So, that is how like we can consider this. So, you, if you feel this is, I mean you can see the same thing without even considering this no this τml or τ_l as a separate kind of quantities, these are just variables to kind of physically understand stuff more nicely.

So, if you see that τ l will be T1 minus T2 into r_l, r_l is a radius of this load disk and r_m was a radius on the motor disk here. So, this ratio between the two will be N. So, one can see this τ_l is equal to N times τ_{ml}. So, see since the load side gear is bigger in size the torque that is applied by the motor, if it is on the motor side is τ_{ml}, then on the load side it gets larger. N is a ratio of radius r_l divided by r_m.

So, r_l is larger here and r_m is lesser here. So, this is like a number which is greater than one. So, your torque enhancement happens when you have a bigger size which you all very well known, the same thing will happen with the gear also same way. So, you also can see that θ_l will be equal to θ_m upon N.

So, the gear or this belt arrangement enhances the torque but reduces the θ value or the displacement value, that displacement value θ_l will get reduced by a factor of N. It is straight forward to see, I do not see this, this is a simple kind of kinematics of this belt drive.

We need to further develop equations of motion based on the motion of this side. So, we have written the equation for motion on this side, we have developed some kinematic relationships between this these two sides in terms of both that torque and the θ, but still we do not have the equation of motion coming up on the load side. So, how do we do that? Again go back to our kinematics and dynamics of machines fundamentals.

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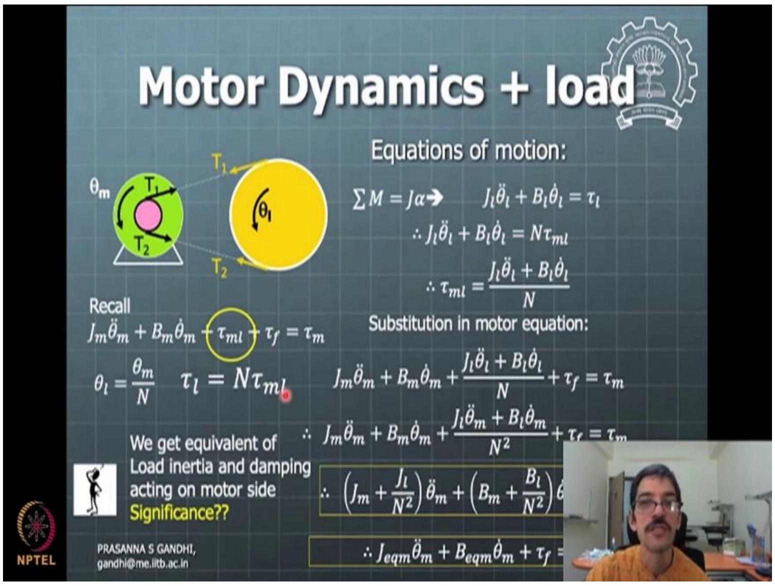

And we can see here that in a summation of moments is equal to J times α will give you equation of motion on the loads side. So, apply this and see what all equations you can get and simplify them and see. So, you pause here for a while and then do this and then we will further see. If your equations are done now, we can see here.

Look, you have $J_l\,\ddot{\theta}$ plus $B_l\,\dot{\theta}$, how this $B_l\,\dot{\theta}$, as we had on the motor side, you have some kind of a bearing friction coming up on the load side also that is what we have written here and that is equal to the torque which is driving the load τ_l. Now, we know this τ_l is nothing but N times τ_{ml} and then now one can gets right equation expression for τ_{ml} in terms of these quantities here.

So, we can substitute that in our original equation. So, this we substitute in our motor side equation and we get now. So, here we are considering still friction torque here. But we are not kind of really considering a model for friction yet, one can assume needs to be zero also to simplify the analysis and see stuff without friction what is that we can see.

So, now one can see that this is load $\ddot{\theta}$ and $\dot{\theta}_1$, now they can be transformed into the same quantity θ_m here. How do you transform that? θ_1 is equal to θ_m upon N, when you use that you get this equation. So, you get this N square term here. And now you collect the like terms which are multiplying the $\ddot{\theta}_1$ and $\dot{\theta}_m$ and then you can see your equation gets to this form.

This is very interesting, you can see now. So, we get something an equivalent inertia up here. So, this this reads like a J_m plus J_l upon N^2. So, see if N is really very large, so say N is three for example, then here inertia J_l is reduced when it is seen by the motor by nine, factor of nine. So, if the gear reduction is really high then as compared to the motor side, the load inertia value will be very as compared to the motor inertia J_m the load inertia seen on the motor side, this term is going to have insignificant kind of value.

So, for this value to be very significant you need to have load inertia much larger than motor inertia, so this is a very interesting physical understanding up here. So, one can kind of say, I will just kind of account for my load side inertia by collecting by saying it may be that at most 4.2 or 4.5 times motor inertia and I can directly say J_m equivalent and is equal to some factor into J_m that can be one simplification one can do depends upon how do you have the inertias on the load side and how much is gear reduction.

So, you can work out some numbers for some specific cases and you will find that this values of low inertia although, it looks very big, this motor is having big inertia when there it is transforming on the motor side it has not much of a new value there. So, this is a very important understanding here. Now, we can reach the same understanding by energy way also. How can you think about that?

So instead of no letting all these free body diagram safe, especially if there are multiple connections. So far we are going to consider only one belt connection and there are gears then there are some rack and pinion or some kind of a gearing system is there, whole of that gearing system would have a degree of freedom still one. So, motor position if it is specified all the positions in the system gets specified.

That is what happens in this case also; you see that when the motor position is specified the loads position is specified time upon N. So, that kinematics is very simple for such a case here. So, in that case we can do some inertia analysis, a little bit of energy analysis and really reach the same conclusion and how we can do that?

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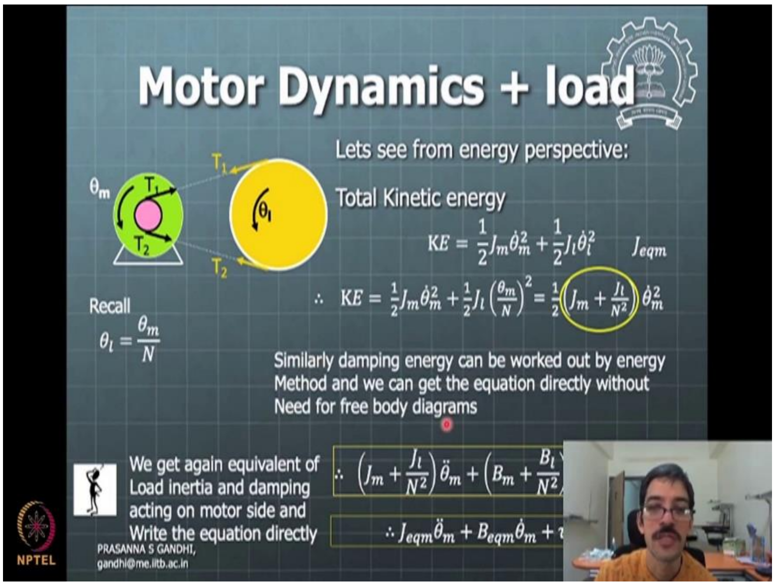

We can see it here. So, total kinetic energy if I want to write here when can say that see from the motor side the kinetic energy is $J_m\ \dot{\theta}_m{}^2$ half and load said it is $J_l\ \dot{\theta}_l{}^2$, there is no translational kinetic energy and there is nothing translating in here So, we are again ignoring the belt effects the belt inertia is ignored here.

So, this kinetic energy again one can see this transformation of load side θ position versus motor side position and after the transformation is applied you can get the same kind of equivalent inertia term here. So, this is equivalent inertia as seen in the motor side. So, by using the same kind of a principle for damping energy which is half times damping factor times $\dot{\theta}_m{}^2$ or $\dot{\theta}^2$ in general…

Then one can write the damping also in a similar way and you get equivalent damping, then one can see that this whole thing translating in to the same kind of a governing equation as we had seen for the case before by using in Newton's analysis by drawing free body diagrams. So, this is a very important kind of consideration that when can directly use total kinetic energy and get to the same point, kinetic energy and of course the damping energy.

So, you can reach this equation in by using the energy way much in a simpler form without drawing the free body diagram. So, this T1, T2 does not come into picture when we when we write this kinetic energy

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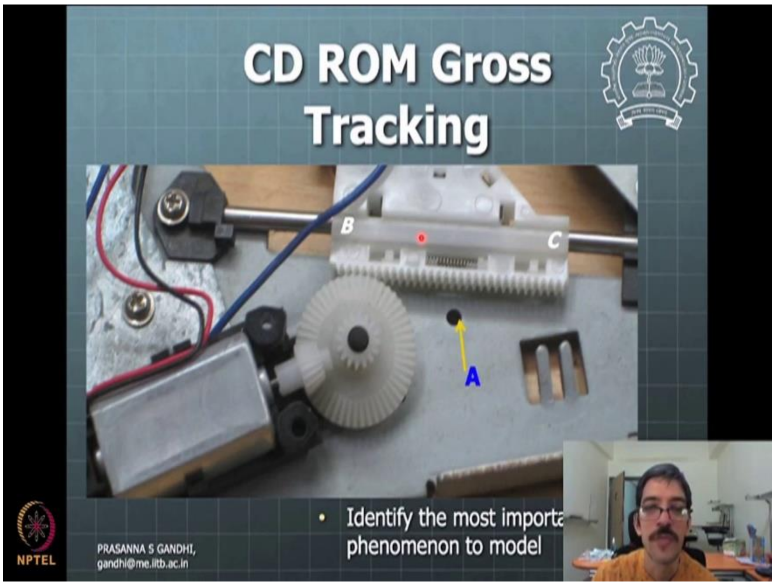

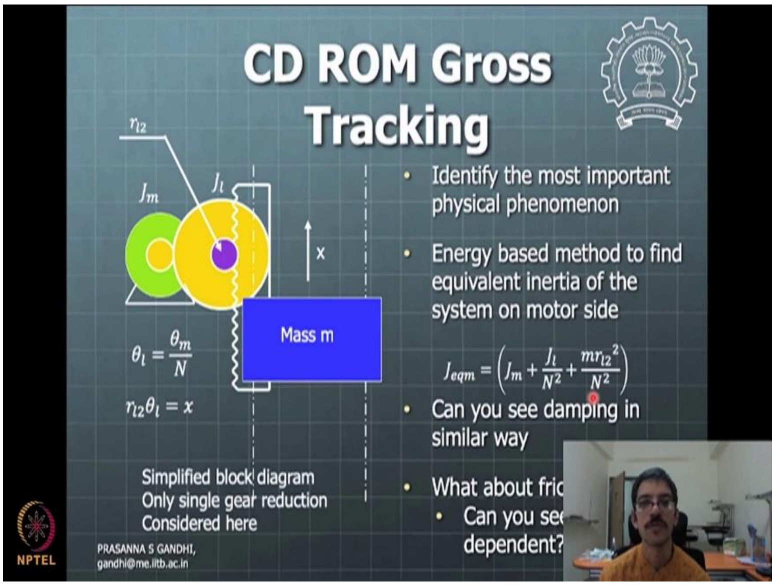

So, now this can be applied to nine little bit more kind of complicated cases also, we have now here one gear driving the other gear, then other gear driving the third gear, then they are driving the rack and pinion and some kind of masses is attached to the rack and pinion which is our CD ROM head mass. So, this is a kind of a system where we can apply this now and no one can see some kind of a simplified form of expressions coming up there. So, you may draw some schematic and try this out yourself if you want to.

I am just kind of giving you some kind of in a small schematic here to just prompt how things will happen. But really you need to draw your own kind of expressions, you derive your own expressions and then know come to the stage and see if they are matching and things like that. So, you can see here there are these motor inertia J_m here, load inertia I am considering only one kind of a gear instead of two gears, two reductions here and then you have the rack and pinion which is driving this mass is the mass of the CD ROM head.

Now, if you see the kinetic energy, again you can add the same kinetic energy for J_m and J_l but the kinetic energy for the mass will be now half m $\ddot{x}^2$. Now we need to know the only the kinetic relationship kinematic relationship between the x and θ. So, θ_l for example θ_l upon N and once the θ_l, then this gear radius r_{l2}, in r_{l2} times θ_l will be x. So, rθ is equal to x that kind of a relationship we can use.

So, this is r_{l2} is a pitch, radius of opinion which is attached to the load side and driving the rack. So, this is how one can get this kinematic relationship, once the kinematic in the relationship is known. One can write a substitute that in the energy expression x dot will be equal to now r_{l2} times now θ_l can substitute here $\dot{\theta}_m$ upon N.

And once you use that into writing the total expression of kinetic energy you are going to get these equivalents inertia on the motor side. So this, now the mass inertia here gets transformed to the motor side in proper dimension form which is required at the motor side is m times r_{l2}^2. So, this is inertia term, this is having same kind of a dimensional quantity as J_l or J_m and divided by N^2.

So, this is very interesting way one can get to the simple system is very easily to resolve, how many different kinds of moving elements are there in the rigid system, kinematically many of mechatronic systems will fall in this kind of a category and one can write nicely this form of equations, one can extend these tool, screw drive also, so you can see that.

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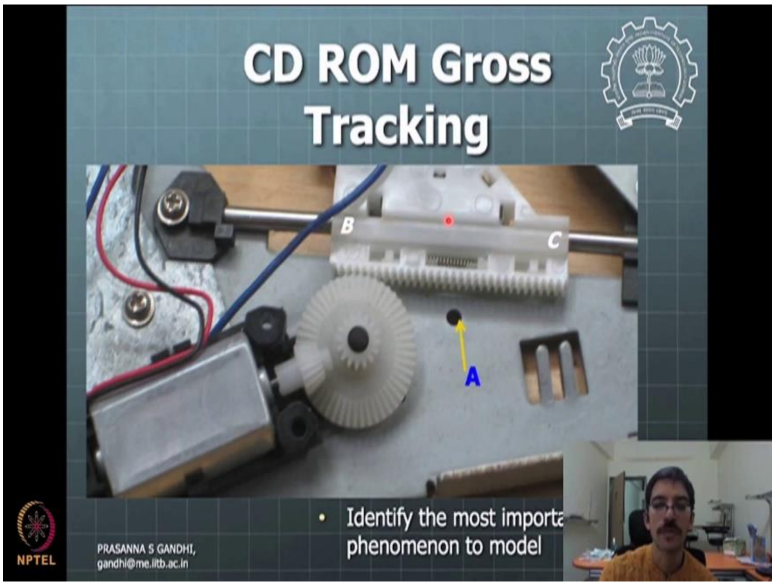

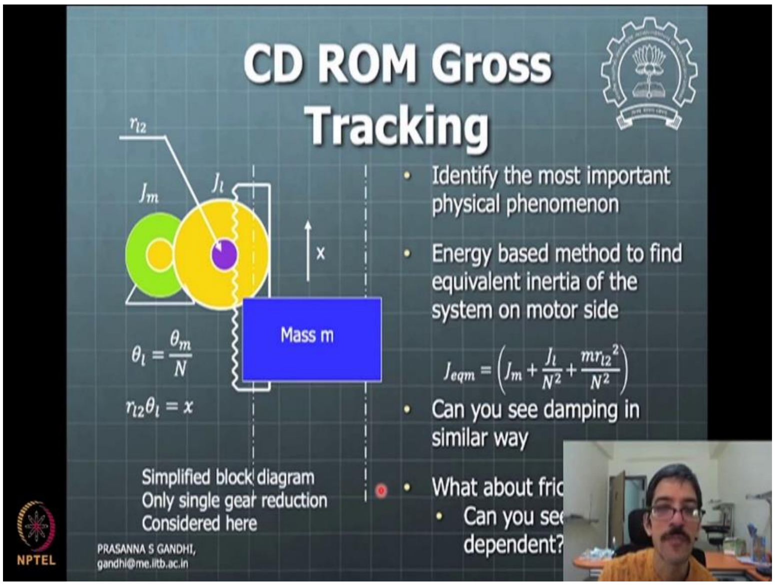

Now, another important part is no friction in this case. So, friction as you see you can see here exists at these points you remember now we had these two points B and C underneath where the contact happens with the rod on the side and there is some kind of contact or happening on the other side as well. So, friction would exist mainly at these points.

Now, if this mass is moving here and mass is having say inertial force x double dot times m in this direction passing to the CG of this mass, this force is going to have some kind of a torque as we see that if we start doing this free body diagram for the for the case of this rack and mass system, rack is connected here and then we assume that these B and C points are somewhere here and here.

So, can you see now how the friction will come up? So, especially is there any connection between friction and inertia is what we want to see here. So for that you really need to draw the free body diagram, I just kind of to give you some small insight into how do we proceed and then maybe you can proceed from there, so let us let us do that.

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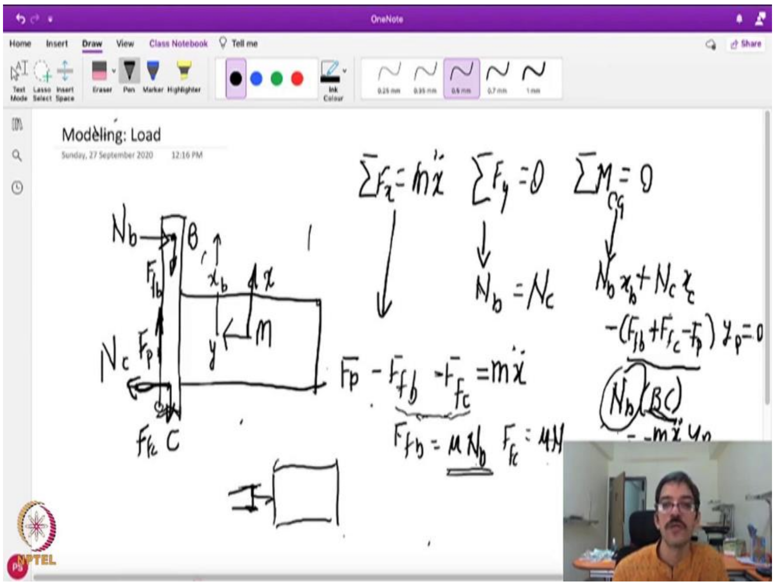

So, if you see here I will draw this rack, so I will just draw some schematics for the rack and this here and then you have this big mass coming up here. Now this mass at the CG of mass you have some x, out on this this mass here and this is x here. Now, these are the points where some kind of a contact is happening with the rod. So there is a reaction that is going to get developed at these points.

So, some reaction will be there at, I call these points B and C and then this reaction here and there it is reaction here. Now, so let us assume this reaction to be saying this direction and this reaction is in this direction. So, if I want to write the equations of motion how will I write here is what you need to think?

So, summation of forces x direction is equal to mass times $\ddot{x}$, what I would, right. Now, at these points where these normal actions are produced, there will be some friction that will be produced, which is in the opposite direction, the motion. So there is some kind of friction which is in the opposite direction with emotion. That is going to happen there.

Then summation of force, say y direction the perpendicular direction are going to be 0, because there is no motion in the y direction. So, let us say this is the y direction here. I just put this right hand y direction that will get. So, can you see now what is happening and I also have equation for the moment, right?

So, this is one rigid body on which some forces are getting applied and for that the summation of moment also is equal to 0 because there is no α, there is no rotation of this body happening here. His body is getting only a translation in the x direction. So, we use these equations of motion and one can see that these reaction, whatever this let say call this N_b here and this is say N_c here.

So, you can get by from this equation you will get N_b is equal to N_c. And then from this equation when can right. So, your friction force here say this is F_b or F_{fb} and at this point we have to say force F_{fc}. So, this friction is always opposing this direction of motion. So, of course, you have this driving force coming from the rack somewhere.

So, let us say driving force point we can assume somewhere here. So, this is a driving force which is coming from the pinion the rack. So, this is F_p. So, F_p is your driving force which is driving the mass in the same direction as x here. So, you see that if you sum the forces up in the direction of F_p minus F_{fb} minus F_{fc} is equal to mass times $\ddot{x}$. Can you see that? Now F_{fb}, F_{fc} are friction forces.

So, F_{fb} is equal to μ times N_b, and F_{fc} is equal to μ times N_c, but we do not know what is N_b and Nc yet. But we just know that they are both equal, so we need to plug in this moment equation here. Now, summation moments you take about what point, say if you take the summation of moments about the centre of mass. So, if you take support centre of mass then you get the equation.

So, M_{cg} here, so then the moment created by N_b is in clockwise direction. So, you have a, some distance between N_b and N_c. So, let us first name the distances here. So, you say this is x_b and x_c. So, x_b is the distance to the point B from the centre of mass, and likewise you will have x_c distance coming up here.

So, you will need to kind of consider N_b times x_b in the clockwise direction plus N_c times x_c and then clockwise direction will be opposed you will have these frictions say moment up here also. So, that is in anti-clockwise direction, so minus from your F_{fb} plus F_{fC}. Now I can consider this F_p also acting a similar kind of a point so minus F_p, p is in the opposite direction, times say y, some distance y_p let say. So, this moment is equal to 0.

This is what I will get from here. Now one can see that F_{fb} plus F_{fc} minus F_p is minus m times $\ddot{x}$. That you can substitute that get this N_b and N_c are equal to this will happen as N_b times say x_b plus x_c which is this total distance between these points B and C. So, this distance BC will be x_b plus x_c and then one can write directly N_b times this distance BC here and this is going to get…

So, you have this y distance y_p still there and then this is equal to minus m times $\ddot{x}$. If F_{fb} plus F_{fc} minus F_p is equal to minus mx double dot this is minus is plus and times y_p. So, this is a kind of relationship you will get for N_b and you substitute that in N_b up here, you get this in this equation and you get some simplification.

So, you will find it here now this friction forces are dependent upon this μ and N value and this N value is itself is dependent upon the $\ddot{x}$ term. So, this reactions are dependent upon $\ddot{x}$ inertia here. So, you can see in this on this block here, see if this distance B and C distance between B and C is very-very small then this N_b and N_c are going to be very large, see this BC value if it is very small, its value is smaller than for the same m times $\ddot{x}$, here, N_b value is going to be very-very large.

See if N_b is higher, this reaction forces are higher than that is not a very good situation because correspondingly your friction forces are going to be higher. So if the friction forces are dependent upon these reaction values, so that is that is why we want to kind of keep this distance BC larger so as to reduce the friction force in the system. You see that?

At some point you may get into a situation that the reactions are so high that you will have very large difficulty in moving the block. So this how one can see the friction coming up in the system and the need for these points B and C to be far away from each other, then your friction forces will be lower. So if you think that we will support only at one-point.

So, say you have this mass and it is guided only at one point on the side here. I had only one-point single point here, then only whatever distance over which it is getting guided, the ends of their loads will get with point B and C. And since they are so close together, you may feel that, there is a situation of jamming coming up there. So, this is how one can think and analyse the friction for such a case. So, clear this part.

So, these are the ways to think about and generate the equations of motion that that you want to use for your system and further use them in the control development later or in a synthesis of a system say in this particular case we want to push these B and C points far away from each other to reduce the friction that is happening in the system and then you get this equation of motion here.

Now, from here it is not very direct connection as in the energy-based connection to the motor side how the friction happens to the motor side that is not the very apparently. You need to write this equation down and then one can get of transform that to the motor side. So, when you have friction forces to be accounted for energies may not help you get it well, but you can still see that the forces are getting still translated in terms of some kind of gear ratio gear ratio and I know rack and pinion kind of drive paramedics.

So, those kinds of parameters may or may scale your friction forces in some way. So, I will leave it to you to explore that further element develop complete model for such a system and see what is what are the equivalent forces coming onto how the friction is coming to the motor side these frictions what we are talking about here. So, that you can derive yourself and look at that in more detail later.

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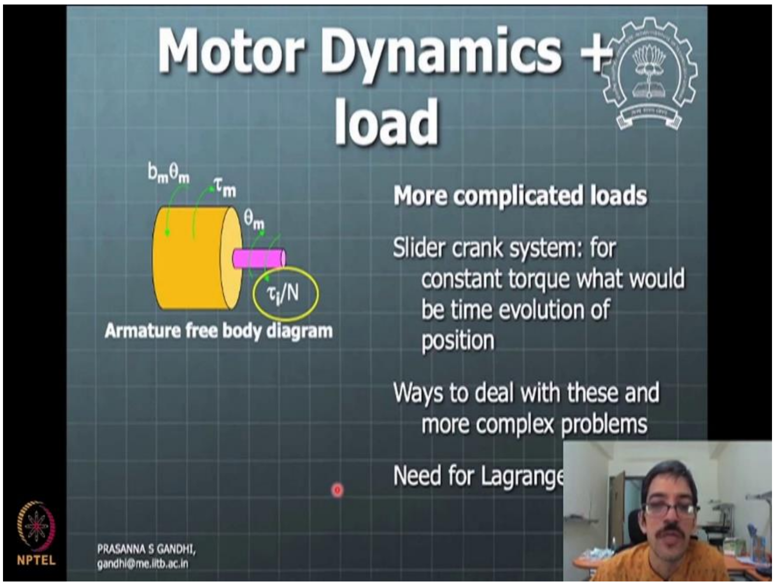

So, now, this friction force whatever you are considering in here can be now considered as one of the models that four models sets we have developed for the friction and then your equations will be fully ready for simulation and analysis purpose or for control development purpose. Now, let us see a little more complicated cases.

So, when you have a slider crank or some kind of a mechanism that is driven by your motor, then you need to see think about say this question, what is the dynamics for such a case? In terms of this question, one can see that for a constant torque, what would we time evolution of the position? So, one of the case problems that we had…

Where you are given us a simple link attached to the motor and it is driven, and what is the time evolution of the position of that second have a system. So, it may, if you have a slide, so this was a simple single link. Now, if you have a slider crank kind of a mechanism, then there will be more complications. Although, again those systems have single degree of freedom, standard crank mechanism, or four bar kind of a system.

Those are again simple systems in terms of degrees of freedom, the only one degree of freedom. But there are complications in terms of non-linear terms that are coming. And one can get to the energy methods then to hundred such things. So, we will then the next topic that we will see is the Lagrangian formulation, which is completely energy base.